Admissible representations of $\mathrm{SL}(2, \mathbb{R})$

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Ringraziamenti

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Today I want to make a link between the representation theory of semisimple Lie groups and a classical question in functional analysis. On Thursday I will discuss some other results about not-necessarily unitary representations.

Introduction

- Representations of $G := \mathrm{SL}(2, \mathbb{R})$;
- the invariant subspace conjecture;
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First, G is a Lie group, that is, a group and a manifold, and the group operations are smooth. We write K for the maximal compact subgroup $\mathrm{SO}(2)$ of G. Tools such as linear algebra and differential geometry are available.

Next, G is a locally compact group. It carries a natural translation invariant measure, and the Lebesgue space $L^1(G)$ of integrable functions is an algebra for convolution:

$$f * g(y) = \int_G f(x)g(x^{-1}y)\,dx.$$

Unitary representations

From early in the 20th century, mathematicians and physicists have studied topological groups and Lie groups and their representations on Hilbert spaces. Hilbert, Weyl, von Neumann, Bargmann, Harish-Chandra, Langlands, ...

We define a (Hilbert) *representation* of G to be a strongly continuous homomorphism ρ from G to the group of bounded linear maps with bounded inverses on a Hilbert space $\mathcal{H}_\rho$. Thus

$$\rho(x^{-1}) = \rho(x)^{-1} \qquad \rho(xy) = \rho(x)\rho(y)$$

$$\text{and} \qquad x_n \to x \quad \Longrightarrow \quad \rho(x_n)\xi \to \rho(x)\xi.$$

If all $\rho(x)$ are unitary operators, then ρ is said to be unitary.

Lie algebras and representations

The Lie algebra $\mathfrak{g}$ of G is the set of 2×2 real matrices X such that $\exp(tX) \in G$ for all $t \in \mathbb{R}$; that is, $\operatorname{trace}(X) = 0$. If $X \in \mathfrak{g}$, then

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To each representation ρ of G we may associate the subspace $\mathcal{H}_\rho^\infty$ of C^∞ vectors: ξ is C^∞ if $x \mapsto \rho(x)\xi$ is C^∞. We define

$$\rho(X)\xi = \frac{d}{dt} \exp(tX)\xi|_{t=0} \qquad \forall \xi \in \mathcal{H}_\rho^\infty;$$

then ρ becomes a representation of $\mathfrak{g}$ in $\mathcal{L}(\mathcal{H}_\rho^\infty)$; that is, ρ is linear and $\rho([X, Y]) = [\rho(X), \rho(Y)]$.

Admissible representations

By Fourier analysis, we may write $\mathcal{H}_m$ for the closed subspace of $\mathcal{H}$ of all ξ such that

$$\rho(k_\theta)\xi = e^{im\theta}\xi \qquad \forall\theta \in \mathbb{R}.$$

Then $\mathcal{H} = \hat{\bigoplus}_{m\in\mathbb{Z}}\mathcal{H}_m$, and the projection $\mathcal{P}_m$ of $\mathcal{H}$ onto $\mathcal{H}_m$ is given by

$$\mathcal{P}_m\xi = \frac{1}{\pi}\int_{-\pi}^{\pi} e^{-im\theta}\rho(k_\theta)\xi \, d\theta.$$

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We say that ρ is *admissible* if

1. each $\mathcal{H}_m$ is finite dimensional, which means that $\mathcal{H}_M \subseteq \mathcal{H}^\infty$;
2. $\bigoplus_{m\in\mathbb{Z}}\mathcal{H}_m$ is finitely generated for (the associative algebra of operators generated by) $\rho(\mathfrak{g})$.

When ρ is admissible, for $X \in \mathfrak{g}$, $\rho(X)\mathcal{H}_m \subseteq \mathcal{H}_{m-1} \oplus \mathcal{H}_m \oplus \mathcal{H}_{m+1}$.

Are irreducible representations admissible?

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Soergel showed that there are irreducible representations on Banach spaces that are not admissible, using the solution of the invariant subspace problem on Banach spaces. His construction generalises to other groups, such as $\mathrm{SO}(n, 1)$.

We (ACD) have recently shown that all irreducible representations of $\mathrm{SL}(2, \mathbb{R})$ on Hilbert spaces are not admissible, using the proposed solution of the invariant subspace problem on Hilbert spaces. Our construction will require more functional analysis to generalise.

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The invariant subspace problem

Let T be a bounded linear operator on a complex Banach space $\mathcal{X}$ of dimension at least 2. The invariant subspace problem is whether $\mathcal{X}$ has a nontrivial closed subspace $\mathcal{X}_0$ such that $T\mathcal{X}_0 \subseteq \mathcal{X}_0$. The question is related to the existence of eigenvectors: if T has an eigenvector ξ, then $\mathbb{C}\xi$ is an invariant subspace.

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We use these proposed solutions to treat representations of G.

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The topological algebra $D(G)_m$

Let ρ be an irreducible Hilbert representation of G. Irreducibility means that, if $\xi \neq 0$, then $\text{span}\{\rho(x)\xi : x \in G\}$ is dense in $\mathcal{H}$, or equivalently, $\{\rho(f)\xi : f \in D(G)\}$ is dense in $\mathcal{H}$. Here

$$\rho(f)\xi = \int_G f(x)\rho(x)\xi \, dx.$$

This is equivalent to saying that the operators $\mathcal{P}_m \rho(f) \mathcal{P}_n$ are (weakly) dense in the space of linear operators from $\mathcal{H}_n$ to $\mathcal{H}_m$.

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This is equivalent to saying that the operators $\mathcal{P}_m\rho(f)\mathcal{P}_n$ are (weakly) dense in the space of linear operators from $\mathcal{H}_n$ to $\mathcal{H}_m$.

In particular, when $n = m$, we see that $\{\rho(f) : f \in D(G)_m\}$ is dense in the space of linear operators on $\mathcal{H}_m$, where $D(G)_m$ is the set of all smooth compactly supported functions on G with the transformation property

$$f(k_\theta x k_\varphi) = e^{-im(\theta+\varphi)} f(x) \qquad \forall x \in G \quad \forall k_\theta, k_\varphi \in K. \qquad (*)$$

The space $D(G)_m$ is a commutative topological algebra for convolution (this is miraculous.)

Why are Hilbert representations admissible?

We shall show that $D(G)_m$, or, more precisely, a suitable completion $\bar{D}(G)_m$ thereof, has a single topological generator f_m.

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We have to show that $\dim(\mathcal{H}_m) \leq 1$ for all m. Suppose for some m that $\dim(\mathcal{H}_m) \geq 2$. Then $\rho(f_M)$ has a nontrivial invariant subspace $\mathcal{H}_m^\circ$, by the proposed solution to the invariant subspace problem. Since f_m generates $\bar{D}(G)_m$, this subspace is invariant for $\{\rho(f) : f \in \bar{D}(G)_m\}$. It then follows that $\mathcal{H}_m^\circ \oplus \sum_{m' \neq m} \mathcal{H}_{m'}$ is a nontrivial invariant subspace for ρ, and this is a contradiction.

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To complete the proof, we need to define "a suitable completion" and show that this space has a single generator. This involves functional analysis, harmonic analysis and complex analysis.

Details

We claim that there exist constants C and α such that

$$\|\rho(x)\| \leq C \|x\|^{\alpha} \qquad \forall x \in G.$$

To see this, observe that $x \mapsto \rho(x)\xi$ is continuous on the compact set $\|\|^{-1}([1, e])$, and hence bounded. The Banach–Steinhaus theorem then implies that there exists a constant C such that

$$\|\rho(x)\| \leq C \qquad \forall x \in \|\|^{-1}([1, e]).$$

If $\|x\| \leq e^k$, then we may write x as a product of at most k factors, each in $\|\|^{-1}([1, e])$, and then $\|\rho(x)\| \leq C^k$.

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Write $\omega(x) := \|x\|^{\beta}$, and $L^1(G, \omega)$ for the space of all f such that

$$\|f\|_{\omega} := \int_G |f(x)| \, \omega(x) \, dx < \infty.$$

Details. 2

When β is big enough, the subspace $L^1(G,\omega)_m$ of $L^1(G,\omega)$ functions with the invariance property $(*)$ is our completion $\bar{D}(G)_m$. It is a commutative Banach algebra, and can be treated using harmonic analysis.

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The Fourier transform (or the Gel′fand transform) maps $L^1(G,\omega)_m$ to a space of bounded even holomorphic functions in a strip

$$\{z \in \mathbb{C} : |\mathrm{Re}(z)| \leq (\beta + 1)/2\}.$$

The dense subspace $D(G)_m$ of $L^1(G,\omega)_m$ maps to a space of such functions that are entire and vanish rapidly at infinity in all vertical strips.

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Let Γ be the vertical straight line contour from $(\beta + 3)/2 - i\infty$ to $(\beta + 3)/2 + i\infty$ in $\mathbb{C}$.

Details. 3

By Cauchy's integral formula, if $f \in D(G)_m$, then

$$\begin{aligned}\hat{f}(z) &= \frac{1}{2\pi i}\left(\int_{\Gamma} \frac{\hat{f}(w)}{w-z}\,dw + \int_{-\Gamma} \frac{\hat{f}(w)}{w-z}\,dw\right)\\ &= \frac{1}{2\pi i}\left(\int_{\Gamma} \frac{\hat{f}(w)}{w-z} + \frac{\hat{f}(w)}{w+z}\,dw\right)\\ &= \frac{1}{\pi i}\left(\int_{\Gamma} \frac{w\,\hat{f}(w)}{w^2-z^2}\,dw\right).\end{aligned}$$

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$$= \frac{1}{2\pi i}\left(\int_\Gamma \frac{\hat{f}(w)}{w-z} + \frac{\hat{f}(w)}{w+z}\,dw\right)$$
$$= \frac{1}{\pi i}\left(\int_\Gamma \frac{w\,\hat{f}(w)}{w^2-z^2}\,dw\right).$$

Formally,

$$f = \frac{1}{\pi i}\left(\int_\Gamma w\,\hat{f}(w)\,r_w\,dw\right),$$

where $\hat{r}_w(z) = (w^2 - z^2)^{-1}$.

Details. 4

We shall show that there are $L^1(G,\omega)_m$ functions r_w such that $\hat{r}_w$ is the function $z \mapsto (w^2 - z^2)^{-1}$ and $\|r_w\|_\omega \leq C\,|w|^4$, and r_w lies in the closed subalgebra of $L^1(G,\omega)_m$ generated by $r_{(\beta+3)/2}$, for all w for which $\mathrm{Re}(w) \geq (\beta+3)/2$. Since $\hat{f}$ is bounded and decays rapidly at infinity on Γ it follows from our formal statement that f is a limit of sums of functions r_w as w varies on Γ, so f is also in the closed subalgebra of $L^1(G,\omega)_m$ generated by $r_{(\beta+3)/2}$.

Details. 5

Let us take the existence and norm estimate for granted, and that if $w_n \to w$, then $r_{w_n} \to r_w$ in $L^1(G,\omega)_m$. Observe that if $|\tilde{w}^2 - w^2| < \|r_w\|_w$, then

$$\hat{r}_{\tilde{w}}(z) = \frac{1}{\tilde{w}^2 - z^2} = \frac{1}{(w^2 - z^2)} \frac{1}{\left(1 - \frac{w^2 - \tilde{w}^2}{w^2 - z^2}\right)} = \sum_{k \in \mathbb{N}} \frac{(w^2 - \tilde{w}^2)^k}{(w^2 - z^2)^{k+1}},$$

whence $r_{\tilde{w}} = \sum_{k \in \mathbb{N}} (w^2 - \tilde{w}^2)^k r_w^{*(k+1)}$. This sum converges in $L^1(G,\omega)_m$, and so $r_{\tilde{w}}$ lies in the algebra generated by r_w.

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We deduce that the set of $\tilde{w}$ such that $\mathrm{Re}(\tilde{w}) \geq (\beta + 3)/2$ and $r_{\tilde{w}}$ is in the closed subalgebra of $L^1(G, \omega)_m$ generated by $r_{(\beta+3)/2}$ is open. The set is also closed, by our assumption that the subalgebra is closed, and hence all $r_{\tilde{w}}$ have the desired property.

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First, a *matrix coefficient* of a representation ρ is a function $x \mapsto \langle \rho(x)\xi, \eta \rangle$, where $\xi, \eta \in \mathcal{H}$. The matrix coefficients of the irreducible representations of G may be written in terms of special functions, and the Fourier transform of a function is given by integration against these.

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Next, there is a Plancherel theorem and an inversion theorem for G, that relate certain functions and their Fourier transforms. At least formally, we can write $f \in L^1(G, w)_m$ in terms of its Fourier transform $\hat{f}$, again with integrals involving special functions.

The behaviour is of the relevant special functions is understood well enough to be able to say that certain $\hat{f}$ really do correspond to f in $L^1(G, \omega)_m$, and to estimate their norms.

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