Bounded representations of $\mathrm{SL}(2,\mathbb{R})$ why are we interested?

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Ringraziamenti

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Today I want to discuss a family of results about not-necessarily unitary (or even uniformly bounded) representations.

Introduction

- Representations of $G := \mathrm{SL}(2, \mathbb{R})$;
- the principal series of representations;
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Next, G is a locally compact group. It carries a natural translation invariant measure, that is unique up to constant multiples, called its *Haar measure*, and the Lebesgue space $L^1(G)$ of integrable functions is an algebra for convolution:

$$f * g(y) = \int_G f(x) g(x^{-1} y) \, dx.$$

The Lie and enveloping algebras

The Lie algebra $\mathfrak{g}$ of G is the set of 2×2 real matrices X such that $\exp(tX) \in G$ for all $t \in \mathbb{R}$; that is, $\mathrm{trace}(X) = 0$. If $X \in \mathfrak{g}$, then

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If X, Y are in $\mathfrak{g}$, then their product need not be in $\mathfrak{g}$, but their commutator $[X, Y]$ is. Lie algebras are algebras with products that abstract the properties of commutators.

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The Lie algebra $\mathfrak{g}$ has a *universal enveloping algebra*; this is an associative algebra $\mathfrak{U}$ and a linear mapping $L : \mathfrak{g} \to \mathfrak{U}$ such that

$$L[X, Y] = [LX, LY] \qquad \forall X, Y \in \mathfrak{g}.$$

Representations of $\mathrm{SL}(2, \mathbb{R})$

A *Hilbert representation* of G is a strongly continuous homomorphism ρ from G to the group of bounded linear maps with bounded linear inverses on a Hilbert space $\mathcal{H}$. That is,

$$\rho(x^{-1}) = \rho(x)^{-1} \qquad \rho(xx') = \rho(x)\rho(x')$$
$$\text{and} \qquad x_n \to x \quad \Longrightarrow \quad \rho(x_n)\xi \to \rho(x)\xi$$

for all x, x', x_n in G and all ξ in $\mathcal{H}$. This implies that all $\rho(x)$ are bounded, and

$$\|\rho(x)\| \leq C \|x\|^{\alpha} \qquad \forall x \in G.$$

We say that ρ is *unitary* if all $\rho(x)$ are unitary (or equivalently isometric), and that ρ is *uniformly bounded* if $\alpha = 0$.

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We say that ρ is *reducible* if there is a nontrivial invariant subspace $\mathcal{H}^\circ$ of $\mathcal{H}$; that is, $\rho(G)\mathcal{H}^\circ \subseteq \mathcal{H}^\circ$, and *irreducible* otherwise.

Lifting representations to a group algebra

We let $D(G)$ denote the linear space of all smooth (C^∞) functions on G with compact support. This is a subalgebra of $L^1(G)$ under convolution.

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We may also lift a representation ρ of G to a representation of the algebra $D(G)$, by the formula

$$\langle \rho(f)\xi, \eta \rangle = \int_G f(x) \langle \rho(x)\xi, \eta \rangle \, dx.$$

Then $\rho(f) \in \mathcal{L}(\mathcal{H})$. The representation ρ lifts to $L^1(G)$ if and only if it is uniformly bounded. If ρ is unitary, then $\rho(D(G))$ is closed under the adjoint operation.

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A closed subspace $\mathcal{H}^\circ$ is $\rho(G)$-invariant if and only if it is $\rho(D(G))$-invariant.

Lifting representations to the Lie algebra

To the representation ρ of G we may associate the subspace $\mathcal{H}^\infty$ of C^∞ vectors: ξ is C^∞ if $x \mapsto \rho(x)\xi$ is C^∞. We define

$$\rho(X)\xi = \frac{d}{dt}\rho(\exp(tX))\xi|_{t=0} \qquad \forall \xi \in \mathcal{H}_\rho^\infty;$$

then ρ becomes a representation of $\mathfrak{g}$ in $\mathcal{L}(\mathcal{H}^\infty)$; that is, ρ is linear and $\rho([X, Y]) = [\rho(X), \rho(Y)]$.

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Connecting the irreducibility of a representation of $\mathfrak{g}$ and of a representation of G is much trickier, because $\mathcal{H}^\infty$ and $\mathcal{H}$ may be very different spaces.

Unitary representations are good

There are several reasons why unitary representations are better than other Hilbert representations.

First, if ρ is unitary and $\mathcal{H}^\circ$ is an invariant subspace, then so is $(\mathcal{H}^\circ)^\perp$; indeed, for all $x \in G$, all $\xi \in (\mathcal{H}^\circ)^\perp$ and all $\eta \in \mathcal{H}^\circ$,

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Next, if ρ is unitary and irreducible, it is *completely irreducible*, in the sense that, given $\xi_1, \ldots, \xi_J, \eta_1, \ldots, \eta_J \in \mathcal{H}$, we can find a sequence $(f_n)_n$ of elements of $D(G)$ such that

$$\rho(f_n)\xi_j \to \eta_j \qquad \forall j \in \{1, \ldots, J\}.$$

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This in turn leads to the conclusion that the centre of the enveloping algebra maps to scalars.

Unitary representations are good. 2

Perhaps the key reason why unitary representations are good is that there are enough of these to give a Plancherel and an inversion formula. We may write

$$\int_G \bar{f}(x)\, f(x)\, dx = \int_{\hat{G}_r} \mathrm{trace}(\pi_\lambda(f)^* \pi_\lambda(f))\, d\mu(\lambda),$$

where $\hat{G}$ is the set of all irreducible unitary representations of G (modulo unitary equivalence) and $\hat{G}_r$ is a subset thereof, each π_λ is an irreducible unitary representation of G, and μ is the *Plancherel measure*.

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This is the typical form of a Plancherel theorem for a nonabelian locally compact group, and an analogue of the Plancherel theorem in Fourier analysis.

The work of Bargmann

The representations of G were found by Bargmann. He identified four types of irreducible unitary representations of G, called the principal series, the discrete series, the complementary series, and the trivial representation.

We are going to define the principal series using linear algebra.

G acts on $\mathbb{T}$

Consider $\mathbb{R}^2$ as a space of row vectors, and let G act on $\mathbb{R}^2$ by right multiplication. Then G fixes the origin o and acts transitively on $\mathbb{R}^2 \setminus \{o\}$. We obtain an action $t \mapsto t \circ x$ of G on the circle $\mathbb{T}$ by renormalising:

$$t \mapsto tx \mapsto \|tx\|^{-1} tx := t \circ x \qquad \forall t \in \mathbb{T} \quad \forall x \in G.$$

This map respects antipodal points on the circle.

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This map respects antipodal points on the circle.

The Jacobian $J(t, x)$ of this map from $\mathbb{T}$ to itself is $\|tx\|^{-2}$, and

$$\max\{\|tx\|^{-1} : t \in \mathbb{T}\} = \|x\| .$$

Formulae for the principal series representations

We define the representations $\rho_{s,\pm}$ for $s \in \mathbb{C}$:

$$\rho_{s,\pm}(x)f(t) = J(t,x)^{(s+1)/2} f(t \circ x) \qquad \forall t \in \mathbb{T},$$

where we restrict to even or odd functions according to the sign. All $\rho_{s,\pm}(x)$ preserve $D(\mathbb{T})$, and

$$\begin{aligned}
\|\rho_{s,\pm}(x)f\|_{L^2(\mathbb{T})} &= \left(\int_{\mathbb{T}} |\rho_{s,\pm}(x)f(t)|^2 \, dt\right)^{1/2} \\
&= \left(\int_{\mathbb{T}} \left|J(t,x)^{(s+1)/2} f(t)\right|^2 \, dt\right)^{1/2} \\
&\leq \|x\|^{|\mathrm{Re}(s)|/2} \left(\int_{\mathbb{T}} J(t,x) \, |f(t)|^2 \, dt\right)^{1/2} \\
&= \|x\|^{|\mathrm{Re}(s)|/2} \, \|f\|_{L^2(\mathbb{T})} .
\end{aligned}$$

If $\mathrm{Re}(s) = 0$, we have equality.

The principal series

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It is possible to do better than this.

The complementary series

Bargmann proved the Plancherel formula for G; this involves the two principal series of representations that we have just discussed, together with a countable family of representations, called the discrete series, that we shall not discuss.

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Bargmann also showed that when $s \in (-1/2, 1/2) \setminus \{0\}$, it is possible to complete $C^\infty(\mathbb{T})$ in a norm different to the $L^2(\mathbb{T})$ norm, and the representations $\rho_{s,\pm}$ act unitarily in the new norms. More precisely, he showed that, for a suitable family of Hilbert spaces $H^s(\mathbb{T})$,

$$\|\rho_{s,\pm}(x)f\|_{H^s(\mathbb{T})} = \|f\|_{H^s(\mathbb{T})} \qquad \forall x \in G.$$

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This was surprising because these representations do not appear in the Plancherel formula.

Doing better

Ehrenpreis and Mautner showed that, when $|\mathrm{Re}(s)| < 1/2$, the representations $\rho_{s,\pm}$ act uniformly boundedly in the $H^s(\mathbb{T})$ norms. More precisely,

$$\|\rho_{s,\pm}(x)f\|_{H^s(\mathbb{T})} \leq C(s)\,\|f\|_{H^s(\mathbb{T})} \qquad \forall x \in G.$$

This solved a problem in the theory of representations posed by Dixmier. This is like having a strip of positive width in $\mathbb{C}$ of complex exponentials that are bounded.

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Kunze and Stein then showed that it is possible to realise all these unitary and uniformly bounded representations on the same Hilbert space, and they used this realisation to prove what is now known as the Kunze–Stein phenomenon. This is now used in disparate areas including ergodic theory and PDE.

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We (ACD) found the best value of the constant $C(s)$.

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The Baum–Connes conjecture

"In mathematics, specifically in operator K-theory, the Baum–Connes conjecture suggests a link between the K-theory of the reduced C*-algebra of a group and the K-homology of the classifying space of proper actions of that group. The conjecture sets up a correspondence between different areas of mathematics, with the K-homology of the classifying space being related to geometry, differential operator theory, and homotopy theory, while the K-theory of the group's reduced C*-algebra is a purely analytical object."

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This is a difficult conjecture to understand, even for the experts in the field. It was loosely formulated in about 1982, but not made precise until 1994!

The Baum–Connes conjecture. 2

We (ACD) wrote a paper about uniformly bounded representations in 2004, proving the following result for the simple Lie groups of split rank 1.

Theorem

When $|\mathrm{Re}(s)| > 1/2$ *and* $\varepsilon > 0$,

$$\|\rho_{s,\pm}(x)f\|_{H^s(\mathbb{T})} \leq C(s,\varepsilon)\,\|x\|^{|\mathrm{Re}(s)|-1/2+\varepsilon}\,\|f\|_{H^s(\mathbb{T})}$$

for all $x \in G$.

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for all $x \in G$.

At the time we thought this was what was needed to proved the conjecture "with coefficients" for such groups. It then turned out that this was not quie correct, and a 2019 paper now formulates a slightly different result that we should prove to establish the conjecture.

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The nature of mathematics

"We have not succeeded in answering all our problems—indeed we sometimes feel we have not completely answered any of them. The answers we have found have only served to raise a whole set of new questions. In some ways we feel that we are as confused as ever, but we think we are confused on a higher level, and about more important things."

Variously attributed.