Geography, Mathematics, Art, etc.

World map, Ṭūsī Salmānī, The marvels of creation (1175). Copied in Baghdad, in 1388.

A map from al-Khwārizmı ’s version of Ptolemy’s *Geography*, representing the Nile, from its sources until its Delta. When the drawing is placed vertically, the upper horizontal line represents the equator. From a manuscript dated 1036.

Al-Idrīsī's world map, from a 1456 copy of his Kitāb Rujar (Book of Roger of Sicily; the Tabula Rogeriana), (a command of King Roger II of Sicily (1105-1154)). Manuscript was copied in Cairo in 1456.

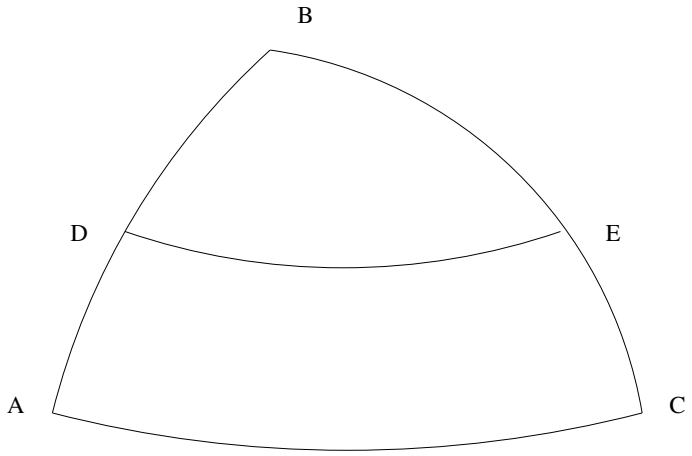
1. The problem of drawing geographical maps

There is no mapping from a region of the sphere to the Euclidean plane which preserves distances up to a scale.

A consequence of the following

Menelaus of Alexandria, Prop. 27 of the *Spherics*:

In a spherical triangle ABC, let D and E be the middles of the sides AB and BC, and DE the segment joining them. Then, $DE > AC/2$.



The problem of geography: To find mappings from the sphere to the plane which minimize distance distortion.

A more general problem:

Top study the distorsion of maps between surfaces: Euclidean, non-Euclidean (spherical, hyperbolic, of variable curvature, etc.) mixed (from Euclidean to spherical, etc.) that minimize distorsion.

Darboux, from a talk at the Rome 1908 ICM titled *Les origines, les méthodes et les problèmes de la géométrie infinitésimale* (IVth ICM, Rome, 1908)

> Like many other branches of human knowledge, infinitesimal geometry was born in the study of practical problems. The Ancients were already busy in obtaining plane representations of the various parts of the Earth, and they had adopted the idea, which was so natural, of projecting onto a plane the surface of our globe. [...]

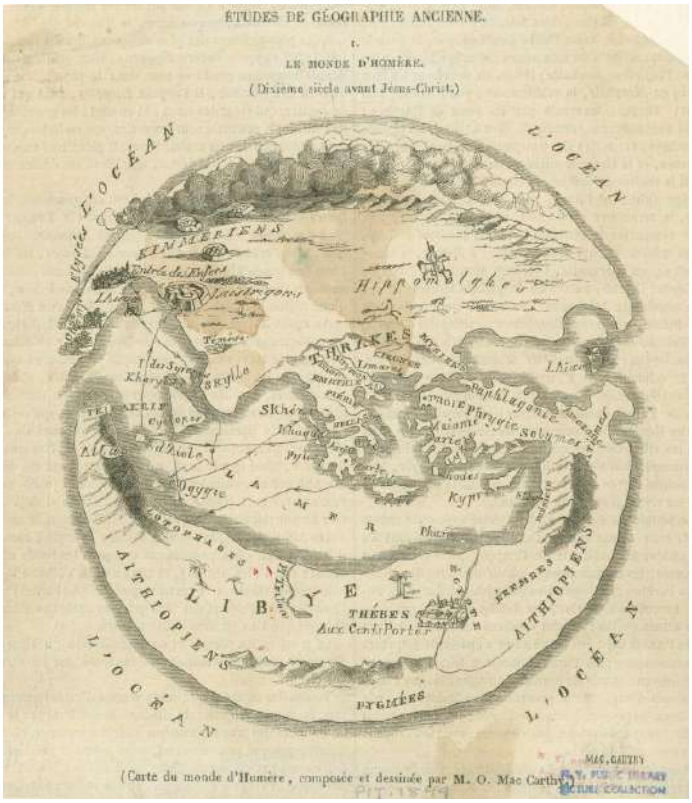
2. A list of some of the mathematicians who worked on these problems

Thales, Eratosthenes, Hipparchus, Ptolemy, Euler, Lagrange, Lambert, Gauss, Chebyshev, Bonnet, Laplace, Darboux, Milnor, Thurston, etc.

3. Geography as an old subject

Homer's Odyssey: Odysseus' 10 years wanderings on his way back to Ithaca, after the end of the Trojan war:

descriptions of lands and regions of the Ancient World, names of cities, mountains, coasts, harbors, distances between them, etc.



A reconstruction of a Homeric map by O. Mac-Carthy, 1849, *Studies of ancient geography*.

Thales of Miletus (c. 624–548 BC).

N. Halma, in the introduction to his edition of Ptolemy's *Geography*: Thales is the real father of Geography and Astronomy.

— Predicted the date of a solar eclipse which took place around 585 AD;

— assembled information collected by travelers and compared it with information he gathered by observing the celestial sphere;

– concluded that the Earth is spherical and that the stars describe circles around the pole;

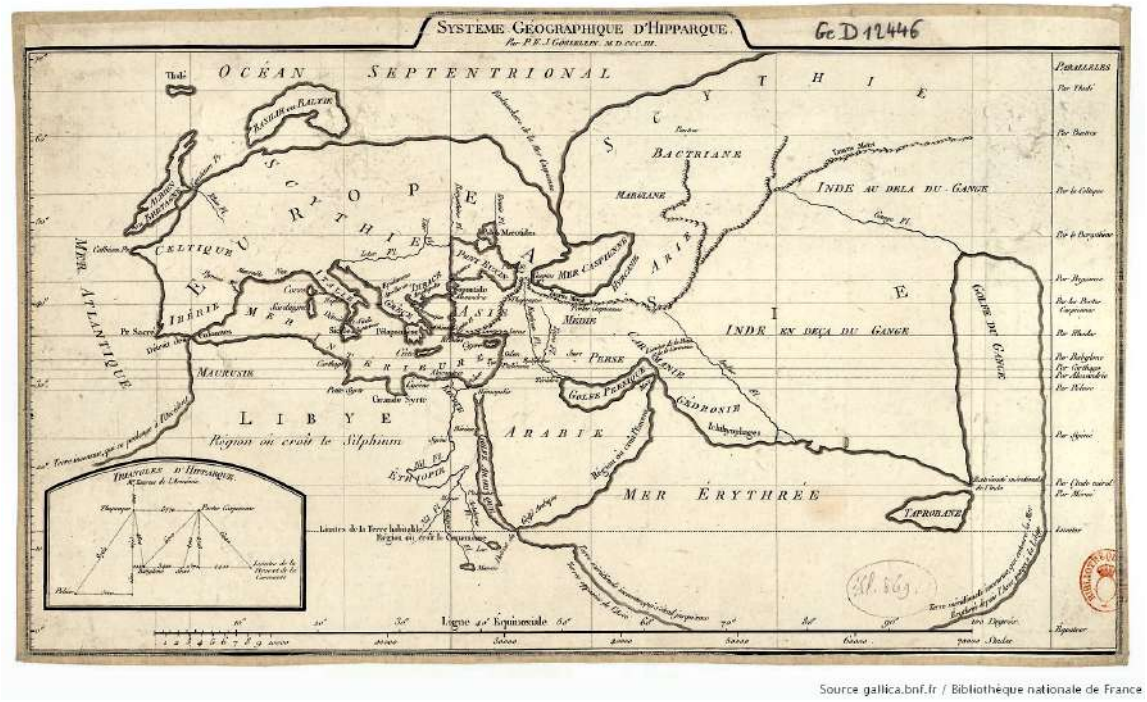
– a precise measure of the height of Egypt's pyramids using their shadows.

etc.

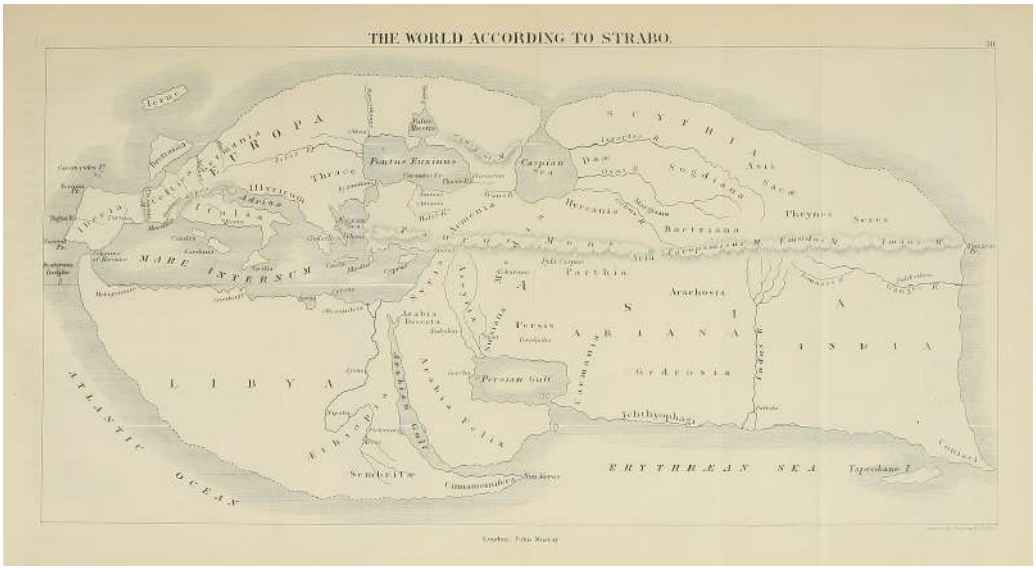
Anaximander of Miletus (c. 610–546 BC), a disciple of Thales.

Strabo, in his Geography, reports that Anaximander was the first to circulate a geographical map, an information he got from Eratosthenes.

Diogenes Laërtius (Lives of the eminent philosophers): "Anaximander held that the Earth, which is of spherical shape, lies in the midst, occupying the place of a centre; that the moon, shining with borrowed light, derives its illumination from the sun; further, that the sun is as large as the Earth and consists of the purest fire."



A map drawn according to Hipparchus' geographical system (P.-F.-J. Gossellin, 1803).



A reconstruction of Strabo's map representing the Ecumene, by Edward Bunbury (1811–1895).

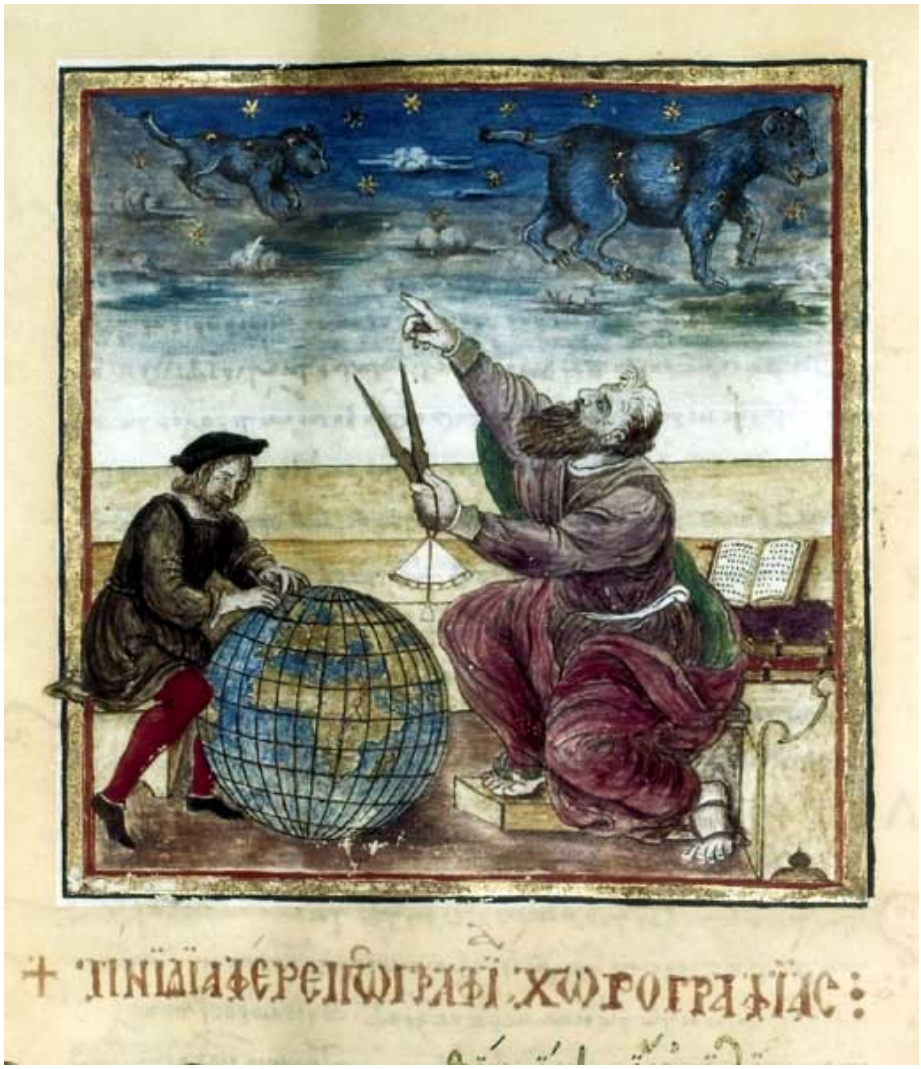
Strabo (1st c. BC-1st c. AD), at several places of his *Geography*, says that the general form of the known world is that of a chlamys (a great mantle), and he gives indications on the shape of each country within it.

4. Ptolemy on goography

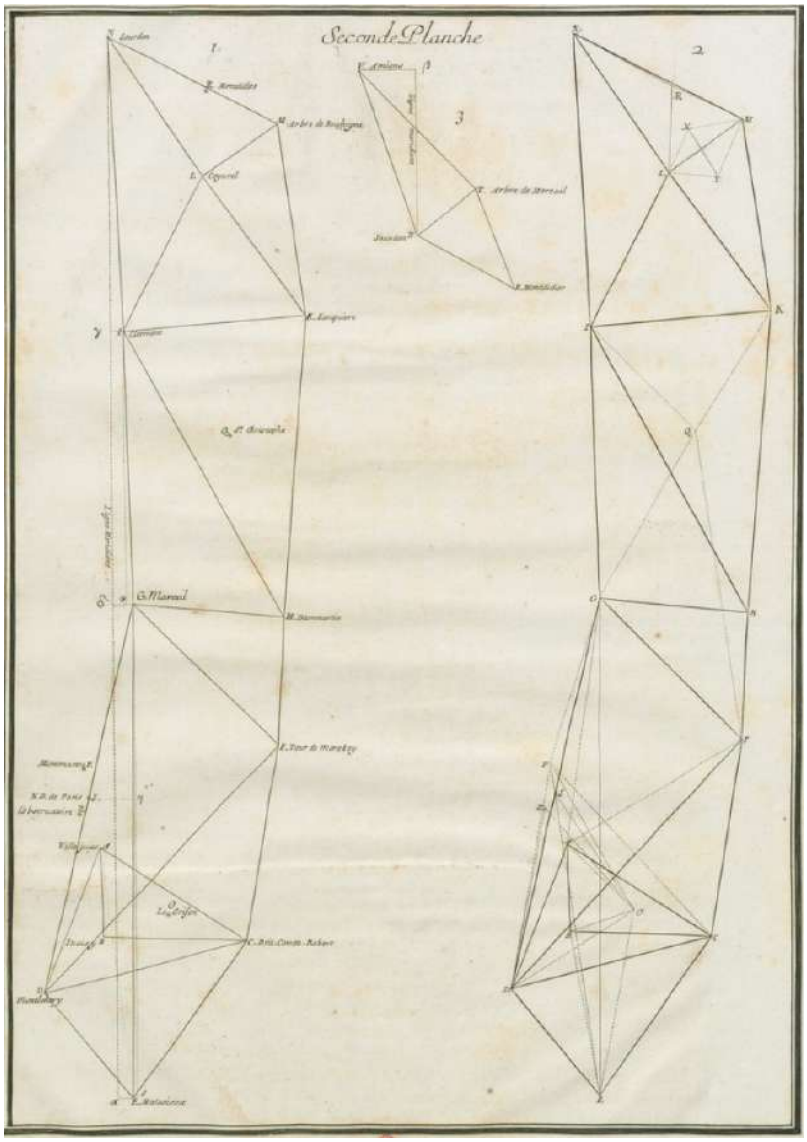
Ptolemy of Alexandria (100-168):
the *Geography*.
Written after the *Almagest* (Syntaxis Mathematica).

Ptolemy uses, for the calculation of distances on the surface of the Earth:

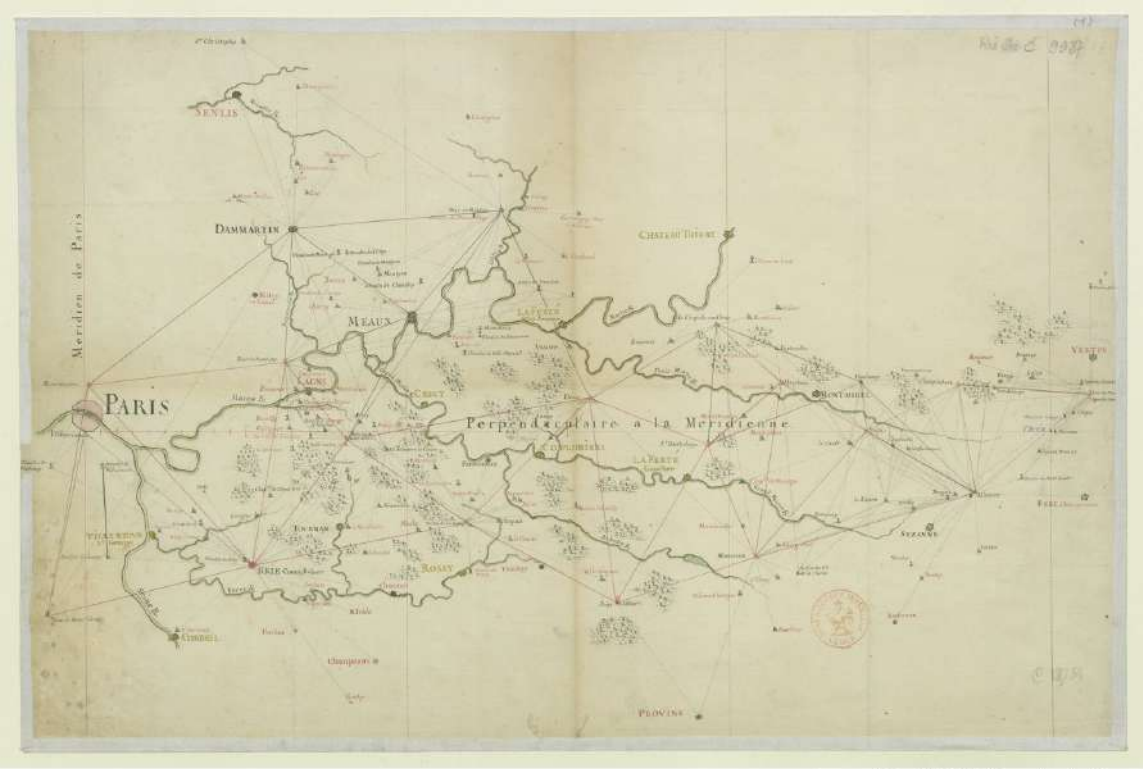
- spherical trigonometry,
- astronomy,
- the method of triangulations.



From a manuscript of Ptolemy's *Geography*, 15th c.

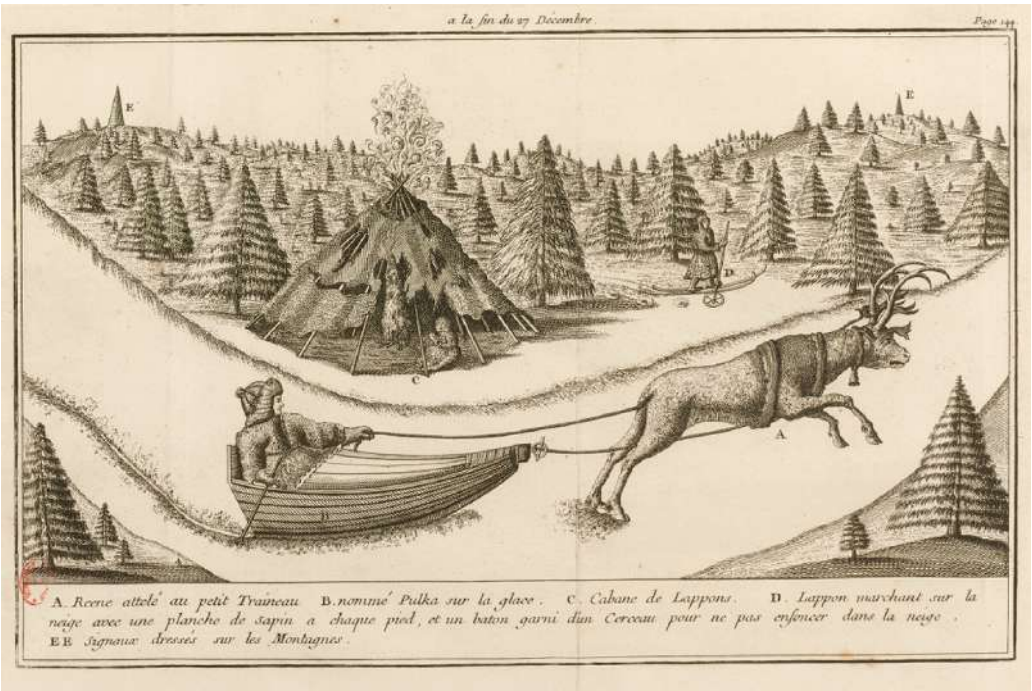


From Abbé Picard (1620 – 1682), *Mesure de la Terre*



Source gallica.bnf.fr / Bibliothèque nationale de France

A triangulation of a region "perpendicular to the Paris meridian," by César-François Cassini de Thury and Jean-Dominique Maraldi.



A. Renne attelé au petit Traineau B. nommé Pulka sur la glace. C. Cabane de Lappons. D. Lappon marchant sur la neige avec une planche de sapin a chaque pied, et un baton garni dun Cerceau pour ne pas enfoncer dans la neige.
EE. Signeaux dressés sur les Montagnes.

Chap. 20 of the *Geography*: Ptolemy mentions a method for drawing geographical maps, which he attributes to Marinus of Tyre (c. 70–130 AD), in which meridians and parallels are sent to perpendicular lines in the plane.

‘

Problem

To study maps of minimal distortion from the sphere to the plane, where meridians and parallels are sent to two perpendicular systems of lines.

Problem

To study the maps of minimal distortion from the sphere onto the plane, where distances are preserved along two parallels

Colbert presenting to Louis XIV the members of the Royal Academy of Sciences (1666). Maps are everywhere.

5. Euler

- Elements of Spheroidal Trigonometry Drawn from the Method of Maxima and Minima (1755);
- On the Representation of the Spherical Surface on the Plane (1778);
- On the Geographical Projection of the Surface of the Sphere (1778);
- On Delisle's Geographical Projection Used for a General Map of the Russian Empire (1778);

6. Euler et Delisle

Projection de Delisle.

Comme dans Ptolémée, les proportions des distances sont préservées le long de deux parallèles (dans le cas de l'Empire russe, ce sont ceux qui bordent cet empire.)

De plus,

- les méridiens sont représentés par des lignes droites,
- les méridiens et les parallèles s'intersectent perpendiculairement,
- la projection est fidèle le long des méridiens.

Atlas russicus

20 maps: 19 maps of the Russian provinces and on map of the entire Russian Empire

Academiae Imperialis Scientiarum Petropolitanae

St. Petersbourg, 1745

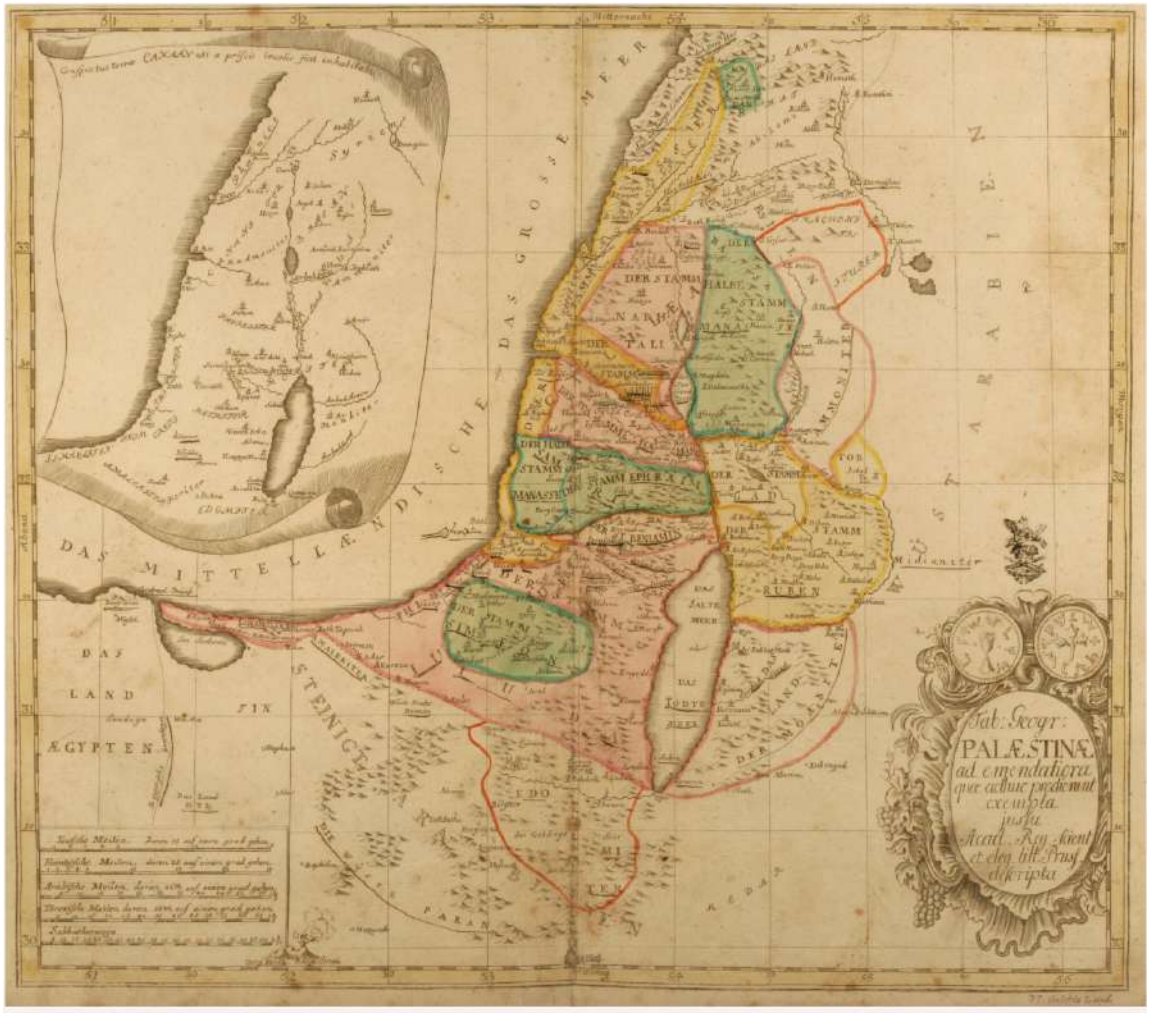
Atlas Geographicus omnes orbis terrarum regiones in XLIV tabulis exhibens

44 maps of all the regions of the Earth

Acad. Roy. des Sc. et Belles Lettres de Prusse, Berlin, 1753.

44 maps. Edited by Euler, who also wrote the preface.

Map of the Russian Empire: No. 20 of Euler's Russian atlas.



Map of Palestine, from Euler's *Atlas Geographicus* (Berlin, 1753)

7. Lagrange and Euler

Lagrange, *Sur la construction des cartes géographiques, I* et *II*, 1779.

The general problem:

Study the conformal maps (angle-preserving) such that:

the meridians and the parallels are sent to perpendicular systems of lines (foliations) that can be chosen arbitrarily.

Euler, in the *On the Representation of the Spherical Surface on the Plane* (1778) had studied a similar problem.

Problem

Study the conformal geographical maps, where the images of the meridians and parallels are sent to a given pair of foliations of the plane.

J. Milnor, *A problem in cartography* (1969):

Milnor defines the
distorsion ("Lipschitz distortion") os a map $f : \Omega \to \mathbb{R}^2$, where Ω is an open subset of the sphere S^2.
He solves the problem in the case where Ω is a disc.

- σ_1 and σ_2 are the best constancs such that

$$\sigma_1 d(x, y) \leq d(f(x), f(y)) \leq \sigma_2 d(x, y).$$

The *distortion* of f:

$$\delta = \log(\sigma_2/\sigma_1).$$

The problem of cartography: To construct maps where δ is the smallest possible.

Preliminary Theorem (Milnor) For any Ω such that $\mathrm{Card}(\Omega) \geq 3$, there a map of minimal distorsion.

Questions:

• Uniqueness, differntiability, constructibility, and an estimate of the distortion of such a map of minimal distortion?

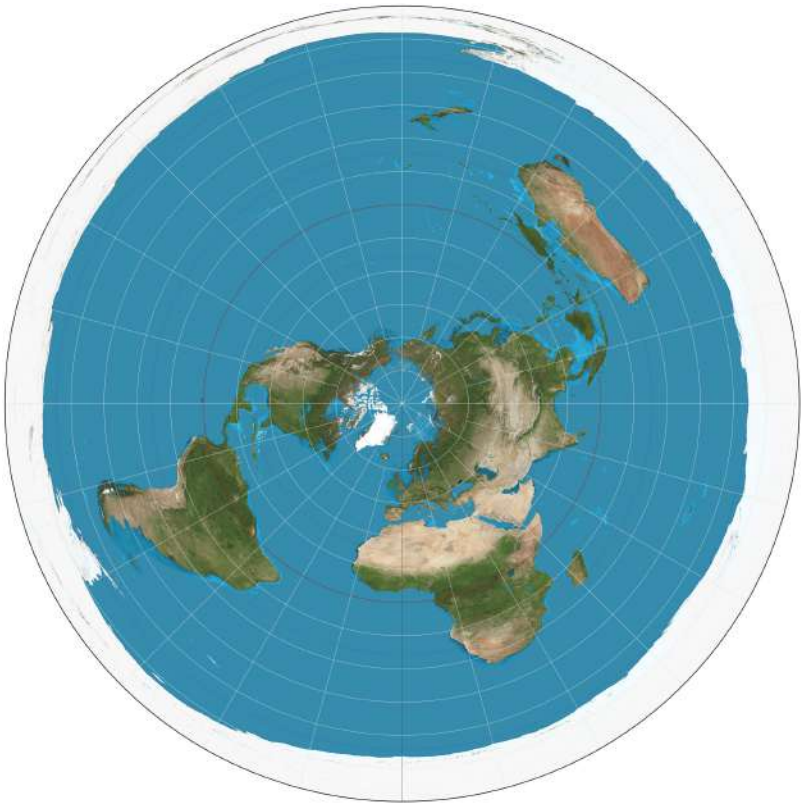
Answer is unknow,

except in a particular case:

Theorem (Milnor): *The response to all these questions is positive if* Ω *is a spherical disc.*

Unknown in the more general case.

The map was already known, and used by cartographers: “equidistant azimutal projection.”:



8. Chebyshev

1856: *Sur la construction des cartes géographiques, I* and *II*.

Chebyshev, Like Lagrange (and Euler in some memoirs) studied *conformal* geographical maps

If f is conformal, then at each point there is a well-defined infinitesimal scale $\sigma(x)$.

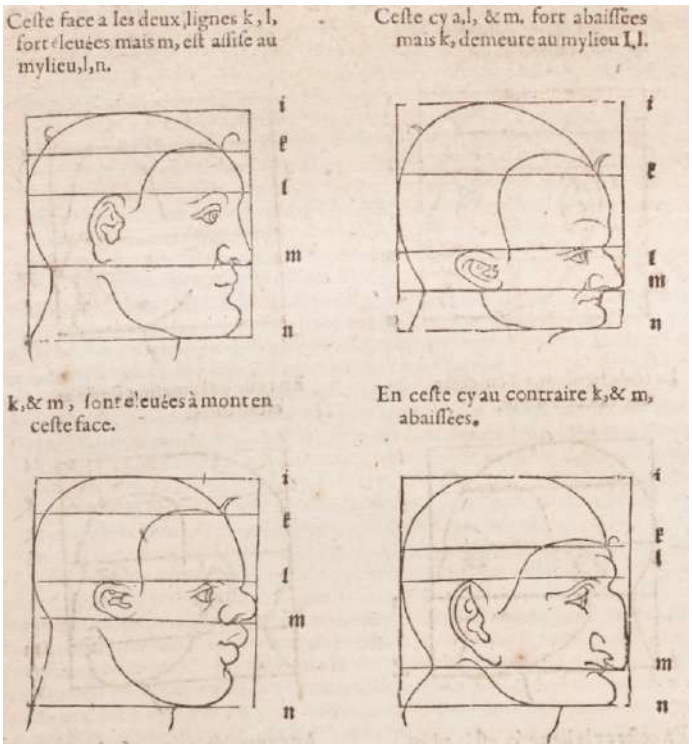
Theorem (Chebyshev) Ω simply connected, which boundary of class C^2. Then there is (up to similarity at the target) a unique best conformal map f. It is characterized by the fact that the infinitesimal scale function is constant on the boundary.

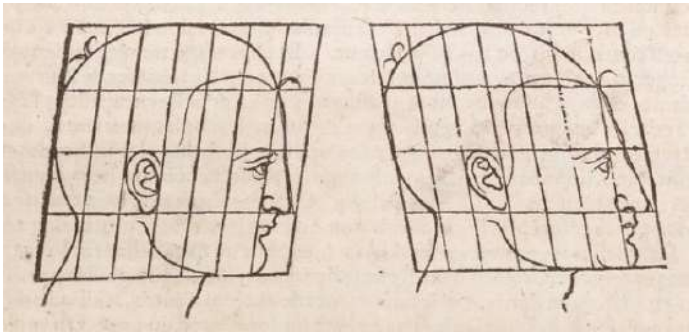
Proofs by Gravé, Darboux (1911) and Milnor (1969), using the Laplace equation.

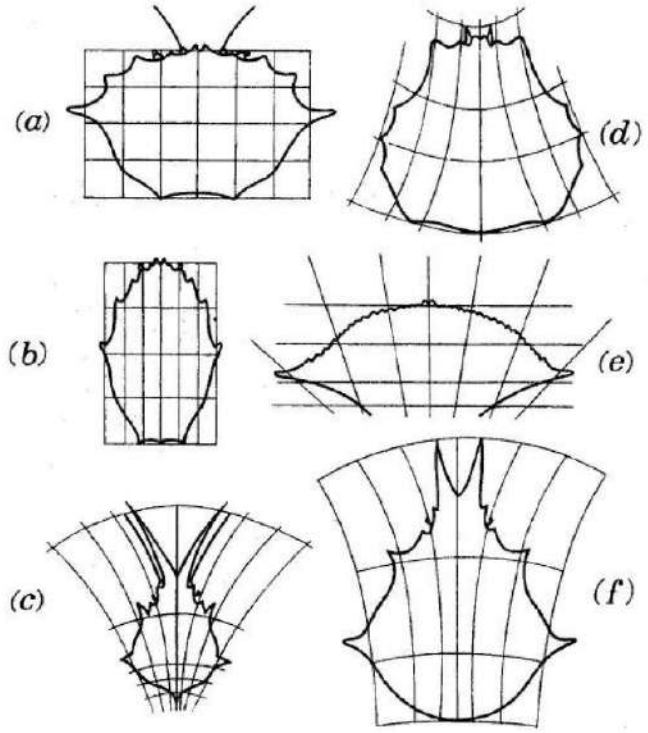
9. Maps between surfaces

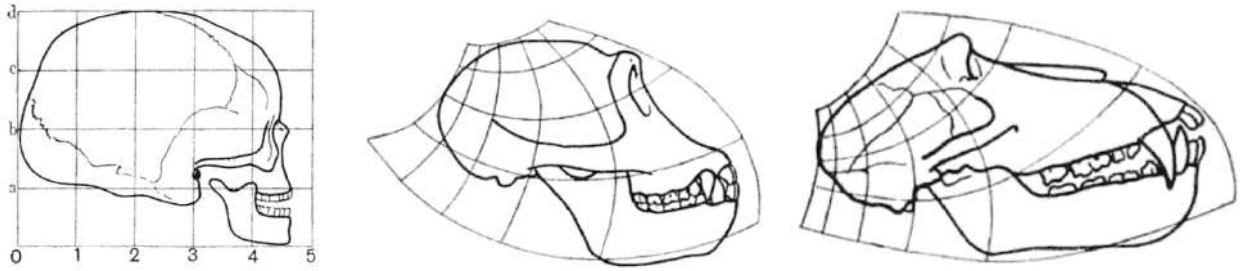
Cf. A. Papadopoulos, Maps with least distortion between surfaces: from geography to brain warping, Notices of the AMS, November 2019, p. 1628-1639.

From Dürer's Four Books of Human Proportions (Nuremberg, 1528)









From D'Arcy Thompson's *Growth and form.*

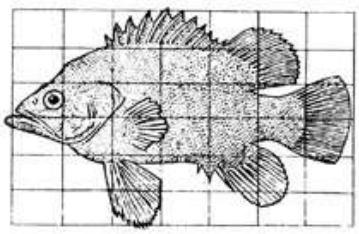


Fig. 150. *Polyprion.*

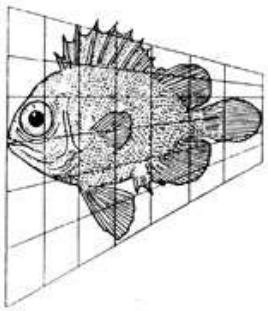


Fig. 151. *Pseudopriacanthus altus.*

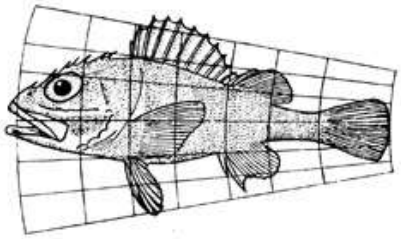


Fig. 152. *Scorpaena* sp.

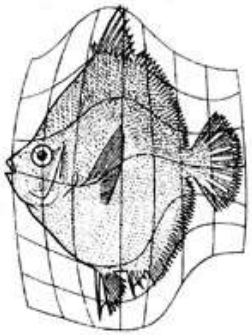
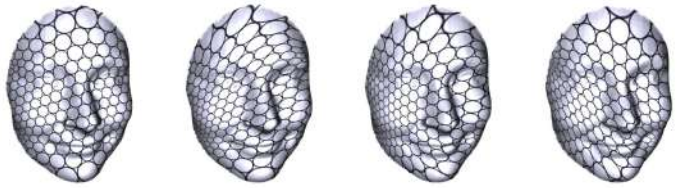
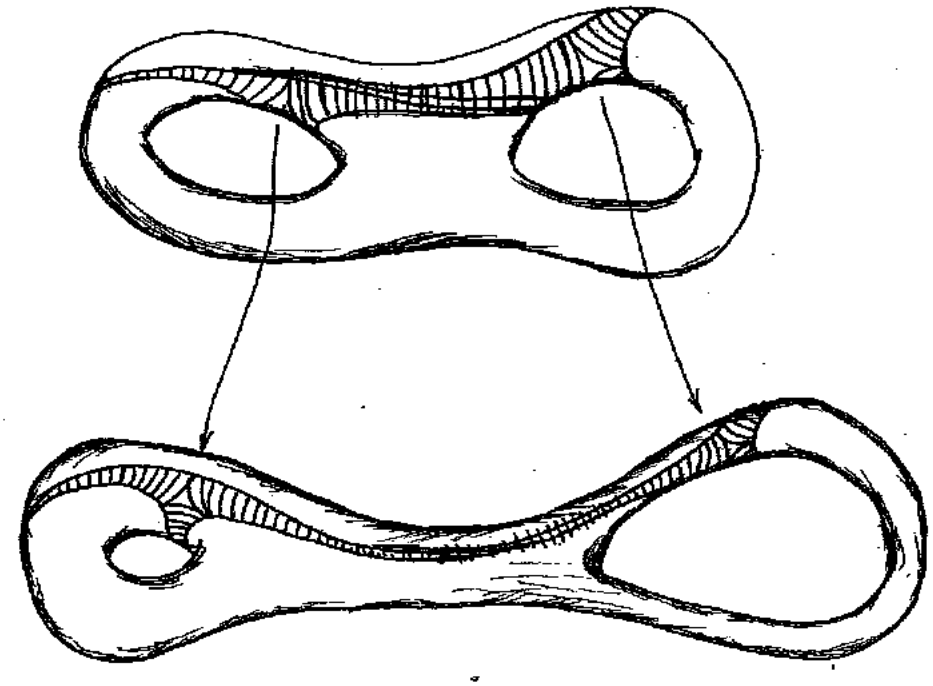


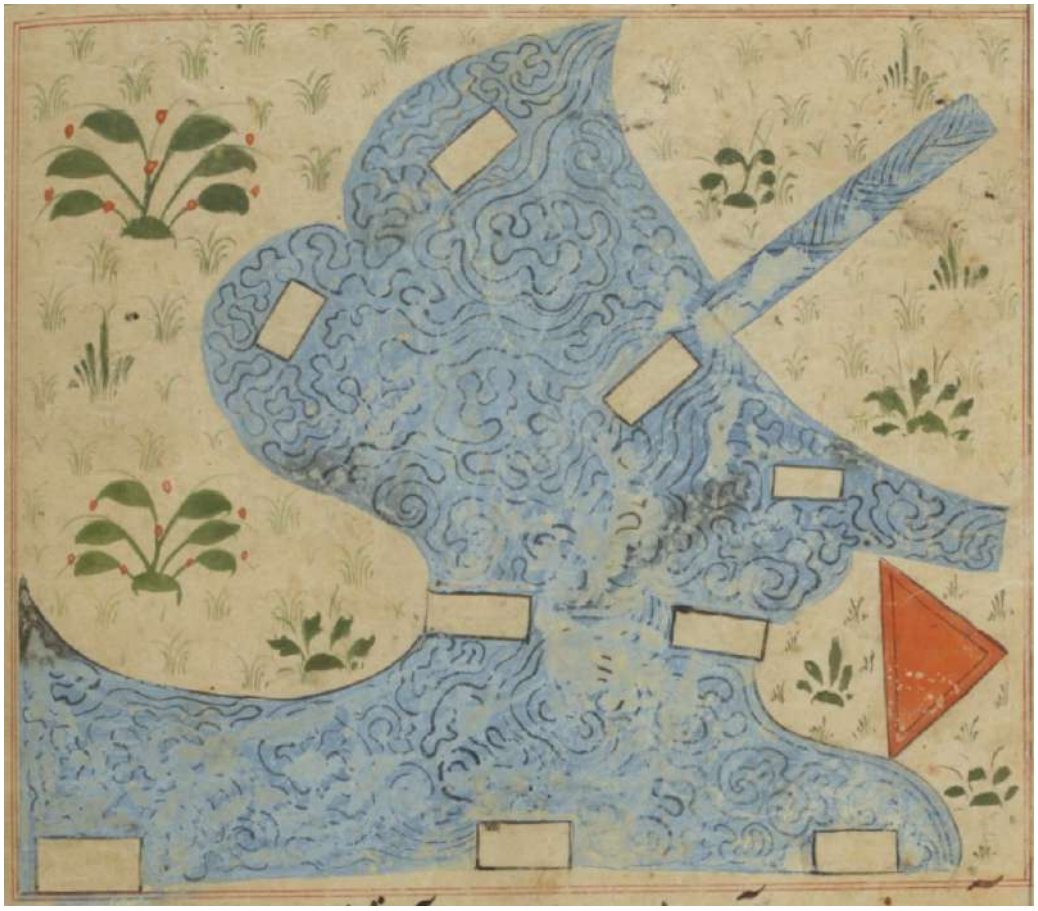
Fig. 153. *Antigonia capros.*



Circle packings used instead of rectangular coordinates as maps between surfaces in the computation of discrete quasiconformal mappings



World map, Ṭūsī Salmānī, The marvels of creation (1175). Copied in Baghdad, in 1388.

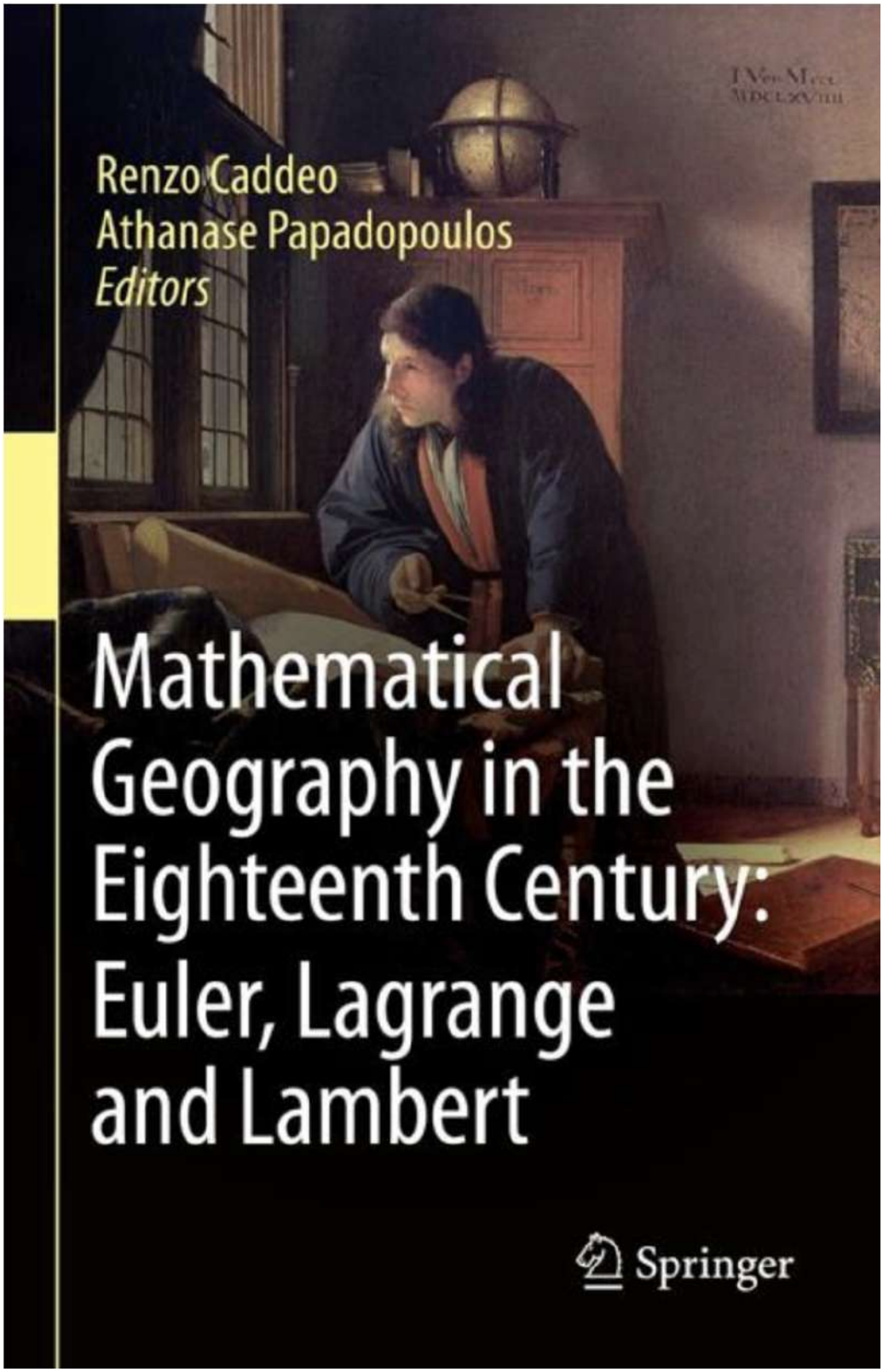


Map of the Persian Gulf, Ṭūsī Salmānī, The marvels of creation, 1175. Copied in Baghdad, in 1388.

10. Conclusion

A quote from Chebyshev:

> Mathematics already traversed two epochs: one where the problems were set by the gods (for instance, the problem of the duplication of the cube) and another one where the semi-gods, like Fermat, Pascal and others, set them. Today we have entered the third period, where the questions to be solved are raised by the needs of humanity.


Renzo Caddeo
Athanase Papadopoulos
Editors
Mathematical Geography in the Eighteenth Century: Euler, Lagrange and Lambert
Springer