THE UNIVERSAL ℓ-FRATTINI COVER OF $\mathrm{PSL}_2(q)$

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Abstract. In this note we study the universal ℓ-Frattini cover $\pi\colon \mathrm{Fr}_\ell(G) \to G$ of the finite group $G = \mathrm{PSL}_2(q)$. In case that the ℓ-Sylow subgroups of G are cyclic or if $\ell = 2$ and q is odd we determine the ℓ-Lefschetz number $\mathcal{L}_\ell(\mathrm{Fr}_\ell(G))$ of $\mathrm{Fr}_\ell(G)$ (Thm.A). From this number one can compute the rank of the pro-ℓ Sylow subgroups of $\mathrm{Fr}_\ell(G)$ (Cor.B) which will be called the projective ℓ-rank of G. The formula for the projective ℓ-rank of a direct product of finite groups shows (Prop.C) that the dimension of the minimal projective cover of the trivial module $\mathbb{F}_\ell$ divided by the order of an ℓ-Sylow subgroup of G plays the role of an extra ordinary multiplier. This number will be calculated for all finite groups $G = \mathrm{PSL}_2(q)$ and all primes ℓ (Thm. D).

1. Introduction

Let G be a finite group and let ℓ be a prime number. A surjective homomorphism $\eta\colon X \to G$ of profinite groups is called an *ℓ-Frattini extension*, if the kernel of η is contained in the pro-ℓ Sylow subgroup $\Phi_\ell(X)$ of the Frattini subgroup of X. It is well-known that every finite group G has a *universal ℓ-Frattini cover* $\pi\colon \mathrm{Fr}_\ell(G) \to G$ (cf. [7], [9], [10], [11, §20.7], [16]). It has the fundamental property that for any ℓ-Frattini extension $\eta\colon X \to G$ there exists a surjective map of profinite groups $\pi_\circ\colon \mathrm{Fr}_\ell(G) \to X$ making the diagram

$$\begin{tikzcd} & \mathrm{Fr}_\ell(G) \arrow[dl, dashed, "\pi_\circ"'] \arrow[d, "\pi"] \\ X \arrow[r, "\eta"] & G \end{tikzcd} \tag{1.1}$$

commute. Of course, if ℓ does not divide the order of G, $\pi\colon \mathrm{Fr}_\ell(G) \to G$ is just an isomorphism. However, in general it is quite difficult to describe the isomorphism type of $\mathrm{Fr}_\ell(G)$ explicitly. One knows that $\pi\colon \mathrm{Fr}_\ell(G) \to G$ coincides the the minimal ℓ-projective cover (cf. [11, Prop.20.33]). In particular, $\mathrm{Fr}_\ell(G)$ is a finitely generated ℓ-projective profinite group, and thus its cohomological ℓ-dimension is less or equal to 1 (cf. [14]). Hence every pro-ℓ Sylow subgroup $\hat{L}$ of $\mathrm{Fr}_\ell(G)$ is a finitely generated free pro-ℓ group (cf. [21, §I.4.2, Cor.2]). We define the *projective ℓ-rank* of G by

$$\mathrm{proj}_\ell(G)\colon = \mathrm{rk}_\ell(\hat{L}) = \dim_{\mathbb{F}_\ell}(\mathrm{Hom}(\hat{L}, \mathbb{F}_\ell)), \tag{1.2}$$

where $\mathbb{F}_\ell$ denotes the finite field with ℓ elements. The finitely generated left $\mathbb{F}_\ell[G]$-module

$$A(G,\ell)\colon = \ker(\pi)/\Phi_\ell(\ker(\pi)) \tag{1.3}$$

Date: February 11, 2007.

2000 *Mathematics Subject Classification.* 20J05.

is called the *ℓ-Frattini module* of G. Several aspects of this module have been studied by R. L. Griess and P. Schmid in [13]. In [24] we introduced the *ℓ-Lefschetz number* $\mathcal{L}_\ell(\hat{G})$ for a finitely generated ℓ-projective virtual pro-ℓ group $\hat{G}$. One can think of the ℓ-Lefschetz number as a type of ℓ-Euler-Poincaré characteristic which is supported on all ℓ-regular[1] $\hat{G}$-conjugacy classes. For the universal ℓ-Frattini cover $\mathrm{Fr}_\ell(G)$ of some finite group G one can compute the ℓ-Lefschetz number from the Brauer character of the ℓ-Frattini module $A(G,\ell)$, i.e., one has

$$\mathcal{L}_\ell(\mathrm{Fr}_\ell(G)) = \frac{1}{|G|}\sum_{g\in G_{\ell'}} (1-\chi_{A(G,\ell)}(g^{-1}))[\tilde{g}], \tag{1.4}$$

where $G_{\ell'}$ denotes the set of all elements of G of ℓ'-order, $\chi_{A(G,\ell)}$ denotes the Brauer character of $A(G,\ell)\otimes_{\mathbb{F}_\ell}\bar{\mathbb{F}}_\ell$, and $[\tilde{g}]$ denotes the unique ℓ-regular $\mathrm{Fr}_\ell(G)$-conjugacy class which image $\pi([\tilde{g}])$ under π contains $g\in G$ (cf. [24, Prop.5.5]). The class function

$$\Lambda_{G,\ell}\colon = \frac{1}{|G|}(1-\bar{\chi}_{A(G,\ell)}) \in \mathfrak{F}({}^{G}G_{\ell'},\mathbb{C}), \tag{1.5}$$

carries much information on the universal ℓ-Frattini cover of the finite group G. Its degree $\Lambda_{G,\ell}(1)$ just coincides with the ℓ-Euler-Poincaré characteristic of $\mathrm{Fr}_\ell(G)$. In particular,

$$\mathrm{proj}_\ell(G) = 1 - \frac{|G|}{|L|}\Lambda_{G,\ell}(1), \qquad L\in\mathrm{Syl}_\ell(G). \tag{1.6}$$

Moreover, the irrationality of this class function implies that $\mathrm{Fr}_\ell(G)$ cannot act co-compactly on a locally finite pro-ℓ tree (cf. [24, Thm.D]).

In this note we study the universal ℓ-Frattini cover of the finite group $G = \mathrm{PSL}_2(q)$, $q=p^f$. In case that the ℓ-Sylow subgroup of G is cyclic or if $\ell=2$ and q is odd one has an explicit description of the $\bar{\mathbb{F}}_\ell[G]$-module

$$\Omega^2(\bar{\mathbb{F}}_\ell) \simeq A(G,\ell)\otimes_{\mathbb{F}_\ell}\bar{\mathbb{F}}_\ell \tag{1.7}$$

thanks to the work of R. Burkhardt [5], K. Erdmann [8] and J. E. Humphreys [19] on the structure of projective $\bar{\mathbb{F}}_\ell[G]$-modules. Here $\bar{\mathbb{F}}_\ell$ denotes the algebraic closure of $\mathbb{F}_\ell$ and $\Omega(\_)$ denotes the *Heller translate* (cf. [1, §20], [17]). In section 3 we will establish the following theorem.

Theorem A. *Let $G\colon = \mathrm{PSL}_2(q)$, $q=p^f$, let ℓ be a prime dividing the order of G, and let ℓ^n be the order of an ℓ-Sylow subgroup of G. If $\ell=p$, assume also that $q=p$. Then*

$$|G|\cdot\Lambda_{G,\ell} = \begin{cases} 1_* - \mathrm{St}_* & \text{if } \ell\neq\{2,p\},\ \ell|(q-1),\\ 1_* - m.(\mathrm{St}_* - 1_*) & \text{if } \ell\neq\{2,p\},\ \ell|(q+1),\ m\colon=\tfrac{1}{2}(\ell^n-1),\\ 1_* - \mathrm{St}_* & \text{if } \ell=2,\ q\equiv 1 \mod 4,\\ 2^{n-1}(1_* - \mathrm{St}_*) & \text{if } \ell=2,\ q\equiv 3 \mod 4,\\ 1_* - \bar{\chi}_* & \text{if } \ell=p=q,\ q\notin\{2,3\},\\ 0 & \text{if } \ell=p=q,\ q\in\{2,3\}, \end{cases} \tag{1.8}$$

where χ is an irreducible character of G of degree $q+1$ which is described explicitly in subsection 3.3, and $\_{}_*$ denotes restriction to ${}^{G}G_{\ell'}$.

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[1] A $\hat{G}$-conjugacy class ${}^{\hat{G}}g$ is called *ℓ-regular*, if the order of the closed subgroup generated by g is a supernatural number co-prime to ℓ.

Here 1 denotes the trivial character and St denotes the Steinberg character. As a consequence one obtains the following result for the projective ℓ-rank of G.

Corollary B. *Let* $G\colon = \mathrm{PSL}_2(q)$, $q = p^f$, *let* ℓ *be a prime dividing the order of* G, *and let* ℓ^n *be the order of an* ℓ*-Sylow subgroup of* G. *Then*

$$\mathrm{proj}_\ell(G) = \begin{cases} 1 + \frac{1}{\ell^n}(q-1), & \textit{if } \ell \neq \{2,p\},\ \ell|(q-1), \\ \frac{1}{2}(1-\frac{1}{\ell^n})(q+1), & \textit{if } \ell \neq \{2,p\},\ \ell|(q+1), \\ 1 + \frac{1}{2^n}(q-1), & \textit{if } \ell = 2,\ q \equiv 1 \mod 4, \\ \frac{1}{2}(q+1), & \textit{if } \ell = 2,\ q \equiv 3 \mod 4, \\ 2 + (f-1)2^f, & \textit{if } \ell = p \neq 2,\ q \neq 3, \\ 2 + (f-2)2^{f-1}, & \textit{if } \ell = p = 2,\ q \neq 2, \\ 1, & \textit{if } \ell = p,\ q \in \{2,3\}. \end{cases} \tag{1.9}$$

In particular, $\mathrm{proj}_\ell(G) = 1$ *if and only if* $(q,\ell) = (2,2)$, $(2,3)$ *or* $(3,3)$, *and* $\mathrm{proj}_\ell(G) = 2$ *if and only if one of the following holds:*

(i) $\ell = p = q$ *is odd, and* $q \neq 3$,
(ii) $\ell = p = 2$ *and* $q = 4$,
(iii) $\ell \neq p$ *and* (ℓ^n, p^f) *are consecutive prime powers, i.e.,* $p^f = \ell^n + 1$.

Note that consecutive prime powers (ℓ^n, p^f), $p^f = \ell^n + 1$, can be easily classified. One has either (i) $\ell = 2$, $f = 1$ and p is a Fermat prime, or (ii) $p = 2$, $n = 1$ and ℓ is a Mersenne prime, or (iii) $(\ell^n, p^f) = (2^3, 3^2)$. By Corollary B there exist non-abelian finite simple groups G for which the projective ℓ-rank coincides with the rank of an ℓ-Sylow subgroup of G. For example, this happens for $\ell = 2$ and $G = A_5$, $\mathrm{PSL}_2(p)$, p a Fermat prime, or $\mathrm{PSL}_2(9)$. This phenomenon can be interpreted also in terms of the ℓ-modular representation theory of G. Let ℓ be a prime number and let $k \geq 1$. The finite group G will be called (ℓ, k)-*minimal* if

$$\mathrm{res}^G_L(\Omega^r(G, \bar{\mathbb{F}}_\ell)) \simeq \Omega^r(L, \bar{\mathbb{F}}_\ell), \qquad L \in \mathrm{Syl}_\ell(G). \tag{1.10}$$

Hence the projective ℓ-rank of G equals the number of generators of an ℓ-Sylow subgroup of G if and only if G is $(\ell, 2)$-minimal (cf. subsection 2.5). It is certainly not surprising that also $(\ell, 1)$-minimality plays a role in this context. Let $\tau_{\mathbb{F}_\ell} : P_{\mathbb{F}_\ell} \to \mathbb{F}_\ell$ denote a minimal projective cover of the trivial left $\mathbb{F}_\ell[G]$-module $\mathbb{F}_\ell$. Then there exists a positive integer $c_\ell(G)$ such that

$$\dim_{\mathbb{F}_\ell}(P_{\mathbb{F}_\ell}) = c_\ell(G) \cdot |L|, \qquad L \in \mathrm{Syl}_\ell(G). \tag{1.11}$$

Hence G is $(\ell, 1)$-minimal if and only if $c_\ell(G) = 1$. For the projective ℓ-rank of a direct product of finite groups has the following property (cf. Prop.2.6).

Proposition C. *Let* G *and* H *be finite groups, and let* ℓ *be a prime number. Then*

$$\mathrm{proj}_\ell(G \times H) = c_\ell(G) \cdot \mathrm{proj}_\ell(H) + c_\ell(H) \cdot \mathrm{proj}_\ell(G) + (c_\ell(G) - 1) \cdot (c_\ell(H) - 1). \tag{1.12}$$

In particular, if G *and* H *are* $(\ell, 1)$*-minimal one has*

$$\mathrm{proj}_\ell(G \times H) = \mathrm{proj}_\ell(G) + \mathrm{proj}_\ell(H). \tag{1.13}$$

The property of being $(\ell, 1)$-minimal can be seen as a weaker form of ℓ-solvability[2], i.e., every finite ℓ-solvable group is $(\ell, 1)$-minimal (cf. Prop.2.3). For the groups $G\colon = \mathrm{PSL}_2(q)$ one has the following (cf. §3).

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[2] A finite group G is called *ℓ-solvable*, if it has a composition series $G = X_1 \gneqq X_2 \cdots \gneqq X_n = 1$ such that X_k/X_{k+1} is either an ℓ-group, or an ℓ'-group.

Theorem D. *Let $G = \mathrm{PSL}_2(q)$, $q = p^f$, let ℓ be a prime number dividing the order of G, and let ℓ^n denote the order of an ℓ-Sylow subgroup of G. Then*

$$c_\ell(G) = \begin{cases} 1 + \frac{1}{2}(1 - \frac{1}{\ell^n})(q-1), & \text{if } \ell \neq \{2,p\},\ \ell|(q-1), \\ \frac{1}{\ell^n}(q+1), & \text{if } \ell \neq \{2,p\},\ \ell|(q+1), \\ \frac{1}{2}(q+1), & \text{if } \ell = 2,\ q \equiv 1 \mod 4, \\ \frac{1}{2^n}(q+1), & \text{if } \ell = 2,\ q \equiv 3 \mod 4, \\ 2^f - 1, & \text{if } \ell = p. \end{cases} \tag{1.14}$$

In particular, G is $(\ell, 1)$-minimal if and only if one of the following holds:

(i) *$\ell = p$, $q = p$.*

(ii) *$\ell \neq p$ and G has an ℓ'-Hall subgroup. This happens precisely when (p^f, ℓ^n) are consecutive prime powers.*

Note that for a non-abelian finite simple group G and a prime divisor ℓ of $|G|$ the property of being $(\ell, 1)$-minimal is not very common. Indeed, the only examples known to the author of $(\ell, 1)$-minimal non-abelian finite simple groups which do not contain an ℓ'-Hall subgroup are of type (i) of Theorem D.

Explicit knowledge of the ℓ-Frattini module $A(G, \ell)$ cannot only be applied for the analysis of the universal ℓ-Frattini cover, but also for the study of very particular ℓ-Frattini extensions. In [22] and [23] we investigated ℓ-Frattini extension $\pi\colon \hat{G} \to G$ for a finite group G with the property that $\hat{G}$ is a weakly orientable ℓ-Poincaré duality group of dimension 2. These investigations and the discussion in section 3 show the following property.

Proposition E. *Let $G\colon = \mathrm{PSL}_2(q)$ and $q \equiv 3 \mod 4$. Then there exists an embedding*

$$\alpha\colon \Omega^{-1}(\mathbb{F}_2) \longrightarrow \Omega^2(\mathbb{F}_2). \tag{1.15}$$

of left $\mathbb{F}_2[G]$-modules. In particular, there exists a 2-Frattini extension $\pi\colon \hat{G} \to G$ such that $\hat{G}$ is an orientable 2-Poincaré duality group of dimension 2.

2. The ℓ-Frattini module of a finite group

Throughout this section we assume that G is a finite group and that ℓ is a prime number dividing the order of G. By ${}_{\bar{\mathbb{F}}_\ell[G]}\mathrm{Mod}$ we denote the abelian category whose objects are finitely generated left $\bar{\mathbb{F}}_\ell[G]$-modules. The set $\bar{\mathfrak{S}}_\ell(G)$ will denote the set of isomorphism types $[S]$ of irreducible left $\bar{\mathbb{F}}_\ell[G]$-modules S.

A surjective morphism $\alpha\colon G \to H$ of finite groups induces canonically two covariant additive functors

$$\begin{aligned} \inf^H_G\colon {}_{\bar{\mathbb{F}}_\ell[H]}\mathrm{Mod} &\longrightarrow {}_{\bar{\mathbb{F}}_\ell[G]}\mathrm{Mod}, \\ \mathrm{def}^G_H\colon {}_{\bar{\mathbb{F}}_\ell[G]}\mathrm{Mod} &\longrightarrow {}_{\bar{\mathbb{F}}_\ell[H]}\mathrm{Mod}, \end{aligned} \tag{2.1}$$

where def^G_H is given by $\mathrm{def}^G_H(M)\colon = \bar{\mathbb{F}}_\ell[H] \otimes_{\bar{\mathbb{F}}_\ell[G]} M$, $M \in ob({}_{\bar{\mathbb{F}}_\ell[G]}\mathrm{Mod})$. Obviously, def^G_H is the left adjoint functor of $\inf^H_G$. Moreover, $\inf^H_G$ is exact and def^G_H is right exact. Here we used the same notation as introduced by J-P. Serre (cf. [21]), i.e., the upper index of a functor indicates the domain category and the lower index indicates the co-domain category. In particular, for an injective morphism $\beta\colon A \to G$ of finite groups we denote by

$$\begin{aligned} \mathrm{res}^G_A\colon {}_{\bar{\mathbb{F}}_\ell[G]}\mathrm{Mod} &\longrightarrow {}_{\bar{\mathbb{F}}_\ell[A]}\mathrm{Mod}, \\ \mathrm{ind}^A_G\colon {}_{\bar{\mathbb{F}}_\ell[A]}\mathrm{Mod} &\longrightarrow {}_{\bar{\mathbb{F}}_\ell[G]}\mathrm{Mod}, \end{aligned} \tag{2.2}$$

the respective restriction and induction functor.

2.1. **Minimal projective covers.** Every finitely generated $\mathbb{F}_\ell[G]$-module (resp. $\bar{\mathbb{F}}_\ell[G]$-module) M has a *minimal projective cover* $\tau_M\colon P_M \to M$. For our purpose the following two fundamental properties will be important which easy proof is left to the reader.

Proposition 2.1. *Let G and H be finite groups.*

(a) *Let $\tau_M\colon P_M \to M$ be the minimal projective cover of the finitely generated $\mathbb{F}_\ell[G]$-module M. Then*

$$\tau_M \otimes \bar{\mathbb{F}}_\ell\colon P_M \otimes_{\mathbb{F}_\ell} \bar{\mathbb{F}}_\ell \longrightarrow M \otimes_{\mathbb{F}_\ell} \bar{\mathbb{F}}_\ell \tag{2.3}$$

is a minimal projective cover of the $\bar{\mathbb{F}}_\ell[G]$-module $M \otimes_{\mathbb{F}_\ell} \bar{\mathbb{F}}_\ell$.

(b) *Let $\tau_X\colon P_X \to X$ be a minimal projective cover of the finitely generated $\bar{\mathbb{F}}_\ell[G]$-module X, and let $\tau_Y\colon P_Y \to Y$ be a minimal projective cover of the finitely generated $\bar{\mathbb{F}}_\ell[H]$-module Y. Then*

$$\tau_X \otimes \tau_Y\colon P_X \otimes_{\bar{\mathbb{F}}_\ell} P_Y \longrightarrow X \otimes_{\bar{\mathbb{F}}_\ell} Y \tag{2.4}$$

is a minimal projective cover of the $\bar{\mathbb{F}}_\ell[G \times H]$-module $X \otimes_{\bar{\mathbb{F}}_\ell} Y$.

Let $\tau_M\colon P_M \to M$ be a minimal projective cover of the finitely generated $\bar{\mathbb{F}}_\ell[G]$-module (resp. $\mathbb{F}_\ell[G]$-module) M. The Heller translate $\Omega(M)$ of M is given by $\Omega(M)\colon = \ker(\tau_M)$. Inductively one defines

$$\Omega^{k+1}(M)\colon = \Omega(\Omega^k(M)), \quad k \geq 2. \tag{2.5}$$

It is well-known that the ℓ-Frattini module $A(G,\ell)$ is isomorphic to the left $\mathbb{F}_\ell[G]$-module $\Omega^2(\mathbb{F}_\ell)$ (cf. [12], [15, p.208]). Hence by Proposition 2.1(a), one has an isomorphism

$$\Omega^2(\bar{\mathbb{F}}_\ell) \simeq A(G,\ell) \otimes_{\mathbb{F}_\ell} \bar{\mathbb{F}}_\ell. \tag{2.6}$$

2.2. **The integer c_M.** Let L be an ℓ-Sylow subgroup of G, and let M be a finitely generated $\bar{\mathbb{F}}_\ell[G]$-module. Every finitely generated projective $\bar{\mathbb{F}}_\ell[L]$-module is free. As res^G_L is mapping projective $\bar{\mathbb{F}}_\ell[G]$-modules to projective $\bar{\mathbb{F}}_\ell[L]$-modules, there exists a positive integer c_M such that

$$\dim_{\bar{\mathbb{F}}_\ell}(P_M) = c_M \cdot |L|. \tag{2.7}$$

We define

$$c_\ell(G)\colon = c_{\bar{\mathbb{F}}_\ell}, \tag{2.8}$$

where $\bar{\mathbb{F}}_\ell$ denotes the trivial $\bar{\mathbb{F}}_\ell[G]$-module. Let M be a finitely generated $\bar{\mathbb{F}}_\ell[G]$-module. Then $\tau_{\bar{\mathbb{F}}_\ell} \otimes M\colon P_{\bar{\mathbb{F}}_\ell} \otimes M \to M$ is a projective cover of M. Thus one has

$$c_M \leq c_\ell(G) \cdot \dim_{\bar{\mathbb{F}}_\ell}(M). \tag{2.9}$$

The value of $c_\ell(G)$ is sometimes quite mysterious. E.g., J. E. Humphreys showed that for $G = \mathrm{SL}_2(\ell^f)$ one has $c_\ell(G) = 2^f - 1$ (cf. [18, Thm.5]). We list some of the properties of the function c in the following proposition.

Proposition 2.2. *Let G be a finite group and let ℓ be a prime number.*

(a) *Assume that G has an ℓ'-Hall subgroup. Then $c_M = 1$ for every one-dimensional $\bar{\mathbb{F}}_\ell[G]$-module M.*

(b) *Let N be an ℓ-solvable normal subgroup of G and let S be an irreducible left $\bar{\mathbb{F}}_\ell[G/N]$-module. For $S'\colon = \operatorname{inf}_G^{G/N}(S)$ one has*

$$c_{S'}(G) = c_S(H). \tag{2.10}$$

Proof. (a) By (2.9), it suffices to prove the claim when $M = \bar{\mathbb{F}}_\ell$ is the trivial $\bar{\mathbb{F}}_\ell[G]$-module. Let $A \leq G$ be an ℓ'-Hall subgroup of G. As ind_G^A is mapping projectives to projectives, $P\colon = \operatorname{ind}_G^A(\bar{\mathbb{F}}_\ell)$ with the canonical map $\tau_P\colon P \to \bar{\mathbb{F}}_\ell$ is a projective cover of $\bar{\mathbb{F}}_\ell$ of dimension $|L|$, $L \in \operatorname{Syl}_\ell(G)$.
(b) The deflation functor $\operatorname{def}_{G/N}^G\colon = \bar{\mathbb{F}}_\ell[G/N] \otimes_{\bar{\mathbb{F}}_\ell[G]} \_$ is mapping projectives to projectives. As $\operatorname{def}_{G/N}^G$ is the left adjoint to $\operatorname{inf}_G^{G/N}$, it is also mapping projective indecomposable modules to projective indecomposable modules[3].

By induction, we may assume that N is either an ℓ-group, or an ℓ'-group. If N is an ℓ'-group and $\tau_S\colon P_S \to S$ is a minimal projective cover of S as $\bar{\mathbb{F}}_\ell[G/N]$-module, $\operatorname{inf}_G^{G/N}(P_S)$ is a projective $\bar{\mathbb{F}}_\ell[G]$-module. This follows from the fact that for a finite group G an $\bar{\mathbb{F}}_\ell[G]$-module M is projective, if and only if its restriction to an ℓ-Sylow subgroup is projective (cf. [3, Cor.3.6.18]). Let N be an ℓ-group, and let L be an ℓ-Sylow subgroup of G. In particular, $N \leq L$. For every left $\bar{\mathbb{F}}_\ell[G]$-module M one has isomorphisms

$$\begin{aligned} \operatorname{res}_{L/N}^{G/N}(\bar{\mathbb{F}}_\ell[G/N] \otimes_{\bar{\mathbb{F}}_\ell[G]} M) &\simeq \operatorname{ind}_G^L(\bar{\mathbb{F}}_\ell[L/N]) \otimes_{\bar{\mathbb{F}}_\ell[G]} M \\ &\simeq \bar{\mathbb{F}}_\ell[L/N] \otimes_{\bar{\mathbb{F}}_\ell[L]} \operatorname{res}_L^G(M), \end{aligned} \tag{2.11}$$

where the induction functor is defined on right $\bar{\mathbb{F}}_\ell[L]$-modules. Let S be an irreducible left $\bar{\mathbb{F}}_\ell[G/N]$-module, and let $\tau_{S'}\colon P_{S'} \to S'$ be a minimal projective cover as $\bar{\mathbb{F}}_\ell[G]$-module. Let $c \geq 1$ such that $\operatorname{res}_L^G(P_{S'}) \simeq \bar{\mathbb{F}}_\ell[L]^c$. Then by (2.11), $\operatorname{res}_{L/N}^{G/N}(\operatorname{def}_{G/N}^G(P_{S'})) = \bar{\mathbb{F}}_\ell[L/N]^c$. This yields the claim. ☐

2.3. $(\ell, 1)$-minimal groups.

The following proposition shows that one can interpret $(\ell, 1)$-minimality as a kind of generalization of ℓ-solvability.

Proposition 2.3. *Let G and H be finite groups, and let ℓ be a prime number.*

(a) *If G has an ℓ'-Hall subgroup, G is $(\ell, 1)$-minimal. In particular, every ℓ-solvable group is $(\ell, 1)$-minimal.*

(b) *Let N be an ℓ-solvable normal subgroup of G. Then G is $(\ell, 1)$-minimal if and only if G/N is $(\ell, 1)$-minimal.*

(c) *$G \times H$ is $(\ell, 1)$-minimal if and only if G and H are $(\ell, 1)$-minimal.*

(d) *Let A be an ℓ'-index subgroup of the $(\ell, 1)$-minimal finite group G. Then A is $(\ell, 1)$-minimal.*

Proof. (a) and (b) are immediate consequences of Proposition 2.2. From Proposition 2.1(b) one concludes that

$$c_\ell(G \times H) = c_\ell(G) \cdot c_\ell(H). \tag{2.12}$$

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[3] Here 0 has to be considered as a projective indecomposable module.

This shows (c). Let A be an ℓ'-index subgroup of the $(\ell,1)$-minimal group G. As A contains an ℓ-Sylow subgroup of G, the minimal projective G-cover of $\bar{\mathbb{F}}_\ell$ restricted to A remains directly indecomposable. This yields (d). □

Theorem D shows that $G = \mathrm{PSL}_2(31)$ is $(2,1)$-minimal. Moreover, G has a subgroup isomorphic to A_5 (cf. [20]), but $c_2(A_5) = 3$. Hence Proposition 2.3(d) does not hold for all subgroups.

2.4. **The projective ℓ-rank of a finite group.** The projective ℓ-rank of the finite group G has the following property:

Proposition 2.4. *Let $\pi\colon \mathrm{Fr}_\ell(G) \to G$ be the universal ℓ-Frattini cover of the finite group G and let L be an ℓ-Sylow subgroup of G. Then one has*

$$\dim_{\mathbb{F}_\ell}(A(G,\ell)) - 1 = |L| \cdot (\mathrm{proj}_\ell(G) - 1) = -\varepsilon_\ell(\mathrm{Fr}_\ell(G)) \cdot |G|, \tag{2.13}$$

where $\varepsilon_\ell(\mathrm{Fr}_\ell(G)) = \Lambda_{G,\ell}(1)$ denotes the ℓ-Euler-Poincaré characteristic[4] *of the finitely generated ℓ-projective virtual pro-ℓ group $\mathrm{Fr}_\ell(G)$.*

Proof. Let $\hat{L}$ be a pro-ℓ Sylow subgroup of $\mathrm{Fr}_\ell(G)$. As $\hat{L}$ is a finitely generated free pro-ℓ group, Schreier's formula implies that

$$\dim_{\mathbb{F}_\ell}(A(G,\ell)) - 1 = \mathrm{rk}_\ell(\ker(\pi)) - 1 = |L| \cdot (\mathrm{rk}_\ell(\hat{L}) - 1) = |L| \cdot (\mathrm{proj}_\ell(G) - 1). \tag{2.14}$$

Moreover, $\varepsilon_\ell(\hat{L}) = 1 - \mathrm{proj}_\ell(G)$ (cf. [21, §I.4.1]), and $\varepsilon_\ell(\mathrm{Fr}_\ell(G)) = \frac{1}{|G/L|} \cdot \varepsilon_\ell(\hat{L})$. This yields the claim. □

There is an alternative way for computing the projective ℓ-rank of a finite group G using the integers c_S, $[S] \in \bar{\mathfrak{S}}_\ell(G)$.

Proposition 2.5. *Let G be a finite group and let ℓ be a prime number. Then*

$$\mathrm{proj}_\ell(G) = 1 - c_\ell(G) + \sum_{[S] \in \bar{\mathfrak{S}}_\ell(G)} c_S \cdot \dim_{\bar{\mathbb{F}}_\ell}(H^1(G,S)). \tag{2.15}$$

Proof. Let $P_1 \to P_0 \to \bar{\mathbb{F}}_\ell$ be a partial minimal projective resolution of the trivial $\bar{\mathbb{F}}_\ell[G]$-module $\bar{\mathbb{F}}_\ell$. From the exact sequence $0 \to \Omega^2(\bar{\mathbb{F}}_\ell) \to P_1 \to P_0 \to \bar{\mathbb{F}}_\ell \to 0$ one concludes that

$$\begin{aligned} \dim_{\bar{\mathbb{F}}_\ell}(\Omega^2(\bar{\mathbb{F}}_\ell)) - 1 &= \dim_{\bar{\mathbb{F}}_\ell}(P_1) - \dim_{\bar{\mathbb{F}}_\ell}(P_0) \\ &= |L| \cdot (-c_\ell(G) + \sum_{[S] \in \bar{\mathfrak{S}}_\ell(G)} c_S \cdot \dim_{\bar{\mathbb{F}}_\ell}(H^1(G,S))), \end{aligned} \tag{2.16}$$

where L is an ℓ-Sylow subgroup of G. Dividing both sides by $|L|$ yields the claim. □

2.5. $(\ell,2)$**-minimal groups.** Let L be an ℓ-Sylow subgroup of the finite group G, and let $\hat{L}$ be a pro-ℓ Sylow subgroup of the universal ℓ-Frattini cover $\mathrm{Fr}_\ell(G)$. Then (2.16) applied for L yields that

$$\dim_{\bar{\mathbb{F}}_\ell}(\Omega^2(L, \bar{\mathbb{F}}_\ell)) - 1 = |L| \cdot (\mathrm{rk}(L) - 1). \tag{2.17}$$

Thus one has equality $\mathrm{rk}(L) = \mathrm{proj}_\ell(G)$ if and only if G is $(\ell,2)$-minimal. Examples of $(\ell,2)$-minimal groups are finite groups with a normal ℓ-Sylow subgroup or finite groups with a normal ℓ'-Hall subgroup. However, note that for every finite group G there exists a finite ℓ-Frattini extension $\tau\colon \tilde{G} \to G$ such that $\tilde{G}$ is $(\ell,2)$-minimal.

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[4] cf. [21, §I.4.1].

2.6. **The projective ℓ-rank of a product of finite groups.** The formula (2.13) is not very suitable for computing the projective ℓ-rank of a product of finite groups, but (2.15) can be used quite effectively. Counting irreducible Brauer characters shows that for two finite groups G and H one has

$$\bar{\mathfrak{S}}_\ell(G\times H)=\{\,[S\otimes_{\bar{\mathbb{F}}_\ell}T]\mid [S]\in\bar{\mathfrak{S}}_\ell(G),\ [T]\in\bar{\mathfrak{S}}_\ell(H)\,\}. \tag{2.18}$$

Moreover, from the Künneth formula one concludes that for $[S]\in\bar{\mathfrak{S}}_\ell(G)$, $[T]\in\bar{\mathfrak{S}}_\ell(H)$ one has

$$H^1(G\times H,S\otimes_{\bar{\mathbb{F}}_\ell}T)\simeq\begin{cases}H^1(G,S) & \text{if } [S]\neq[\bar{\mathbb{F}}_\ell],\ [T]=[\bar{\mathbb{F}}_\ell],\\ H^1(H,T) & \text{if } [S]=[\bar{\mathbb{F}}_\ell],\ [T]\neq[\bar{\mathbb{F}}_\ell],\\ H^1(G,\bar{\mathbb{F}}_\ell)\oplus H^1(H,\bar{\mathbb{F}}_\ell) & \text{if } [S]=[\bar{\mathbb{F}}_\ell],\ [T]=[\bar{\mathbb{F}}_\ell],\\ 0 & \text{else.}\end{cases} \tag{2.19}$$

From these facts one concludes the following.

Proposition 2.6. *Let G and H be finite groups, and let ℓ be a prime number. Then*

$$\mathrm{proj}_\ell(G\times H)=c_\ell(G)\cdot\mathrm{proj}_\ell(H)+c_\ell(H)\cdot\mathrm{proj}_\ell(G)+(c_\ell(G)-1)\cdot(c_\ell(H)-1). \tag{2.20}$$

In particular, if G and H are $(\ell,1)$-minimal, one has

$$\mathrm{proj}_\ell(G\times H)=\mathrm{proj}_\ell(G)+\mathrm{proj}_\ell(H). \tag{2.21}$$

Proof. From (2.16) and (2.19) one deduces that

$$\begin{aligned}\mathrm{proj}_\ell(G\times H)=1-c_\ell(G\times H)&+\sum_{[S]\in\bar{\mathfrak{S}}_\ell(G)}c_{S\otimes_{\bar{\mathbb{F}}}\bar{\mathbb{F}}_\ell}(G\times H)\cdot\dim_{\bar{\mathbb{F}}_\ell}(H^1(G,S))\\ &+\sum_{[T]\in\bar{\mathfrak{S}}_\ell(H)}c_{\bar{\mathbb{F}}_\ell\otimes_{\bar{\mathbb{F}}}T}(G\times H)\cdot\dim_{\bar{\mathbb{F}}_\ell}(H^1(H,T)).\end{aligned} \tag{2.22}$$

By (2.4), one has

$$\begin{aligned}c_{S\otimes_{\bar{\mathbb{F}}_\ell}\bar{\mathbb{F}}_\ell}(G\times H)&=c_S(G)\cdot c_\ell(H),\\ c_{\bar{\mathbb{F}}_\ell\otimes_{\bar{\mathbb{F}}_\ell}T}(G\times H)&=c_\ell(G)\cdot c_T(H).\end{aligned} \tag{2.23}$$

This yields

$$\begin{aligned}\mathrm{proj}_\ell(G\times H)=1-c_\ell(G)\cdot c_\ell(H)&+(\mathrm{proj}_\ell(G)+c_\ell(G)-1)c_\ell(H)\\ &+c_\ell(G)(\mathrm{proj}_\ell(H)+c_\ell(H)-1),\end{aligned} \tag{2.24}$$

and thus the claim. □

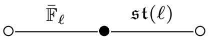
3. THE ℓ-FRATTINI MODULE FOR $G=\mathrm{PSL}_2(q)$

In this section we analyze the $\bar{\mathbb{F}}_\ell[G]$-module $\Omega^2(\bar{\mathbb{F}}_\ell)=A(G,\ell)\otimes_{\mathbb{F}_\ell}\bar{\mathbb{F}}_\ell$ for $G\colon=\mathrm{PSL}_2(q)$, $q=p^f$, using ℓ-modular representation theory.

3.1. **Case I: $\ell\notin\{2,p\}$.** If $\ell\notin\{2,p\}$ the ℓ-Sylow subgroups of G are cyclic, and therefore, one knows that the principal block b_0 of $\bar{\mathbb{F}}_\ell[G]$ is a Brauer tree algebra (cf. [1, Chap.V]). Moreover, the Brauer trees occuring in this context were determined by R.Burkhardt (cf. [5]).

3.1.1. *Case Ia:* $\ell \notin \{2,p\}$, $\ell | (q-1)$. Let $q-1 = r\ell^n$ for some natural number $n \in \mathbb{N}$ and some number r coprime to ℓ. Let $\mathfrak{st} \in ob({}_{\mathbb{C}[G]}\mathrm{Mod})$ denote the Steinberg module and let $\mathfrak{st}(\ell) \in ob({}_{\bar{\mathbb{F}}_\ell[G]}\mathrm{Mod})$ denote a reduction modulo ℓ of $\mathfrak{st}$. Then $\mathfrak{st}(\ell)$ is an irreducible $\bar{\mathbb{F}}_\ell[G]$-module (cf. [5]). The Brauer tree of the principal block b_0 has the form

(3.1)



(cf. [5, case I, III, V]) and the multiplicity of the exceptional vertex is equal to $m\colon = \frac{1}{2}(\ell^n - 1)$. From this explicit description one concludes the following:

Proposition 3.1. *Let $G = \mathrm{PSL}_2(q)$, and let ℓ be an odd prime number such that $\ell | (q-1)$. Then one has the following:*

(a) *$A(G,\ell) \otimes_{\mathbb{F}_\ell} \bar{\mathbb{F}}_\ell = \Omega^2(\bar{\mathbb{F}}_\ell) \simeq \mathfrak{st}(\ell)$. In particular,*

$$\Lambda_{G,\ell} = \frac{1}{|G|}(1_* - \mathrm{St}_*). \tag{3.2}$$

(b) $\mathrm{proj}_\ell(G) = 1 + r = 1 + \frac{1}{\ell^n}(q-1)$.

(c) *For $[S] \in \bar{\mathfrak{S}}_\ell(G)$ one has*

$$H^1(G,S) \simeq \begin{cases} \bar{\mathbb{F}}_\ell & \text{if } [S] = [\mathfrak{st}(\ell)], \\ 0 & \text{if } [S] \neq [\mathfrak{st}(\ell)], \end{cases} \tag{3.3}$$

and

$$H^2(G,S) \simeq \begin{cases} \bar{\mathbb{F}}_\ell & \text{if } [S] = [\mathfrak{st}(\ell)], \\ 0 & \text{if } [S] \neq [\mathfrak{st}(\ell)]. \end{cases} \tag{3.4}$$

(d) *One has*

$$\begin{aligned} c_\ell(G) &= 1 + rm &&= 1 + \tfrac{1}{2}(1 - \tfrac{1}{\ell^n})(q-1), \\ c_{\mathfrak{st}(\ell)} &= 1 + r(m+1) &&= 1 + \tfrac{1}{2}(1 + \tfrac{1}{\ell^n})(q-1). \end{aligned} \tag{3.5}$$

In particular, G is not $(\ell,1)$-minimal.

Proof. From the structure of the Brauer tree one concludes that $P_{\bar{\mathbb{F}}_\ell}$ is a uniserial $\bar{\mathbb{F}}_\ell[G]$-module of Loewy length $2m+1$ with

$$\mathrm{rad}_j(P_{\bar{\mathbb{F}}_\ell}) / \mathrm{rad}_{j+1}(P_{\bar{\mathbb{F}}_\ell}) = \begin{cases} \bar{\mathbb{F}}_\ell, & \text{if } j \leq 2m,\ j \text{ even}, \\ \mathfrak{st}(\ell), & \text{if } j \leq 2m,\ j \text{ odd}, \\ 0, & \text{if } j \geq 2m+1 \end{cases} \tag{3.6}$$

(cf. [1, Chap.V]). Hence $\mathrm{hd}(\Omega_1(G,\bar{\mathbb{F}}_\ell)) = \mathfrak{st}(\ell)$ and this yields (3.3) (cf. [3, Cor.2.5.4]). Furthermore, $P_{\mathfrak{st}(\ell)}$ is also a uniserial $\bar{\mathbb{F}}_\ell[G]$-module of Loewy length $2m+1$ with

$$\mathrm{rad}_j(P_{\mathfrak{st}(\ell)}) / \mathrm{rad}_{j+1}(P_{\mathfrak{st}(\ell)}) = \begin{cases} \mathfrak{st}(\ell), & \text{if } j \leq 2m,\ j \text{ even}, \\ \bar{\mathbb{F}}_\ell, & \text{if } j \leq 2m,\ j \text{ odd}, \\ 0, & \text{if } j \geq 2m+1. \end{cases} \tag{3.7}$$

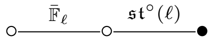
This implies (a) and also (3.4). (b) follows from Proposition 2.4. Moreover,

$$\dim_{\bar{\mathbb{F}}_\ell}(P_{\bar{\mathbb{F}}_\ell}) = m + 1 + mq = \ell^n(1 + r \cdot \tfrac{1}{2}(\ell^n - 1)), \tag{3.8}$$

which yields (d). □

3.1.2. *Case Ib: $\ell \notin \{2, p\}$, $\ell | (q+1)$.* Let $q+1 = r\ell^n$ for some natural number $n \in \mathbb{N}$ and some number r coprime to ℓ. In this case the reduction mod ℓ of the Steinberg module $\mathfrak{st}$ is not irreducible but has two composition factors: $\bar{\mathbb{F}}_\ell$ and another one which we will denote by $\mathfrak{st}^\circ(\ell)$. In particular, $\dim_{\bar{\mathbb{F}}_\ell}(\mathfrak{st}^\circ(\ell)) = q - 1$. The Brauer tree of the principal block b_0 has the form

(3.9)



(cf. [5, case II, IV, VI]). The multiplicity m of the exceptional vertex is again equal to $m\colon = \frac{1}{2}(\ell^n - 1)$. In this case one obtains the following:

Proposition 3.2. *Let $G = \mathrm{PSL}_2(q)$, let ℓ be an odd prime number such that $\ell | (q+1)$, and let $q + 1 = r\ell^n$. Then one has the following:*

(a) *$A(G, \ell) \otimes_{\mathbb{F}_\ell} \bar{\mathbb{F}}_\ell = \Omega^2(G, \bar{\mathbb{F}}_\ell)$ is a uniserial $\bar{\mathbb{F}}_\ell[G]$-module of Loewy length m with each composition factor isomorphic to $\mathfrak{st}^\circ(\ell)$. In particular,*

$$\Lambda_{G,\ell} = \frac{1}{|G|}(1_* - m(\mathrm{St}_* - 1_*)). \tag{3.10}$$

(b) $\mathrm{proj}_\ell(G) = rm = \frac{1}{2}(1 - \frac{1}{\ell^n})(q + 1)$.

(c) *For $[S] \in \tilde{\mathfrak{S}}_\ell(G)$ one has*

$$H^1(G, S) \simeq \begin{cases} \bar{\mathbb{F}}_\ell, & \textit{if } [S] = [\mathfrak{st}^\circ(\ell)], \\ 0, & \textit{if } [S] \neq [\mathfrak{st}^\circ(\ell)], \end{cases} \tag{3.11}$$

and

$$H^2(G, S) \simeq \begin{cases} \bar{\mathbb{F}}_\ell, & \textit{if } [S] = [\mathfrak{st}^\circ(\ell)], \\ 0, & \textit{if } [S] \neq [\mathfrak{st}^\circ(\ell)]. \end{cases} \tag{3.12}$$

(d) *One has*

$$\begin{aligned} c_\ell(G) &= r &&= \tfrac{q+1}{\ell^n}, \\ c_{\mathfrak{st}^\circ(\ell)} &= r(m+1) - 1 &&= \tfrac{1}{2}(1 + \tfrac{1}{\ell^n})(q+1) - 1. \end{aligned} \tag{3.13}$$

In particular, G is $(\ell, 1)$-minimal, if and only if $q = 2^f$ and (i) *$n = 1$ and ℓ a Fermat prime, or* (ii) *$\ell^n = 3^2$ and $q = 2^3$.*

Proof. From the structure of the Brauer tree (3.9) one concludes that $P_{\bar{\mathbb{F}}_\ell}$ is a uniserial $\bar{\mathbb{F}}_\ell[G]$-module of Loewy length 3 with

$$\mathrm{rad}_j(P_{\bar{\mathbb{F}}_\ell})/\,\mathrm{rad}_{j+1}(P_{\bar{\mathbb{F}}_\ell}) = \begin{cases} \bar{\mathbb{F}}_\ell, & \text{if } j = 0, 2, \\ \mathfrak{st}^\circ(\ell), & \text{if } j = 1, \\ 0, & \text{if } j \geq 3. \end{cases} \tag{3.14}$$

This yields (3.11) and the first part of (d). Moreover,

$$\mathrm{rad}(P_{\mathfrak{st}^\circ(\ell)})/\,\mathrm{soc}(P_{\mathfrak{st}^\circ(\ell)}) \simeq \bar{\mathbb{F}}_\ell \oplus Q, \tag{3.15}$$

where Q is a uniserial $\bar{\mathbb{F}}_\ell[G]$-module of Loewy length $m - 1$ and

$$\mathrm{rad}_j(Q)/\,\mathrm{rad}_{j+1}(Q) \simeq \mathfrak{st}^\circ(\ell) \tag{3.16}$$

for $j = 0, .., m - 2$. This implies that $\Omega^2(\bar{\mathbb{F}}_\ell)$ is a uniserial $\bar{\mathbb{F}}_\ell[G]$-module of Loewy length m with all its composition factors isomorphic to $\mathfrak{st}^\circ(\ell)$. This yields (a). (b) is a direct consequence of (a), and (c) follows from the Loewy structure of $\Omega^1(\bar{\mathbb{F}}_\ell)$ and $\Omega^2(\bar{\mathbb{F}}_\ell)$. (d) follows from an elementary calculation. Assume $c_\ell(G) = 1$. From

(d) one concludes that (p^f, ℓ^n) are consecutive prime powers and this yields the claim. □

3.2. **Case II:** $\ell = 2$**,** $p \neq 2$**.** In this case a 2-Sylow subgroup $L \in \mathrm{Syl}_2(G)$ of G is dihedral. For short we put $2^n = |L|$. The principal block b_0 of $\bar{\mathbb{F}}_2[G]$ has 3 isomorphism classes of irreducible modules: $\bar{\mathbb{F}}_2$, S_1, S_2, where $\dim_{\bar{\mathbb{F}}_2}(S_i) = \frac{1}{2}(q-1)$ for $i = 1, 2$. Let P_0, P_1, P_2 denote their minimal projective covers. The Loewy structures of these modules were described explicitly by K.Erdmann (cf. [8]). One has to distinguish two cases depending whether q is congruent to 1 or 3 modulo 4.

Let $M\colon = \mathrm{ind}_G^B(\bar{\mathbb{F}}_2)$, where $B \leq G$ is the normalizer of a p-Sylow subgroup of G. Then M has $\bar{\mathbb{F}}_2[G]$-submodules M_i, $i = 1, 2$, $M_1 \leq M_2$, such that $M/M_1 \simeq M_2 \simeq \bar{\mathbb{F}}_2$ and

$$M_1/M_2 \simeq S_1 \oplus S_2 \tag{3.17}$$

(cf. [8, Lemma 4.3, Lemma 5.1]). For short we put $\mathfrak{st}^\circ(2)\colon = M_1/M_2$.

3.2.1. *Case IIa:* $\ell = 2$*,* $q \equiv 1 \mod 4$.

Proposition 3.3. *Let* $G = \mathrm{PSL}_2(q)$ *with* $q \equiv 1 \mod 4$ *and put* $r\colon = \frac{q-1}{2^n}$.

(a) $\Omega^2(\bar{\mathbb{F}}_2)$ *is an indecomposable* $\bar{\mathbb{F}}_2[G]$*-module of Loewy length* 2 *with*

$$\begin{aligned} \mathrm{hd}(\Omega^2(\bar{\mathbb{F}}_2)) &\simeq \bar{\mathbb{F}}_2, \\ \mathrm{soc}(\Omega^2(\bar{\mathbb{F}}_2)) &\simeq S_1 \oplus S_2. \end{aligned} \tag{3.18}$$

In particular,

$$\Lambda_{G,2} = \frac{1}{|G|}(1_* - \mathrm{St}_*). \tag{3.19}$$

(b) $\mathrm{proj}_\ell(G) = 1 + r = 1 + \frac{q-1}{2^n}$.

(c) *For* $[S] \in \bar{\mathfrak{S}}_\ell(G)$ *one has*

$$H^1(G, S) \simeq \begin{cases} \bar{\mathbb{F}}_\ell, & \text{if } [S] = [S_1],\ [S_2], \\ 0, & \text{if } [S] \neq [S_1],\ [S_2], \end{cases} \tag{3.20}$$

and

$$H^2(G, S) \simeq \begin{cases} \bar{\mathbb{F}}_\ell, & \text{if } [S] = [\bar{\mathbb{F}}_2], \\ 0, & \text{if } [S] \neq [\bar{\mathbb{F}}_2]. \end{cases} \tag{3.21}$$

(d) *One has*

$$\begin{aligned} c_\ell(G) &= \tfrac{1}{2}(q+1), \\ c_{S_i} &= \tfrac{1}{2}(r + \tfrac{q+1}{2}). \end{aligned} \tag{3.22}$$

In particular, G *is not* $(2,1)$*-minimal.*

Proof. By [8, Thm.2], P_1 and P_2 are uniserial modules of Loewy length $2^n + 1$. More precisely,

$$\mathrm{rad}_j(P_i)/\mathrm{rad}_{j+1}(P_i) \simeq \begin{cases} S_i, & \text{if } j \leq 2^n,\ j+1 \equiv 1 \mod 4, \\ S_{i'}, & \text{if } j \leq 2^n,\ j+1 \equiv 3 \mod 4, \\ \bar{\mathbb{F}}_2, & \text{if } j \leq 2^n,\ j+1 \equiv 0 \mod 2, \\ 0, & \text{if } j \geq 2^n+1, \end{cases} \tag{3.23}$$

where $\{i, i'\} = \{1, 2\}$. Moreover,

$$\operatorname{rad}(P_0)/\operatorname{soc}(P_0) = \operatorname{rad}_2(P_1) \oplus \operatorname{rad}_2(P_2). \tag{3.24}$$

Hence the indecomposability of $\Omega^2(\bar{\mathbb{F}}_2)$ yields (a). Moreover, (b) is a direct consequence of (a), and (c) can be deduced from (3.23), (3.24) and (a). (d) follow from an elementary calculation. □

3.2.2. *Case IIb: $\ell = 2$, $q \equiv 3 \mod 4$.*

Proposition 3.4. *Let $G = \mathrm{PSL}_2(q)$ with $q \equiv 3 \mod 4$ and put $r\colon = \frac{q+1}{2^n}$.*

(a) *$\Omega^2(\bar{\mathbb{F}}_2)$ is an indecomposable $\bar{\mathbb{F}}_2[G]$-module with*

$$\begin{aligned} \operatorname{hd}(\Omega^2(\bar{\mathbb{F}}_2)) &\simeq \bar{\mathbb{F}}_2 \oplus S_1 \oplus S_2, \\ \operatorname{soc}(\Omega^2(\bar{\mathbb{F}}_2)) &\simeq S_1 \oplus S_2, \\ \Omega_2(\bar{\mathbb{F}}_2)/\operatorname{soc}(\Omega_2(\bar{\mathbb{F}}_2)) &\simeq \bar{\mathbb{F}}_2 \oplus R_1 \oplus R_2, \end{aligned} \tag{3.25}$$

where R_i are uniserial $\bar{\mathbb{F}}_2[G]$-modules of length $2^{n-1} - 1$ with $\operatorname{hd}(R_i) = S_i$, and each composition factor is isomorphic to either S_1 or S_2. Moreover,

$$\Lambda_{G,2} = \frac{2^{n-1}}{|G|}(1_* - \mathrm{St}_*) \tag{3.26}$$

(b) $\operatorname{proj}_\ell(G) = \frac{1}{2}(q + 1)$.

(c) *For $[S] \in \bar{\mathfrak{S}}_\ell(G)$ one has*

$$H^1(G, S) \simeq \begin{cases} \bar{\mathbb{F}}_\ell, & \text{if } [S] = [S_1],\ [S_2], \\ 0, & \text{if } [S] \neq [S_1],\ [S_2], \end{cases} \tag{3.27}$$

and

$$H^2(G, S) \simeq \begin{cases} \bar{\mathbb{F}}_\ell, & \text{if } [S] = [\bar{\mathbb{F}}_2],\ [S_1],\ [S_2], \\ 0, & \text{if } [S] \neq [\bar{\mathbb{F}}_2],\ [S_1],\ [S_2]. \end{cases} \tag{3.28}$$

(d) *One has*

$$\begin{aligned} c_2(G) &= \tfrac{q+1}{2^n}, \\ c_{S_i} &= \tfrac{1}{2}(r + \tfrac{1}{2}(q - 1)) \end{aligned} \tag{3.29}$$

for $i = 1, 2$. In particular, G is $(2, 1)$-minimal, if and only if $q = p$ and p is a Mersenne prime.

Proof. By [8, Thm.4], one has

$$\operatorname{rad}(P_0)/\operatorname{soc}(P_0) \simeq S_1 \oplus S_2. \tag{3.30}$$

This yields (3.27). Moreover, for $i = 1, 2$,

$$\operatorname{rad}(P_i)/\operatorname{soc}(P_i) \simeq \bar{\mathbb{F}}_2 \oplus R_i, \tag{3.31}$$

where R_i are uniserial $\bar{\mathbb{F}}_2[G]$-modules of Loewy length $2^{n-1} - 1$ with

$$\operatorname{rad}_j(R_i)/\operatorname{rad}_{j+1}(R_i) \simeq \begin{cases} S_{i'} & \text{if } j \leq 2^{n-1} - 2,\ j \text{ even}, \\ S_i & \text{if } j \leq 2^{n-1} - 2,\ j \text{ odd}, \\ 0 & \text{if } j \geq 2^{n-1} - 1, \end{cases} \tag{3.32}$$

where $\{i, i'\} = \{1, 2\}$. This yields (3.25) and (3.28). Note that

$$\dim_{\bar{\mathbb{F}}_2}(\Omega_2(\bar{\mathbb{F}}_2)) = 1 + 2(2^{n-1} - 1)\tfrac{q-1}{2} + 2\tfrac{q-1}{2} = 1 + 2^n\tfrac{q-1}{2}, \tag{3.33}$$

which yields (b) (cf. (2.13)). (3.29) follows from an elementary calculation and implies the final remark. □

The previously mentioned discussion also shows the following interesting phenomenon.

Proposition 3.5. *Let* $G = \mathrm{PSL}_2(q)$ *with* $q \equiv 3 \mod 4$. *Then there exists an injective map*

$$\alpha\colon \Omega^{-1}(\mathbb{F}_2) \longrightarrow \Omega^2(\mathbb{F}_2). \tag{3.34}$$

Proof. By standard arguments it suffices to show that there exists an injective map $\beta\colon \Omega^{-1}(\bar{\mathbb{F}}_2) \to \Omega^2(\bar{\mathbb{F}}_2)$. Let $M \leq \Omega^2(\bar{\mathbb{F}}_2)$ be the $\bar{\mathbb{F}}_2[G]$-submodule containing $\mathrm{soc}(\Omega^2(\bar{\mathbb{F}}_2))$ such that $M/\mathrm{soc}(\Omega^2(\bar{\mathbb{F}}_2)) \simeq \bar{\mathbb{F}}_2$. Then (3.30) implies that $M \simeq \Omega^{-1}(\bar{\mathbb{F}}_2)$. This yields the claim. □

3.3. **Case III:** $\ell = p$. In this case an ℓ-Sylow subgroup $L \in \mathrm{Syl}_\ell(G)$ of $G = \mathrm{PSL}_2(\ell^f)$ is elementary abelian of order ℓ^f. Hence for $f = 1$, L is cylcic and again the modular representation theory will be significantly easier. In this case one can use the explicit description of the projective indecomposable modules given by J.E.Humphreys (cf. [19, §11.1]) in order to determine the structure of $\Omega^2(\bar{\mathbb{F}}_\ell)$. For $f \geq 2$ one has also a description of these modules given by H.H.Anderson, J.Jørgensen and P.Landrock (cf. [2]). However, it will not be sufficient in order to determine the radical filtration of $\Omega^2(\bar{\mathbb{F}}_\ell)$ explicitly.

Let $\bar{G}\colon = \mathrm{SL}_2(\ell^f)$. It was already known to R.Brauer and C.Nesbitt (cf. [4]) that every irreducible $\bar{\mathbb{F}}_\ell[\bar{G}]$-module is isomorphic to some heighest weight module M_λ, $\lambda = \sum_{i=1}^f \lambda_i \ell^{i-1}$, $0 \leq \lambda_i \leq \ell - 1$, for the algebraic group SL_2 defined over $\bar{\mathbb{F}}_\ell$ restricted to $\bar{G}$. By Steinberg's tensor product theorem,

$$M_\lambda \simeq M_{\lambda_1} \otimes M_{\lambda_2}^{[\ell]} \otimes \cdots \otimes M_{\lambda_f}^{[\ell^{f-1}]}, \tag{3.35}$$

where $\_^{[\ell^k]}$ denotes the k^{th} Frobenius twist. For matters of convenience we will identify the weight λ corresponding to an irreducible $\bar{\mathbb{F}}_\ell[\bar{G}]$-module M_λ with the f-tupel $(\lambda_1, .., \lambda_f)$. We also put

$$(\lambda_1, \dots, \lambda_f)^F\colon = (\lambda_2, \dots, \lambda_f, \lambda_1). \tag{3.36}$$

Moreover, M_0 is isomorphic to the trivial $\bar{\mathbb{F}}_\ell[\bar{G}]$-module, M_1 is isomorphic to the natural $\bar{\mathbb{F}}_\ell[\bar{G}]$-module, and provided $\ell \neq 2$, M_2 is isomorphic to the $\bar{\mathbb{F}}_\ell$-Chevalley Lie algebra $\mathfrak{sl}_2$ with the adjoint action of $\bar{G}$. If $0 \leq \lambda \leq \ell - 1$,

$$\dim_{\bar{\mathbb{F}}_\ell}(M_\lambda) = \lambda + 1. \tag{3.37}$$

Hence (3.35) and (3.37) allow to compute the $\bar{\mathbb{F}}_\ell$-dimension of every irreducible $\bar{\mathbb{F}}_\ell[\bar{G}]$-module M_λ.

Obviously, M_λ is an irreducible $\bar{\mathbb{F}}_\ell[G]$-module if $\ell = 2$, or if ℓ is odd and λ is even. Let ℓ be odd and let $z \in Z(\bar{G})$ denote the non-trivial element in the center of $\bar{G}$. Then $e\colon = \frac{1}{2}(1 + z) \in \bar{\mathbb{F}}_\ell[\bar{G}]$ is a central idempotent acting as the identity on every M_λ with λ even. Thus for λ even, the projective indecomposable $\bar{\mathbb{F}}_\ell[\bar{G}]$-module P_{M_λ}, $\mathrm{hd}(P_{M_\lambda}) = M_\lambda$, is in fact a projective indecomposable $\bar{\mathbb{F}}_\ell[G]$-module. Fur further details see [18].

The canonical map $\rho_\ell\colon \mathbb{Z}_\ell \to \mathbb{F}_\ell$ restricted to the set of roots of unity co-prime to ℓ is injective. Thus it induces a bijection $\tau\colon \mathbb{F}_\ell^* \to O_{\ell'}(\mu(\mathbb{Z}_\ell))$ where $\mu(\mathbb{Z}_\ell)$ denotes the subgroup of roots of unity. Hence one has a canonical non-trivial character $\tau\colon \mathbb{F}_\ell^* \to \mathbb{Q}_p^*$. By τ^k we denote the k-fold product of τ by itself.

Let $T \leq \mathrm{SL}_2(\ell)$ be a maximal split torus. The fundamental co-root yields a canonical isomorphism $\alpha^\vee\colon \mathbb{F}_\ell^* \to T$. By χ we denote the irreducible character of $\mathrm{PSL}_2(\ell)$ of degree $\ell + 1$ satisfying

$$\chi(\alpha^\vee(t)) = \tau^2(t) + \tau^2(t^{-1}) \tag{3.38}$$

for all $t \in \mathbb{F}_\ell^*$.

3.3.1. *Case IIIa: $\ell = p = q$.* If ℓ is odd, the minimal projective cover $P_0 \colon = P_{\bar{\mathbb{F}}_\ell}$ of $\bar{\mathbb{F}}_\ell$ is a uniserial $\bar{\mathbb{F}}_\ell[G]$-module of Loewy length 3 with

$$\operatorname{rad}_j(P_0)/\operatorname{rad}_{j+1}(P_0) \simeq \begin{cases} \bar{\mathbb{F}}_\ell, & \text{if } j = 0, 2, \\ M_{\ell-3}, & \text{if } j = 1, \\ 0, & \text{if } j \geq 3. \end{cases} \tag{3.39}$$

Moreover, provided $\ell \neq 3$, $P_{M_{\ell-3}}$ has also Loewy length 3 and

$$\operatorname{rad}(P_{M_{\ell-3}})/\operatorname{rad}_{j+1}(P_{M_{\ell-3}}) \simeq \begin{cases} M_{\ell-3}, & \text{if } j = 0, 2, \\ M_0 \oplus M_2, & \text{if } j = 1, \\ 0, & \text{if } j \geq 3. \end{cases} \tag{3.40}$$

If $\ell = 2$, P_0 is uniserial of Loewy length 2 (cf. [18], [19, §11.1]). This implies:

Proposition 3.6. *Let $G = \mathrm{PSL}_2(\ell)$.*

(a) *For $\ell = 2$ or 3 one has $\Omega^2(\bar{\mathbb{F}}_\ell) \simeq \bar{\mathbb{F}}_\ell$, and thus $\operatorname{proj}_\ell(G) = 1$. Moreover, for $k \geq 0$ and $[S] \in \bar{\mathfrak{S}}_\ell(G)$*

$$H^k(G, S) \simeq \begin{cases} \bar{\mathbb{F}}_\ell, & \text{if } [S] = [\bar{\mathbb{F}}_\ell], \\ 0, & \text{if } [S] \neq [\bar{\mathbb{F}}_\ell]. \end{cases} \tag{3.41}$$

(b) *Let $\ell \notin \{2, 3\}$. Then $\Omega^2(\bar{\mathbb{F}}_\ell)$ is a uniserial $\bar{\mathbb{F}}_\ell[G]$-module of Loewy length 2 with*

$$\operatorname{rad}_j(\Omega^2(\bar{\mathbb{F}}_\ell))/\operatorname{rad}_{j+1}(\Omega^2(\bar{\mathbb{F}}_\ell)) \simeq \begin{cases} \mathfrak{sl}_2, & \text{if } j = 0, \\ M_{\ell-3}, & \text{if } j = 1, \\ 0, & \text{if } j \geq 2. \end{cases} \tag{3.42}$$

In particular,

$$\Lambda_{G,\ell} = \frac{1}{|G|}(1_* - \bar{\chi}_*), \tag{3.43}$$

where χ is given as in (3.38)*.*

(c) *Let $\ell \neq 2, 3$. Then $\operatorname{proj}_\ell(G) = 2$.*

(d) *Let $\ell \neq 2, 3$. For $[S] \in \bar{\mathfrak{S}}_\ell(G)$ one has*

$$H^1(G, S) \simeq \begin{cases} \bar{\mathbb{F}}_\ell, & \text{if } [S] = [M_{\ell-3}], \\ 0, & \text{if } [S] \neq [M_{\ell-3}], \end{cases} \tag{3.44}$$

and

$$H^2(G, S) \simeq \begin{cases} \bar{\mathbb{F}}_\ell, & \text{if } [S] = [\mathfrak{sl}_2], \\ 0, & \text{if } [S] \neq [\mathfrak{sl}_2]. \end{cases} \tag{3.45}$$

(e) *For all primes ℓ one has $c_\ell(G) = 1$, i.e., G is $(\ell, 1)$-minimal.*

Proof. For $\ell = 2, 3$, $G = O_{\ell'}(G) \rtimes L$, and $L \in \mathrm{Syl}_\ell(G)$ is a cyclic of order ℓ. This yields (a). Assume that $\ell \neq 2, 3$. The structure of $P_{\bar{\mathbb{F}}_\ell}$ and $P_{M_{\ell-3}}$ implies (3.42), (d) and (e) (cf. (3.39)). The weight space decomposition of M_2 and $M_{\ell-3}$ shows that the Brauer character of $\Omega^2(\bar{\mathbb{F}}_\ell)$ coincides with χ_*. This yields (3.43). (c) is a direct consequence of (b). □

3.3.2. *Case IIIb: $\ell = p$, $q = \ell^f$, $f \geq 2$.* In this case we cannot determine the structure of $\Omega^2(\bar{\mathbb{F}}_\ell)$ explicitly. Nevertheless, we will prove:

Proposition 3.7. *Let $G = \mathrm{PSL}_2(\ell^f)$, $f \geq 2$.*

(a) *One has*

$$\dim_{\bar{\mathbb{F}}_\ell}(\Omega^2(\bar{\mathbb{F}}_\ell)) = \begin{cases} 1 + ((f-1)2^f + 1)\ell^f, & \ell \neq 2, \\ 1 + ((f-2)2^{f-1} + 1)2^f, & \ell = 2. \end{cases} \tag{3.46}$$

(b)

$$\mathrm{proj}_\ell(G) = \begin{cases} 2 + (f-1) \cdot 2^f, & \text{for } \ell \neq 2, \\ 2 + (f-2) \cdot 2^{f-1}, & \text{for } \ell = 2. \end{cases} \tag{3.47}$$

(c) *Let $M = M_\lambda$ be an irreducible $\bar{\mathbb{F}}_\ell[G]$-module. Then $H^1(G, M) = 0$ unless $\lambda = (\ell - 2, 1, 0, \ldots, \ell)^{F^r}$ for some $r \geq 0$. In this case one has $H^1(G, M) \simeq \bar{\mathbb{F}}_\ell$ for $\ell^f \neq 9$, and*

$$H^1(\mathrm{PSL}_2(9), M_{(1,1)}) \simeq \bar{\mathbb{F}}_3^2. \tag{3.48}$$

(d) *$c_\ell(G) = 2^f - 1$. Hence G is not $(\ell, 1)$-minimal.*

Proof. The statement (c) was proved in [2, Cor.4.5] [5]. For all $[S] \in \bar{\mathfrak{S}}_\ell(G)$ with $H^1(G, S) \neq 0$, one has

$$\dim_{\bar{\mathbb{F}}_\ell}(P_S) = \begin{cases} 2^f \cdot \ell^f & \text{if } \ell \neq 2, \\ 2^{f-1} \cdot 2^f & \text{if } \ell = 2. \end{cases} \tag{3.49}$$

(cf. [18, Thm.5]). Hence (a) follows from (2.16), (c) and (3.49). Moreover, (b) is a direct consequence of (a) and Proposition (2.4), and (d) is a direct consequence of J. E. Humphreys' theorem. □

In [6, Thm.4.3], J. F. Carlson has computed the degree 2 cohomology for every irreducible $\bar{\mathbb{F}}_\ell[G]$-module M. Thus, apart from the dimension one knows also the head of the $\bar{\mathbb{F}}_\ell[G]$-module $\Omega^2(\bar{\mathbb{F}}_\ell)$. We list his result in the following proposition.

Proposition 3.8. *Let $G = \mathrm{PSL}_2(\ell^f)$, $f \geq 2$, and let $M = M_\lambda$ be an irreducible $\bar{\mathbb{F}}_\ell[G]$-module. Then $H^2(G, M) = 0$ unless one of the following holds:*

(a) *$\ell \neq 2$, $f = 2$ and $\lambda = (\ell - 3, \ell - 3)$,*
(b) *$\ell \neq 2$, $f \geq 2$ and $\lambda = (2, 0, \ldots, 0)^{F^k}$ for some $k \geq 0$,*
(c) *$\ell = 3$, $f = 4$ and $\lambda = (1, 1, 1, 1)$,*
(d) *$f \geq 4$ and $\lambda = (\lambda_1, \ldots, \lambda_f)$ is a weight with the following properties: there exist indices $i, j \in \{1, \ldots, f\}$ such that*

$$(\lambda_i, \lambda_{i+1}) = (\lambda_j, \lambda_{j+1}) = (\ell - 2, 1), \tag{3.50}$$

$i, i+1, j, j+1$ are pairwise distinct, and $\lambda_k = 0$ if $k \notin \{i, i+1, j, j+1\}$,

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[5] The case $(\ell, f) = (3, 2)$ has been missed out in [2, Cor.4.5(b)] (cf. [6, Thm.4.1(iii) and (iv)]).

(e) $\ell = 2$, $f = 2$ *and* $\lambda = (0,0)$,

(f) $\ell = 2$, $f \geq 3$ *and* $\lambda = (1,0,\dots,0)^{F^k}$ *for some* $k \geq 0$.

Moreover, in each case apart from (c), *one has* $H^2(G,M) \simeq \bar{\mathbb{F}}_\ell$, *while*

$$H^2(\mathrm{PSL}_2(81), M_{(1,1,1,1)}) \simeq \bar{\mathbb{F}}_3^2. \tag{3.51}$$

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