FRATTINI EXTENSIONS AND CLASS FIELD THEORY

TH. WEIGEL

Abstract. A. Brumer has shown that every profinite group of strict cohomological p-dimension 2 possesses a class field theory - the tautological class field theory. In particular, this result also applies to the universal p-Frattini extension $\tilde{G}_p$ of a finite group G. We use this fact in order to establish a class field theory for every p-Frattini extension $\pi\colon \tilde{G} \to G$ (Thm.A). The role of the class field module will be played by the p-Frattini module. The universal norms of this class field theory will carry important information about the p-Frattini extension $\pi\colon \tilde{G} \to G$. A detailled analysis will lead to a characterization of finite groups G which have a p-Frattini extension $\pi\colon \tilde{G} \to G$ in which $\tilde{G}$ is a weakly-orientable p-Poincaré duality group of dimension 2 (Thm.B).

In section §5 we characterize the p-Frattini extensions $\pi_{A_1}: Sl_2(\mathbb{Z}_p) \to Sl_2(\mathbb{F}_p)$, $p \neq 2, 3, 5$, by some kind of localization technique. This answers a question posed by M.D.Fried and M.Jarden (Thm.C). It is quite likely that such an approach might also be successful for the characterization of the p-Frattini extensions $\pi_D : X_D(\mathbb{Z}_p) \to X_D(\mathbb{F}_p)$, where X_D is the simple simply-connected split $\mathbb{Z}$-Chevalley group scheme with Dynkin diagram D.

1. Introduction

Let G be a finite group and let p be a prime number. An extension of G by a pro-p group A

$$1 \longrightarrow A \xrightarrow{\iota} \tilde{G} \xrightarrow{\pi} G \longrightarrow 1 \tag{1.1}$$

is called a *p-Frattini extension*, if $im(\iota)$ is contained in the Frattini subgroup of $\tilde{G}$. The study of p-Frattini extensions of finite groups has a long history. W.Gaschütz (cf. [8]) showed that every finite group G has a universal elementary p-abelian Frattini extension $\pi_{/p}\colon \tilde{G}_{/p} \to G$ which kernel is - considered as (left) $\mathbb{F}_p[G]$-module - isomorphic to $\Omega_2(G, \mathbb{F}_p)$, where $\Omega_k(G, \_) = \Omega^{-k}(G, \_)$ denotes the k^{th}-Heller translate in the category ${}_G\text{mod}_p$ of finitely generated (left) $\mathbb{F}_p[G]$-modules. Based on this result J.Cossey, L.G.Kovács and O.H.Kegel [3] showed the existence of a universal p-Frattini cover $\pi_p\colon \tilde{G}_p \to G$. As the universal p-Frattini cover coincides with the minimal projective cover (cf. [6, Prop.20.33]), K.Gruenberg's theorem [7] implies that $\tilde{G}_p$ is of cohomological p-dimension less or equal to 1, i.e, $cd_p(\tilde{G}_p) \leq 1$. In particular, $ker(\pi_p)$ is a finitely generated free pro-p group (cf. [12, §I.4.2, Cor.2]).

If p divides the order of G, the profinite group $\tilde{G}_p$ is of strict cohomological p-dimension 2. For these groups A.Brumer [2] showed the existence of a *tautological class field theory*. The goal of this paper is to use this tautological class field theory for the group $\tilde{G}_p$ in order to obtain new result on p-Frattini extensions.

The most efficient way to establish a class field theory is to use the theory of *cohomological Mackey-Functors*. A. Dress introduced this notion in [4]. The

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exposition given by P.Webb in [15] will be particularly useful for our purpose, and therefore we will follow it closely as far as possible.

The following theorem can be seen as a "structure theorem for p-Frattini extensions", which combines W.Gaschütz theorem with the fact that the inflation mapping $H^1(\pi, S)$ is bijective for a p-Frattini extension π as in (1.1) and an irreducible (left) $\mathbb{F}_p[G]$-module S [16, Prop.3.1]. Its proof can be found in section 3.3 (cf. Thm.3.1, Cor.3.2).

Theorem A. *Let G be a finite group, let p be a prime number and let $\pi\colon \tilde{G} \to G$ be a p-Frattini extension. Let $\mathcal{F}(\tilde{G})$ be the set of all open normal subgroups of $\tilde{G}$ being contained in $ker(\pi)$. Then there exists a p-class field theory $(\mathbf{C}, \gamma)$ for $(\tilde{G}, \mathcal{F})$, i.e.,*

(i) *$\mathbf{C}$ is a cohomological $\mathcal{F}(\tilde{G})$-Mackey functor of type H^0 (this is a short form to say that it has Galois descent),*

(ii) *$\mathbf{C}_U = \Omega_2(\tilde{G}/U, \mathbb{Z}_p)$ for all $U \in \mathcal{F}(\tilde{G})$,*

(iii) *$\gamma\colon \mathbf{C} \to \mathbf{Ab}^p$ is a surjective morphism of cohomological $\mathcal{F}(\tilde{G})$-Mackey functors, where $\mathbf{Ab}^p$ denotes the cohomological $\mathcal{F}(\tilde{G})$-Mackey functor of maximal p-abelian quotients (cf. §3.1),*

(iv) *for all $U, V \in \mathcal{F}(\tilde{G})$, $V \leq U$, γ induces an isomorphism*

$$\mathbf{C}_U / im(N^{\mathbf{C}}_{V,U}) \simeq (U/V)^{ab}_p, \tag{1.2}$$

(v) *let $U, V, W \in \mathcal{F}(\tilde{G})$, $V, W \leq U$, such that U/V and U/W are abelian p-groups. Then $im(N^{\mathbf{C}}_{V,U}) = im(N^{\mathbf{C}}_{W,U})$ implies $V = W$.*

The class field theory $(\mathbf{C}, \gamma)$ has also two further properties one would usually require from a class field theory: (vi) There exists a canonical class $c \in \mathbf{nat}^2(\mathfrak{X}(\mathbb{Z}_p), \mathbf{C})$, (vii) $H^1(\tilde{G}/V, \mathbf{C}_V) = H^1(U/V, \mathbf{C}_V) = 0$ for all $U, V \in \mathcal{F}(\tilde{G})$, $V \leq U$ (cf. Rem.3.3). However, this will not be of importance for our purpose.

The kernel of γ will be called *the universal norms* (of $\mathbf{C}$). Its analysis will finally enable us to characterize finite groups G possessing a p-Frattini cover $\pi\colon \tilde{G} \to G$ in which $\tilde{G}$ is a weakly-orientable profinite p-Poincaré duality group of dimension 2 (cf. Cor.4.6). Here we call a profinite p-Poincaré duality group $\tilde{G}$ of dimension d *weakly-orientable*, if $H^d(\tilde{G}, \mathbb{F}_p[\![\tilde{G}]\!]) \simeq \mathbb{F}_p$ is the trivial module.

Theorem B. *Let G be a finite group, and let p be a prime number. Then the following are equivalent:*

(i) *There exist a p-Frattini extension $\pi\colon \tilde{G} \to G$, where $\tilde{G}$ is a profinite weakly-orientable p-Poincaré duality group of dimension 2.*

(ii) *There exists an injective map*

$$\alpha\colon \Omega^1(G, \mathbb{F}_p) \longrightarrow \Omega_2(G, \mathbb{F}_p) \tag{1.3}$$

which is not an isomorphism.

Remark 1.1. Theorem B raises the following two questions: (1) For which finite groups G and prime numbers p does there exist an injective but not surjective map $\alpha\colon \Omega^1(G, \mathbb{F}_p) \longrightarrow \Omega_2(G, \mathbb{F}_p)$? (2) Provided such a mapping exists, how many isomorphism types of p-Frattini covers $\pi\colon \tilde{G} \to G$ exist, where $\tilde{G}$ is a weakly-orientable p-Poincaré duality group of dimension 2?

Unfortunately, we cannot say anything about the second question. Explicit computations using the work of K.Erdmann [5] show that for $q \equiv 3 \mod 4$, such a mapping α exists for $G\colon = PSl_2(q)$ and $p = 2$ (cf. [16], [17]). However, it seems a very difficult problem to characterize or classify the tuples (G,p) for which such a mapping exists.

Let $\mathfrak{S}_p(G)$ denote the set of isomorphism types of irreducible (left) $\mathbb{F}_p[G]$-modules, and let $\Delta \subseteq \mathfrak{S}_p(G)$ be a subset of $\mathfrak{S}_p(G)$. For short we call a p-Frattini extension $\pi\colon \tilde{G} \to G$ a Δ-*Frattini extension*, if the isomorphism type of every G-composition factor of $ker(\pi)$ is contained in Δ. From the existence of the universal p-Frattini extension one deduces easily the existence of a universal Δ-Frattini extension $\pi_\Delta\colon \tilde{G}_\Delta \to G$ (cf. §5.2). Obviously, $\tilde{G}_{\mathfrak{S}_p(G)}$ coincides with $\tilde{G}_p$, and $\tilde{G}_\emptyset$ coincides with G itself. For our purpose it will be useful that the universal Δ-Frattini extension can be charcterized by vanishing of second degree cohomology in a similar way as it is known for the universal p-Frattini extension (cf. Prop.5.1).

It is well-known that for $p \neq 3$, the extension

$$\pi_{A_1}\colon Sl_2(\mathbb{Z}_p) \longrightarrow Sl_2(\mathbb{F}_p) \tag{1.4}$$

is indeed a p-Frattini extension (cf. [18]). However, it remained an open problem to characterize the extension π_{A_1} among all p-Frattini extension (cf. [6, Problem 20.40]).

For $p \neq 2,3$, M.Lazard's theorem implies that $Sl_2(\mathbb{Z}_p)$ is an orientable p-Poincaré duality group of dimension 3 (cf. [13]). From this fact we will deduces the following characterization:

Theorem C. *Let p be a prime different from 2, 3 and 5. Let M_k, $k = 0,\dots,p-1$, denote the simple $\mathbb{F}_p[Sl_2(\mathbb{F}_p)]$-module of weight k and $\mathbb{F}_p$-dimension $k+1$. Then for every subset $\Delta \subset \mathfrak{S}_p(Sl_2(\mathbb{F}_p))$ satisfying*

(i) $[M_2] \in \Delta$,
(ii) $[M_{p-3}] \notin \Delta$,

the universal Δ-Frattini extension π_Δ of $Sl_2(\mathbb{F}_p)$ coincides with π_{A_1}, i.e., one has an isomorphism

$$\phi\colon \widetilde{Sl}_2(\mathbb{F}_p)_\Delta \longrightarrow Sl_2(\mathbb{Z}_p) \tag{1.5}$$

satisfying $\pi_{A_1} \circ \phi = \pi_\Delta$.

For a given Dynkin diagram D let X_D be the simple simply-connected $\mathbb{Z}$-Chevalley group scheme associated to D. It has been proved in [18] that apart from finitely many (more or less explicitly known) values of (D,p),

$$\pi_D\colon X_D(\mathbb{Z}_p) \longrightarrow X_D(\mathbb{F}_p) \tag{1.6}$$

is a p-Frattini extension. Therefore, one wonders whether one can characterize $X_D(\mathbb{Z}_p)$ in a similar fashion as $Sl_2(\mathbb{Z}_p)$ answering the problem raised in [6, Prob.20.40] in a wider context:

Question 1.2. *Assume that p is large with respect to the Coxeter number of D. Let $\mathfrak{L}_D(\mathbb{F}_p)$ denote the $\mathbb{F}_p$-Chevalley Lie algebra associated to D considered as (left) $\mathbb{F}_p[X_D(\mathbb{F}_p)]$-module and put $\Delta_D\colon = \{[\mathfrak{L}_D(\mathbb{F}_p)]\}$. Are π_D and π_{Δ_D} isomorphic p-Frattini covers?*

Remark 1.3. Proposition 5.1 shows that Question 1.2 is equivalent to the question whether

$$H^2(X_D(\mathbb{Z}_p), \mathfrak{L}_D(\mathbb{F}_p)) = 0. \tag{1.7}$$

2. Cohomological Mackey functors

2.1. **Profinite modules of profinite groups.** Let p be a prime number, and let $\hat{G}$ be a profinite group. The *completed* $\mathbb{Z}_p$*-group algebra* of $\hat{G}$ is given by

$$\mathbb{Z}_p[\![\hat{G}]\!] := \varprojlim_U \mathbb{Z}_p[\hat{G}/U], \tag{2.1}$$

where the inverse system is running over all open normal subgroups of $\hat{G}$. By ${}_{\hat{G}}\mathbf{prf}_p$ we denote the abelian category the objects of which are abelian pro-p groups with continuous left $\hat{G}$-action. The morphisms from M to N, $M, N \in ob({}_{\hat{G}}\mathbf{prf}_p)$, are defined to be the continuous morphisms of profinite groups commuting with the action of $\hat{G}$. The abelian group of morphisms from M to N will be denoted by $\mathbf{Hom}_{\hat{G}}(M, N)$. This category can be identified with the full subcategory of the category of topological left $\mathbb{Z}_p[\![\hat{G}]\!]$-modules, the objects of which are also abelian pro-p groups. It is well-known that ${}_{\hat{G}}\mathbf{prf}_p$ has enough projectives, and in particular minimal projective covers. If $\hat{G}$ is the trivial group, then ${}_{\hat{G}}\mathbf{prf}_p$ coincides with the category of abelian pro-p groups, which we will denote by $\mathbf{prf}_p$.

By ${}_{\hat{G}}\mathbf{prf}_{/p}$ we denote the abelian category the objects of which are profinite $\mathbb{F}_p$-vector spaces with continuous left $\hat{G}$-action. It is a full subcategory of ${}_{\hat{G}}\mathbf{prf}_p$, and objects can be considered as topological modules for the *completed* $\mathbb{F}_p$*-group algebra*

$$\mathbb{F}_p[\![\hat{G}]\!] := \varprojlim_U \mathbb{F}_p[\hat{G}/U]. \tag{2.2}$$

For further details the reader may wish to consult [2], [11] or [13].

2.2. **Cohomological Mackey functors.** There are several equivalent ways to define a cohomological Mackey functor. Here we will follow more or less the approach chosen by P.Webb (cf. [15, §2]).

Let $\hat{G}$ be a profinite group and let $\mathcal{N}$ be a set of open normal subgroups of $\hat{G}$. For short we call $\mathcal{N}$ a *normal Mackey system*, if $\mathcal{N}$ is closed with respect to products and intersections, and if $\bigcap_{U\in\mathcal{N}} U = 1$.

Let $\mathcal{N}$ be a normal Mackey system of the profinite group $\hat{G}$. A *cohomological* $\mathcal{N}$*-Mackey functor* $\mathbf{X}$ with coefficients in $\mathbf{prf}_p$ is a collection $(\mathbf{X}_U)_{U\in\mathcal{N}}$ of $\hat{G}$-modules $\mathbf{X}_U \in ob({}_{\hat{G}/U}\mathbf{prf}_p)$, together with two series of mappings $i^{\mathbf{X}}_{U,V}$ and $N^{\mathbf{X}}_{V,U}$ for $U, V \in \mathcal{N}$, $V \leq U$, where

$$\begin{aligned} i^{\mathbf{X}}_{U,V} &\in \mathbf{Hom}_{\hat{G}/V}(\mathbf{X}_U, \mathbf{X}_V), \\ N^{\mathbf{X}}_{V,U} &\in \mathbf{Hom}_{\hat{G}/V}(\mathbf{X}_V, \mathbf{X}_U), \end{aligned} \tag{2.3}$$

and which satisfy the following relations:

$$i^{\mathbf{X}}_{U,U} = N^{\mathbf{X}}_{U,U} = id_{\mathbf{X}_U} \qquad \text{for all } U \in \mathcal{N}, \tag{2.4}$$

$$i^{\mathbf{X}}_{U,W} = i^{\mathbf{X}}_{V,W} \circ i^{\mathbf{X}}_{U,V} \qquad \text{for all } U, V, W \in \mathcal{N},\ U \leq V \leq W, \tag{2.5}$$

$$N^{\mathbf{X}}_{W,U} = N^{\mathbf{X}}_{V,U} \circ N^{\mathbf{X}}_{W,V} \qquad \text{for all } U, V, W \in \mathcal{N},\ U \leq V \leq W, \tag{2.6}$$

$$i^{\mathbf{X}}_{UV,V} \circ N^{\mathbf{X}}_{U,UV} = N^{\mathbf{X}}_{U\cap V,V} \circ i^{\mathbf{X}}_{U,U\cap V} \qquad \text{for all } U, V \in \mathcal{N}, \tag{2.7}$$

$$i^{\mathbf{X}}_{U,V} \circ N^{\mathbf{X}}_{V,U} = \sum_{x\in U/V} x \qquad \text{for all } U, V \in \mathcal{N},\ U \leq V, \tag{2.8}$$

$$N^{\mathbf{X}}_{V,U} \circ i^{\mathbf{X}}_{U,V} = |U : V|.id_{\mathbf{X}_U} \qquad \text{for all } U, V \in \mathcal{N},\ U \leq V, \tag{2.9}$$

The notation we have chosen is closer related to number theory than the one introduced in [15]. One can easily verify that the role of I_V^U in [15] is played by $N_{V,U}^{\mathbf{X}}$, and $i_{U,V}^{\mathbf{X}}$ plays the role of R_V^U. Our axioms (2.3) and (2.4)-(2.6) are obviously equivalent to the axioms (0)-(5) in [15, §2]. The axioms (2.7) and (2.8) are reformulating axiom (6) in [15], as we assumed that all open subgroups of $\hat{G}$ under consideration are normal in $\hat{G}$. Axiom (2.9) characterizes cohomological Mackey functors among all Mackey functors (cf. [15, §7]).

By $\mathfrak{CM}_{\mathcal{N}}(\hat{G}, \mathbf{prf}_p)$ we denote the category of cohomological $\mathcal{N}$-Mackey functors of $\hat{G}$ with coefficients in $\mathbf{prf}_p$. A morphism between cohomological $\mathcal{N}$-Mackey functors $\eta\colon \mathbf{X} \to \mathbf{Y}$ is a sequence of mappings $(\eta_U)_{U\in\mathcal{N}}$, $\eta_U \in \mathbf{Hom}_{\hat{G}/U}(\mathbf{X}_U, \mathbf{Y}_U)$, for which the diagrams

$$\begin{array}{ccc} \mathbf{X}_U & \xrightarrow{\eta_U} & \mathbf{Y}_U \\ {\scriptstyle i^{\mathbf{X}}_{U,V}}\downarrow & & \downarrow{\scriptstyle i^{\mathbf{Y}}_{U,V}} \\ \mathbf{X}_V & \xrightarrow{\eta_V} & \mathbf{Y}_V \end{array} \qquad \begin{array}{ccc} \mathbf{X}_U & \xrightarrow{\eta_U} & \mathbf{Y}_U \\ {\scriptstyle N^{\mathbf{X}}_{V,U}}\uparrow & & \uparrow{\scriptstyle N^{\mathbf{Y}}_{V,U}} \\ \mathbf{X}_V & \xrightarrow{\eta_V} & \mathbf{Y}_V \end{array} \tag{2.10}$$

commute for all $U, V \in \mathcal{N}$, $V \le U$. By $\mathbf{nat}(\mathbf{X}, \mathbf{Y})$ we denote the abelian group of morphisms of cohomological $\mathcal{N}$-Mackey functors from $\mathbf{X}$ to $\mathbf{Y}$.

Using the interpretation of $\mathfrak{CM}_{\mathcal{N}}(\hat{G}, \mathbf{prf}_p)$ as the category of additive $\mathbb{Z}_p$-linear functors from the category of $\hat{G}$-permutation modules of discrete $\hat{G}$-sets with isotropy group being contained in $\mathcal{N}$ to the category $\mathbf{prf}_p$ of abelian pro-p groups (cf. [15, Prop.7.2]), one sees easily that $\mathfrak{CM}_{\mathcal{N}}(\hat{G}, \mathbf{prf}_p)$ is an abelian category. Kernels and cokernels are defined in the obvious way.

2.3. **From cohomological Mackey functors to $\hat{G}$-modules and vice versa.** Taking the inverse limit over the norm maps $N_{V,U}$ defines a covariant left exact functor

$$\begin{aligned} m\colon \mathfrak{CM}_{\mathcal{N}}(\hat{G}, \mathbf{prf}_p) &\longrightarrow {}_{\hat{G}}\mathbf{prf}_p, \\ m(\mathbf{X})\colon = \varprojlim_{U\in\mathcal{N}} \mathbf{X}_U, &\quad \text{for } \mathbf{X} \in ob(\mathfrak{CM}_{\mathcal{N}}(\hat{G}, \mathbf{prf}_p)). \end{aligned} \tag{2.11}$$

In case $\mathcal{N}$ contains a countable basis of neighbourhoods of $1 \in \hat{G}$, $\varprojlim^1$ vanishes, since all modules $\mathbf{X}_U$ are compact. Hence in this case m is exact.

Let $M \in ob({}_{\hat{G}}\mathbf{prf}_p)$ be an abelian pro-p group with continuous left $\hat{G}$-action. For an open normal subgroup $U \in \mathcal{N}$ we denote by

$$M_U\colon = \mathbb{Z}_p[\hat{G}/U]\hat{\otimes}_{\hat{G}} M = M/cl(\langle\, (1-u).M | u \in U\,\rangle) \tag{2.12}$$

the U-coinvariants of M. Here $\hat{\otimes}$ denotes the pro-p tensor product as defined by A.Brumer (cf. [2, §2]), and cl denotes the closure operation. The assignment $\mathfrak{X}(M)$ which assigns $U \in \mathcal{N}$ the U-coinvariants $\mathfrak{X}(M)_U\colon = M_U$ together with the natural map $N_{V,U}^{\mathfrak{X}(M)}\colon M_V \to M_U$, $V \le U$, and the mapping $i_{U,V}^{\mathfrak{X}(M)}\colon M_U \to M_V$, $V \le U$,

$$i_{U,V}^{\mathfrak{X}(M)}(m + cl(\langle\, (1-u).M | u \in U\,\rangle)\colon = \sum_{x\in V/U} x.m + cl(\langle\, (1-v).M | v \in V\,\rangle), \tag{2.13}$$

defines a cohomological $\mathcal{N}$-Mackey functor $\mathfrak{X}(M) \in ob(\mathfrak{CM}_{\mathcal{N}}(\hat{G}, \mathbf{prf}_p))$. It induces a covariant additive right exact functor

$$\mathfrak{X}(\_)\colon {}_{\hat{G}}\mathbf{prf}_p \longrightarrow \mathfrak{CM}_{\mathcal{N}}(\hat{G}, \mathbf{prf}_p), \tag{2.14}$$

which will be in general not exact. As we will see in the next subsection, the cohomological $\mathcal{N}$-Mackey functors obtained this way have a particular property which characterizes them.

2.4. **Cohomology and homology of cohomological $\mathcal{N}$-Mackey functors.** Let $\mathbf{X}$ be a cohomological $\mathcal{N}$-Mackey functor for $\hat{G}$ with coefficients in $\mathbf{prf}_p$. For short we call $\mathbf{X}$ *i-injective*, if all maps $i^{\mathbf{X}}_{U,V}$, $U,V \in \mathcal{N}$, $V \leq U$, are injective. Similarly, $\mathbf{X}$ is called *N-surjective*, if $N^{\mathbf{X}}_{V,U}$ is surjective for all $U,V \in \mathcal{N}$, $V \leq U$.

Assume that $\mathbf{X} \in ob(\mathfrak{CM}_{\mathcal{N}}(\hat{G}, \mathbf{prf}_p))$ is i-injective. Then we call $\mathbf{X}$ of *type* H^0, if

$$im(i^{\mathbf{X}}_{U,V}) = \mathbf{X}_V^{U/V} \tag{2.15}$$

for all $U,V \in \mathcal{N}$, $V \leq U$. Here $\mathbf{X}_V^{U/V}$ denotes the abelian group of U/V-fixed points on $\mathbf{X}_V$. Cohomological $\mathcal{N}$-Mackey functors of type H^0 are sometimes also called to have *Galois descent*. The N-surjective cohomological $\mathcal{N}$-Mackey functor is called of *type* H_0, if

$$ker(N^{\mathbf{X}}_{V,U}) = \sum_{x \in U/V} (x-1).\mathbf{X}_V \tag{2.16}$$

for all $U,V \in \mathcal{N}$, $V \leq U$. From this definition it is straight forward, that a cohomological $\mathcal{N}$-Mackey functor is of type H_0, if and only if it is isomorphic to a functor $\mathfrak{X}(M)$ for some $M \in ob({}_{\hat{G}}\mathbf{prf}_p)$. The cohomological $\mathcal{N}$-Mackey functors being oy type H_0 are sometimes also called to have *Galois codescent*.

It is possible to interprete the definitions of being of type H_0 or of type H^0 in a more general homological context. For a cohomological $\mathcal{N}$-Mackey functor $\mathbf{X}$ we define for $U,V \in \mathcal{N}$, $V \leq U$,

(2.17)

$$\mathbf{k}^0(U/V,\mathbf{X})\colon = ker(i^{\mathbf{X}}_{U,V}), \qquad \mathbf{k}^1(U/V,\mathbf{X})\colon = \mathbf{X}_V^{U/V}/im(i^{\mathbf{X}}_{U,V}),$$

(2.18)

$$\mathbf{c}_0(U/V,\mathbf{X})\colon = coker(N^{\mathbf{X}}_{V,U}), \qquad \mathbf{c}_1(U/V,\mathbf{X})\colon = ker(N^{\mathbf{X}}_{U,V})/\sum_{x \in U/V}(x-1)\mathbf{X}_V.$$

Let $0 \to \mathbf{X} \to \mathbf{Y} \to \mathbf{Z} \to 0$ be a short exact sequence of cohomological $\mathcal{N}$-Mackey functors. Then the snake lemma implies that one has exact sequences

$$\begin{aligned} 0 \to \mathbf{k}^0(U/V,\mathbf{X}) \to \mathbf{k}^0(U/V,\mathbf{Y}) \to \mathbf{k}^0(U/V,\mathbf{Z}) \dots \\ \to \mathbf{k}^1(U/V,\mathbf{X}) \to \mathbf{k}^1(U/V,\mathbf{Y}) \to \mathbf{k}^1(U/V,\mathbf{Z}), \end{aligned} \tag{2.19}$$

$$\begin{aligned} \mathbf{c}_1(U/V,\mathbf{X}) \to \mathbf{c}_1(U/V,\mathbf{Y}) \to \mathbf{c}_1(U/V,\mathbf{Z}) \to \dots \\ \mathbf{c}_0(U/V,\mathbf{X}) \to \mathbf{c}_0(U/V,\mathbf{Y}) \to \mathbf{c}_0(U/V,\mathbf{Z}) \to 0. \end{aligned} \tag{2.20}$$

One can therefore think of $\mathbf{k}^{0/1}(U/V,\_)$ as the 0- and 1-dimensional *section cohomology* of cohomological $\mathcal{N}$-Mackey functors, and of $\mathbf{c}_{0/1}(U/V,\_)$ as the 0- and 1-dimensional *section homology* of cohomological $\mathcal{N}$-Mackey functors. It is possible to extend these functors to cohomological and homological functors, respectively. Since we will not make use of the higher derived functors we omit a detailed discussion here. However, we would like to remark, that these functors are not unrelated.

Proposition 2.1. *Let* $\mathbf{X} \in \mathfrak{CM}_{\mathcal{N}}(\hat{G}, \mathbf{prf}_p)$ *be a cohomological* $\mathcal{N}$*-Mackey functor and let* $U, V \in \mathcal{N}$*,* $V \leq U$*. Then one has an exact sequence of* $\hat{G}/U$*-modules*

$$\begin{aligned} 0 \longrightarrow \mathbf{c}_1(U/V, \mathbf{X}) \xrightarrow{\alpha_1} \hat{H}^{-1}(U/V, \mathbf{X}_V) \xrightarrow{\alpha_2} \mathbf{k}^0(U/V, \mathbf{X}) \xrightarrow{\alpha_3} \dots \\ \mathbf{c}_0(U/V, \mathbf{X}) \xrightarrow{\alpha_4} \hat{H}^0(U/V, \mathbf{X}_V) \xrightarrow{\alpha_5} \mathbf{k}^1(U/V, \mathbf{X}) \longrightarrow 0, \end{aligned} \tag{2.21}$$

where $\hat{H}^\bullet(U/V, \_)$ *denotes Tate cohomology.*

Proof. The mapping $\alpha_1 \colon \mathbf{c}_1(U/V, \mathbf{X}) \to \hat{H}^{-1}(U/V, \mathbf{X}_V)$ is clearly injective. Since α_2 is induced by the norm map $N^{\mathbf{X}}_{V,U}$, one has

$$ker(\alpha_2) = ker(N^{\mathbf{X}}_{V,U}) / \sum_{x \in U/V} (x-1)\mathbf{X}_V = im(\alpha_1). \tag{2.22}$$

Furthermore, by axiom (2.9)

$$ker(\alpha_3) = ker(i^{\mathbf{X}}_{U,V}) \cap im(N_{V,U}) = N_{V,U}(ker(\sum_{x \in U/V} x)) = im(\alpha_2). \tag{2.23}$$

The mapping α_4 is induced by $i^{\mathbf{X}}_{U,V}$. Hence

$$ker(\alpha_4) = \big(ker(i^{\mathbf{X}}_{U,V}) + im(N^{\mathbf{X}}_{V,U})\big) / im(N^{\mathbf{X}}_{V,U}) = im(\alpha_3). \tag{2.24}$$

The mapping α_5 is the canonical map and thus surjective. Furthermore,

$$ker(\alpha_5) = im(i^{\mathbf{X}}_{U,V}) / (\textstyle\sum_{x \in U/V} x).\mathbf{X}_V = im(\alpha_4). \tag{2.25}$$

This yields the claim. □

Remark 2.2. Let $\hat{G}$ be a finite cyclic group and let $\mathcal{N} \colon = \{1, \hat{G}\}$. Using an alternative approach for the definition of $\mathbf{c}_\bullet(\hat{G}, \_)$ and $\mathbf{k}^\bullet(\hat{G}, \_)$ one sees that there exist connecting homomorphisms making the sequence

$$(\mathbf{k}^0(\hat{G}, \_), \mathbf{k}^1(\hat{G}, \_), \mathbf{c}_1(\hat{G}, \_), \mathbf{c}_0(\hat{G}, \_)) \tag{2.26}$$

a (co)homological functor. Let $M \in ob({}_{\hat{G}}\mathbf{prf}_p)$ be a finitely generated $\mathbb{Z}_p[\hat{G}]$-module. Then (2.21) says that the Herbrand quotient (cf. [10, Kap.IV, §7])

$$h(\hat{G}, M) \colon = \frac{|\hat{H}^0(\hat{G}, M)|}{|\hat{H}^{-1}(\hat{G}, M)|} \tag{2.27}$$

can be interpreted as a kind of multiplicative Euler characteristic, i.e., one has

$$h(\hat{G}, M) = \frac{|\mathbf{c}_0(\hat{G}, \mathfrak{X}(M))| \cdot |\mathbf{k}^1(\hat{G}, \mathfrak{X}(M))|}{|\mathbf{c}_1(\hat{G}, \mathfrak{X}(M))| \cdot |\mathbf{k}^0(\hat{G}, \mathfrak{X}(M))|} =: \chi(\mathfrak{X}(M)). \tag{2.28}$$

For short we say that a cohomological $\mathcal{N}$-Mackey functor $\mathbf{X}$ is *cohomologically trivial*, if $\mathbf{X}$ is of type H^0 and H_0. From Proposition 2.1 follows that such a functor satisfies

$$\hat{H}^{-1}(U/V, \mathbf{X}_V) = \hat{H}^0(U/V, \mathbf{X}_V) = 0 \tag{2.29}$$

for all $U, V \in \mathcal{N}$, $V \leq U$.

Proposition 2.3. *Let* $P \in ob({}_{\hat{G}}\mathbf{prf}_p)$ *be projective. Then for* $V \in \mathcal{N}$*,* $\mathfrak{X}(P)_V$ *(cf. 2.3) is a projective* $\mathbb{Z}_p[\hat{G}/V]$*-module. In particular,* $\mathfrak{X}(P)$ *is a cohomologically trivial cohomological* $\mathcal{N}$*-Mackey functor.*

Proof. The first statement follows from the fact that deflation from ${}_{\hat{G}}\mathbf{prf}_p$ to ${}_{\hat{G}/V}\mathbf{prf}_p$ is mapping projectives to projectives. Since restriction to closed subgroups is mapping projectives to projectives, it suffices to prove the second claim for $U = \hat{G}$. Since $\mathfrak{X}(P)$ is of type H_0, $\mathbf{c}_{0/1}(\hat{G}/V, \mathfrak{X}(P)) = 0$. As $P_V \in ob({}_{\hat{G}/V}\mathbf{prf}_p)$ is projective, $\hat{H}^{-1}(\hat{G}/V, P_V) = \hat{H}^0(\hat{G}/V, P_V) = 0$. Hence Proposition 2.1 yields the claim. □

3. Class field theories

Throughout this section let $\hat{G}$ be a profinite group, and let p be a prime number. We also assume that $\mathcal{N}$ is a normal Mackey system for $\hat{G}$.

For a finite group G we denote by $\mathfrak{S}_p(G)$ the set of isomorphism types of irreducible (left) $\mathbb{F}_p[G]$-modules. For an irreducible $\mathbb{F}_p[G]$-module S we use the symbol $[S] \in \mathfrak{S}_p(G)$ to denote its isomorphism type.

3.1. The cohomological Mackey functors $\mathbf{Ab}^p$ and $\mathbf{Ab}^{/p}$. For $U \in \mathcal{N}$, let

$$\mathbf{Ab}^p_U \colon = U^{ab}_p = U/cl([U,U])/O_{p'}(U/cl([U,U]) \tag{3.1}$$

denote the largest continuous homomorphic image of U which is an abelian pro-p group. Here $[\_,\_]$ stands for the commutator subgroup, and cl denotes the closure operation. Then for $U, V \in \mathcal{N}$, $V \leq U$, one has a canonical map $N^{\mathbf{Ab}^p}_{V,U} : V^{ab}_p \to U^{ab}_p$. This map together with the *transfer map* (cf. [10, p.312])

$$i^{\mathbf{Ab}^p}_{U,V} \colon = tr^U_V \colon U^{ab}_p \to V^{ab}_p \tag{3.2}$$

makes $\mathbf{Ab}^p \in ob(\mathfrak{CM}_{\mathcal{N}}(\hat{G}, \mathbf{prf}_p))$ a cohomological $\mathcal{N}$-Mackey functor. By $\mathbf{Ab}^{/p}$ we denote its reduction modulo p, i.e., for $U \in \mathcal{N}$ one has

$$\mathbf{Ab}^{/p}_U \colon = U^{ab}_{/p} = \mathbf{Ab}^p_U/p.\mathbf{Ab}^p_U, \tag{3.3}$$

and the maps $i^{\mathbf{Ab}^{/p}}_{U,V}$ and $N^{\mathbf{Ab}^{/p}}_{V,U}$, $U, V \in \mathcal{N}$, $V \leq U$, are the maps induced from $i^{\mathbf{Ab}^p}_{U,V}$ and $N^{\mathbf{Ab}^p}_{V,U}$, respectively. It is obviously a cohomological $\mathcal{N}$-Mackey functor.

3.2. Weak p-class field theories. We define a *weak p-class field theory* $(\mathbf{X}, \eta)$ (for $(\hat{G}, \mathcal{N})$) to be a cohomological $\mathcal{N}$-Mackey functor $\mathbf{X} \in ob(\mathfrak{CM}_{\mathcal{N}}(\hat{G}, \mathbf{prf}_p))$, together with a surjective morphism $\eta \colon \mathbf{X} \to \mathbf{Ab}^p$ of cohomological $\mathcal{N}$-Mackey functors with the following properties:

(i) $\mathbf{X}$ is of type H^0,
(ii) $\mathbf{c}_0(U/V, \eta) \colon \mathbf{c}_0(U/V, \mathbf{X}) \to (U/V)^{ab}_p$ is an isomorphism for all $U, V \in \mathcal{N}$, $V \leq U$.

The property (i) implies that $\mathbf{k}^{0/1}(U/V, \mathbf{X}) = 0$ for all $U, V \in \mathcal{N}$, $V \leq U$. In particular, one has an isomorphism $\mathbf{c}_0(U/V, \mathbf{X}) = \hat{H}^0(U/V, \mathbf{X}_V)$. The property (ii) is one of the properties one would expect from a p-class field theory. However, in order to state the other property, one has also to require some structure on the normal Mackey system $\mathcal{N}$.

3.3. **p-Class field theories.** For short we call a normal Mackey system *p-closed*, if it satisfies the following property: Assume that W is an open normal subgroup of $\hat{G}$ which is contained in an open normal subgroup in $U \in \mathcal{N}$, such that U/W is a finite abelian p-group. Then W is also contained in $\mathcal{N}$.

Let $\mathcal{N}$ be a p-closed normal Mackey system of $\hat{G}$. Then we call the weak p-class field theory $(\mathbf{X}, \eta)$ a *p-class field theory*, if it satisfies additionally the following property:

(iii) Let $U \in \mathcal{N}$ and let $V, W \leq U$ be open and normal in $\hat{G}$, such that U/V and U/W are finite abelian p-groups. Assume that $im(N^{\mathbf{X}}_{V,U}) = im(N^{\mathbf{X}}_{W,U})$. Then $V = W$.

In a similar fashion one defines a *$/p$-class field theory*: Let $\mathcal{N}$ be a p-closed normal Mackey system of $\hat{G}$. A cohomological $\mathcal{N}$-Mackey functor $\mathbf{X}$ together with a surjective morphism of $\mathcal{N}$-Mackey functors $\eta\colon \mathbf{X} \to \mathbf{Ab}^{/p}$ is called a $/p$-class field theory, if the following properties hold:

(i) $\mathbf{X}$ is of type H^0,
(ii) $\mathbf{c}_0(U/V, \eta)\colon \mathbf{c}_0(U/V, \mathbf{X}) \to (U/V)^{ab}_{/p}$ is an isomorphism for all $U, V \in \mathcal{N}$, $V \leq U$,
(iii) Let $U \in \mathcal{N}$ and let $V, W \leq U$ be open and normal in $\hat{G}$, such that U/V and U/W are finite elementary abelian p-groups. Assume that $im(N^{\mathbf{X}}_{V,U}) = im(N^{\mathbf{X}}_{W,U})$. Then $V = W$.

3.4. **The p-Frattini class field theory and the $/p$-Frattini class field theory.** Let G be a finite group, and let $\pi_p\colon \tilde{G}_p \to G$ denote its universal p-Frattini cover. We are considering the normal Mackey system

$$\mathcal{F}\colon = \{\, U \leq ker(\pi_p) \mid U \text{ open and normal in } \tilde{G}_p \,\}. \tag{3.4}$$

As $ker(\pi_p)$ is a pro-p group, it is obviously p-closed.

Let

$$0 \longrightarrow P_1 \xrightarrow{\delta} P_0 \xrightarrow{\varepsilon} \mathbb{Z}_p \longrightarrow 0 \tag{3.5}$$

be a minimal projective resolution of the trivial $\mathbb{Z}_p[\![\tilde{G}_p]\!]$-module $\mathbb{Z}_p$ in ${}_{\tilde{G}_p}\mathbf{prf}_p$. In particular, $\varepsilon\colon P_0 \to \mathbb{Z}_p$ and $\delta'\colon P_1 \to ker(\varepsilon)$ are minimal projective covers in ${}_{\tilde{G}_p}\mathbf{prf}_p$.

Let $\mathfrak{S}_p(G)$ denote the set of isomorphism types of irreducible $\mathbb{F}_p[G]$-modules, and let $\tau_S\colon P_S \to S$ denote a minimal projective cover in ${}_{\tilde{G}_p}\mathbf{prf}_p$, $[S] \in \mathfrak{S}_p(G)$. As (3.5) is minimal, one has isomorphisms

$$\mathbf{Hom}_{\tilde{G}_p}(P_1, S) \simeq H^1(\tilde{G}_p, S) \tag{3.6}$$

for all $[S] \in \mathfrak{S}_p(G)$. In particular, $P_1 \simeq \coprod_{[S]\in\mathfrak{S}_p(G)} P_S^{\mu_S}$, where

$$\mu_S\colon = \frac{\dim_{\mathbb{F}_p}(H^1(\tilde{G}_p, S))}{\dim_{\mathbb{F}_p}(\mathrm{End}_G(S))}. \tag{3.7}$$

Let $U \in \mathcal{F}$. As $\_{U}$ is right exact, one has an exact sequence

$$(P_1)_U \xrightarrow{\delta_U} (P_0)_U \xrightarrow{\varepsilon_U} \mathbb{Z}_p \longrightarrow 0. \tag{3.8}$$

As $\tilde{G}_p \to \tilde{G}_p/U$ is a p-Frattini extension, inflation induces isomorphisms

$$H^1(\tilde{G}_p, S) \simeq H^1(\tilde{G}_p/U, S) \tag{3.9}$$

for all $[S] \in \mathfrak{S}_p(G)$ (cf. [16, Prop.3.1]). This yields that

$$H^1(\tilde{G}_p/U, S) \simeq \mathbf{Hom}_{\tilde{G}_p/U}((P_1)_U, S) \tag{3.10}$$

for all $[S] \in \mathfrak{S}_p(G)$, and from this one concludes easily that (3.8) is a partial minimal projective resolution. In particular, $ker(\delta_U) = \Omega_2(\tilde{G}_p/U, \mathbb{Z}_p)$.

Let $\mathbf{\Omega}_2 \colon = ker(\mathfrak{X}(\delta))$. Then one has an exact sequence of cohomological $\mathcal{F}$-Mackey functors

$$0 \longrightarrow \mathbf{\Omega}_2 \longrightarrow \mathfrak{X}(P_1) \xrightarrow{\mathfrak{X}(\delta)} \mathfrak{X}(P_0) \xrightarrow{\mathfrak{X}(\varepsilon)} \mathfrak{X}(\mathbb{Z}_p) \longrightarrow 0, \tag{3.11}$$

and $\mathbf{\Omega}_{2,U} = \Omega_2(\tilde{G}_p/U, \mathbb{Z}_p)$.

From the Eckmann-Shapiro lemma for $\mathbf{Tor}_\bullet$ (cf. [13, Lemma 3.3.4]), and the canonical isomorphism $\mathbf{H}_1(U, \mathbb{Z}_p) \simeq U_p^{ab} = \mathbf{Ab}_U^p$, where $\mathbf{H}_\bullet$ denotes homology as defined by A.Brumer (cf. [2, §2]), one obtains an isomorphism

$$\eta \colon \mathbf{\Omega}_2 \longrightarrow \mathbf{Ab}^p \tag{3.12}$$

of cohomological $\mathcal{F}$-Mackey functors.

By $\mathbf{\Omega}_2^{/p}$ we denote the reduction mod p of $\mathbf{\Omega}_2$, i.e., one has a short exact sequence in $\mathfrak{CM}_{\mathcal{F}}(\tilde{G}_p, \mathbf{prf}_p)$

$$0 \longrightarrow \mathbf{\Omega}_2 \xrightarrow{p.id} \mathbf{\Omega}_2 \longrightarrow \mathbf{\Omega}_2^{/p} \longrightarrow 0. \tag{3.13}$$

By $\eta^{/p} \colon \mathbf{\Omega}_2^{/p} \to \mathbf{Ab}^{/p}$ we denote the induced isomorphism.

Theorem 3.1. *Let G be a finite group, $\pi_p \colon \tilde{G}_p \to G$ its universal p-Frattini cover, and let $\mathcal{F}$ be given as in* (3.4).

(a) *The tuple $(\mathbf{\Omega}_2, \eta)$ is a p-class field theory for $(\tilde{G}_p, \mathcal{F})$.*
(b) *The tuple $(\mathbf{\Omega}_2^{/p}, \eta^{/p})$ is a /p-class field theory for $(\tilde{G}_p, \mathcal{F})$.*

We call $(\mathbf{\Omega}_2, \eta)$ the *p-Frattini class field theory* for $(\tilde{G}_p, \mathcal{F})$, and $(\mathbf{\Omega}_2^{/p}, \eta^{/p})$ the */p-Frattini class field theory* for $(\tilde{G}_p, \mathcal{F})$.

Proof. (a) One has to verify the axioms (i)-(iii). Axiom (ii) is obviously satisfied. Consider the short exact sequence

$$0 \longrightarrow \mathbf{\Omega}_2 \xrightarrow{\iota} \mathfrak{X}(P_1) \longrightarrow coker(\iota) \longrightarrow 0. \tag{3.14}$$

Since $coker(\iota)$ is a cohomological $\mathcal{F}$-subMackey functor of $\mathfrak{X}(P_0)$, $\mathbf{k}^0(coker(\iota)) = 0$ (cf. (2.19), Prop.2.3). The long exact sequence (2.19) applied to (3.14) and the cohomological triviality of $\mathfrak{X}(P_0)$ and $\mathfrak{X}(P_1)$ yields that $\mathbf{\Omega}_2$ is of type H^0. Hence axiom (i) is satisfied. It remains to verify (iii). We may assume that p divides the order of the finite group G, since otherwise $\mathbf{\Omega}_2 = 0$, and there is nothing to prove. In this case $\tilde{G}_p$ is of cohomological p-dimension 1, and thus of strict cohomological p-dimension 2 (cf. [12, §I.3.2]). In particular, by Brumer's theorem (cf. [2], [10, Kap.IV, §6, Aufg.6]) $\hat{G}$ possesses a *tautological class field theory.* Let $(\mathfrak{H}, \rho)$ denote its restriction to the Mackey system $\mathcal{F}$, i.e., $\mathfrak{H}_U = \mathbf{Ab}_U^p$ and ρ_U is the identity on $\mathbf{Ab}_U^p$. In particular $(\mathfrak{H}, \rho)$ and $(\mathbf{\Omega}_2, \eta)$ essentially coincide, i.e., one has a commutative diagram in $\mathfrak{CM}_{\mathcal{F}}(\tilde{G}_p, \mathbf{prf}_p)$

$$\begin{array}{ccc} \mathbf{\Omega}_2 & \xrightarrow{\eta} & \mathfrak{H} \\ {\scriptstyle\eta}\downarrow & & \| \\ \mathbf{Ab}^p & = & \mathbf{Ab}^p \end{array} \tag{3.15}$$

The property (iii) is well-known for $(\mathfrak{H},\rho)$ (cf. [10, Kap.IV, Thm.6.7]). Thus it also holds for $(\mathbf{\Omega}_2,\eta)$.
(b) It suffices to prove that $\mathbf{\Omega}_2^{/p}$ is of type H^0. The axiom (ii) is obvious, and axiom (iii) follows from axiom (iii) for $(\mathbf{\Omega}_2,\eta)$.

Let $\mathfrak{X}(P_{0/1})^{/p}$ denote the reduction mod p of $\mathfrak{X}(P_0)$ and $\mathfrak{X}(P_1)$, respectively. Then one has a short exact sequence

$$0 \longrightarrow \mathbf{\Omega}_2^{/p} \xrightarrow{\iota^{/p}} \mathfrak{X}(P_1)^{/p} \longrightarrow coker(\iota^{/p}) \longrightarrow 0, \tag{3.16}$$

and $coker(\iota^{/p})$ is a cohomological $\mathcal{F}$-sub Mackey functor of $\mathfrak{X}(P_0)^{/p}$. From Proposition 2.1 one concludes that $\mathfrak{X}(P_0)^{/p}$ and $\mathfrak{X}(P_1)^{/p}$ are cohomologically trivial. Hence the long exact sequence (2.19) yields the claim. □

Let $\pi\colon \tilde{G} \to G$ be any p-Frattini extension, finite or infinite. By universality, there exists a mapping $\tau\colon \tilde{G}_p \to \tilde{G}$, such that $\pi_p = \pi \circ \tau$. Since π is a p-Frattini extension, τ is surjective. For short we put $N\colon = ker(\tau)$.

The morphism τ induces a canonical bijection of sets $\tau_*\colon \mathcal{F}_N \to \mathcal{F}(\tilde{G})$, where $\mathcal{F}$ is given as in (3.4) and

$$\begin{aligned} \mathcal{F}_N\colon &= \{\, U \in \mathcal{F} \mid N \le U \,\}, \\ \mathcal{F}(\tilde{G})\colon &= \{\, U' \le ker(\pi) \mid U' \text{ open and normal in } \tilde{G} \,\}. \end{aligned} \tag{3.17}$$

Let $\mathbf{C} \in ob(\mathfrak{CM}_{\mathcal{F}(\tilde{G})}(\tilde{G},\mathbf{prf}_p))$ denote the cohomological $\mathcal{F}(\tilde{G})$-Mackey functor given by

$$\mathbf{C}_U\colon = \mathbf{\Omega}_{2,\tau_*^{-1}(U)}, \quad U \in \mathcal{F}(\tilde{G}) \tag{3.18}$$

equipped with the obvious maps $i^{\mathbf{C}}_{U,V}$, $N^{\mathbf{C}}_{V,U}$, $U,V \in \mathcal{F}(\tilde{G})$, $V \le U$. Let $\gamma\colon \mathbf{C} \to \mathbf{Ab}^p$ denote the morphism of $\mathcal{F}(\tilde{G})$-Mackey functors induced by η. In particular, γ is surjective, but if $\tilde{G}$ does not coincide with the universal p-Frattini cover, γ will not be an isomorphism.

Similarly, we define the reduction mod p $\mathbf{C}^{/p}$ of $\mathbf{C}$, i.e., one has

$$\mathbf{C}_U^{/p}\colon = \mathbf{\Omega}^{/p}_{2,\tau_*^{-1}(U)}, \quad U \in \mathcal{F}(\tilde{G}), \tag{3.19}$$

and by $\gamma^{/p}\colon \mathbf{C}^{/p} \to \mathbf{Ab}^{/p}$ we denote the surjective morphism induced by $\eta^{/p}$. Again, apart from the case $\tilde{G} \simeq \tilde{G}_p$, $\gamma^{/p}$ will not be surjective. From Theorem 3.1 one concludes:

Corollary 3.2. *Let G be a finite group, and let $\pi\colon \tilde{G} \to G$ be any p-Frattini extension. Then*

(a) *The tuple $(\mathbf{C},\gamma)$ is a p-class field theory for $(\tilde{G},\mathcal{F}(\tilde{G}))$.*
(b) *The tuple $(\mathbf{C}^{/p},\gamma^{/p})$ is a $/p$-class field theory for $(\tilde{G},\mathcal{F}(\tilde{G}))$.*

Remark 3.3. The definition of a p or a $/p$-class field theory we have given here is very much adapted to our main purpose, which is to prove Theorem B. Nevertheless, $(\mathbf{\Omega}_2,\eta)$ satisfies all class field theory axioms, which are usually required in number theory, i.e., using Tate cohomology one sees easily that for all $U,V \in \mathcal{F}$, $V \le U$,

$$H^1(U/V,\mathbf{\Omega}_{2,V}) = H^1(\tilde{G}_p/V,\mathbf{\Omega}_{2,V}) = 0. \tag{3.20}$$

Moreover, (3.11) defines a *canonical class* $c \in \mathbf{nat}^2(\mathfrak{X}(\mathbb{Z}_p),\mathbf{\Omega}_2)$, where $\mathbf{nat}^\bullet(\_,\_)$ denote the derived functors of $\mathbf{nat}(\_,\_)$ (cf. [9, Chap.XII]). This also applies to the p-class field theory $(\mathbf{C},\gamma)$ defined for any p-Frattini cover $\pi\colon \tilde{G} \to G$. However, as the reader might verify by himself, (3.20) does not hold for the $/p$-class field theories $(\mathbf{\Omega}_2^{/p}, \eta^{/p})$ or $(\mathbf{C}^{/p}, \gamma^{/p})$. Nevertheless, as we will see in the next section, these are the class field theories which are easiest to deal with.

4. p-Poincaré duality groups of dimension 2 as p-Frattini extensions

Throughout this section we assume that G is a finite group, and that $\pi\colon \tilde{G} \to G$ is a p-Frattini extension. By

$$
\begin{aligned}
&P_1 \xrightarrow{\delta} P_0 \xrightarrow{\varepsilon} \mathbb{Z}_p,\\
&Q_1 \xrightarrow{\delta^{/p}} P_0 \xrightarrow{\varepsilon^{/p}} \mathbb{F}_p
\end{aligned}
\tag{4.1}
$$

we denote partial minimal projective resolutions in ${}_{\tilde{G}}\mathbf{prf}_p$ and ${}_{\tilde{G}}\mathbf{prf}_{/p}$, respectively.

4.1. **Universal norms.** Let $\pi\colon \tilde{G} \to G$ be a p-Frattini extension, and let $(\mathbf{C}, \gamma)$ denote its p-Frattini class field theory. We call the cohomological $\mathcal{F}(\tilde{G})$-Mackey functor $\mathfrak{N}\colon = ker(\gamma)$ the *universal norms* of $(\mathbf{C}, \gamma)$. Similarly, $\mathfrak{N}^{/p}\colon = ker(\gamma^{/p})$ will be called the *universal norms* of $(\mathbf{C}^{/p}, \gamma^{/p})$. One has:

Proposition 4.1. *Let $\pi\colon \tilde{G} \to G$ be a p-Frattini extension. Then:*

(a) *$\mathfrak{N}$ is N-surjective. Let $P_1 \xrightarrow{\delta} P_0 \longrightarrow \mathbb{Z}_p$ be a partial minimal projective resolution of $\mathbb{Z}_p$ in ${}_{\tilde{G}}\mathbf{prf}_p$. Then $ker(\delta) \simeq m(\mathfrak{N})$.*

(b) *$\mathfrak{N}^{/p}$ is N-surjective. Let $Q_1 \xrightarrow{\delta} Q_0 \longrightarrow \mathbb{F}_p$ be a partial minimal projective resolution of $\mathbb{F}_p$ in ${}_{\tilde{G}}\mathbf{prf}_{/p}$. Then $ker(\delta) \simeq m(\mathfrak{N}^{/p})$.*

Proof. (a) For simplicity let us assume that $\iota\colon \mathfrak{N} \to \mathbf{C}$ is given by inclusion. Let $\{U_k\}_{k\in\mathbb{N}} \subseteq \mathcal{F}(\tilde{G})$ be a linearly ordered basis of neighbourhoods of $1 \in \tilde{G}$. We have to show that for $x \in \bigcap_{m\geq n} im(N^{\mathbf{C}}_{U_m,U_n})$, there exists a sequence $(y_k)_{k\in\mathbb{N}_0}$, $y_k \in \mathbf{C}_{U_{n+k}}$, such that $y_0 = x$ and $y_k = N_{U_{n+k+1},U_{n+k}}(y_{k+1})$.

Let $Z\colon = \prod_{k\in\mathbb{N}_0} \mathbf{C}_{U_{n+k}}$. Then Z is compact by Tychonoff's theorem. Let

$$
Z_{x,r}\colon = \{\, (z_k)_{k\in\mathbb{N}_0} \in Z \mid z_0 = x,\ N_{U_{k+1},U_k}(z_{k+1}) = z_k \text{ for all } k \leq r.\,\}. \tag{4.2}
$$

Then $Z_{x,r+1} \subseteq Z_{x,r}$ and all sets $Z_{x,r}$ are closed. By definition, any finite intersection of sets $Z_{x,r}$ is non-empty. Hence $Z_{x,\infty}\colon = \bigcap_{r\in\mathbb{N}} Z_{x,r}$ is non-empty. Any element $(y_k)_{k\in\mathbb{N}_0} \in Z_{x,\infty}$ will have the desired property.

By construction, $ker(\mathfrak{X}(\delta)) = \mathbf{C}$. Moreover, one has a short exact sequence of $\mathcal{F}(\tilde{G})$-Mackey functors $0 \to \mathfrak{N} \to \mathbf{C} \to \mathbf{Ab}^p \to 0$. Obviously, $m(\mathbf{Ab}^p) = 0$. Thus the claim follows from the exactness of m. The assertion (b) follows by a similar argument. □

4.2. **Weakly oriented p-Poincaré duality groups.** Let $\hat{G}$ be a profinite group of cohomological p-dimension d, $d \in \mathbb{N}$. Then $\hat{G}$ is called a *p-Poincaré duality group of dimension d*, if

(i) for every finite discrete left $\hat{G}$-module of p-power order X and for all $k \in \mathbb{N}_0$ one has

$$
|H^k(\hat{G}, X)| < \infty, \tag{4.3}
$$

(ii) the p-dualizing module $\mathbb{I}_{\hat{G},p}$ of $\hat{G}$ is isomorphic to $\mathbb{Q}_p/\mathbb{Z}_p$ as abelian group,

(iii) for every finite discrete left $\hat{G}$-module of p-power order X, cup-product induces a non-degenerate pairing

$$H^k(\hat{G}, X') \times H^{d-k}(\hat{G}, X) \overset{H^d(ev_X)\circ(.\cup.)}{\longrightarrow} H^d(\hat{G}, \mathbb{I}_{\hat{G},p}) \overset{i}{\longrightarrow} \mathbb{Q}_p/\mathbb{Z}_p, \tag{4.4}$$

where $X' := \mathrm{Hom}(X, \mathbb{I}_{\hat{G},p})$, $ev_X : X' \times X \to \mathbb{I}_{\hat{G},p}$ is the evaluation map and i is given as in [12, §I.3.5].

The p-Poincaré duality group $\hat{G}$ of dimension d is called *orientable*, if $\mathbb{I}_{\hat{G},p}$ is a trivial $\hat{G}$-module, and *weakly-orientable*, if the socle of $\mathbb{I}_{\hat{G},p}$ is a trivial $\hat{G}$-module, i.e., $\mathrm{soc}(\mathbb{I}_{\hat{G},p}) \simeq \mathbb{F}_p$.

One can charcterize these groups by continuous cochain cohomology as introduced by J.Tate (cf. [14]) with coefficients in $\mathbb{F}_p[\![\hat{G}]\!]$ as follows:

Proposition 4.2. *Let $\hat{G}$ be a profinite group of cohomological p-dimension d, $d \in \mathbb{N}$, and assume* (4.3) *holds for every finite discrete left $\hat{G}$-module of p-power order X. Then the following are equivalent:*

(i) *$\hat{G}$ is a weakly-orientable p-Poincaré duality group of dimension d,*

(ii)

$$\mathbf{H}^k(\hat{G}, \mathbb{F}_p[\![\hat{G}]\!]) = \begin{cases} \mathbb{F}_p & \text{for } k = d, \\ 0 & \text{for } k \neq d, \end{cases} \tag{4.5}$$

where $\mathbb{F}_p$ denotes the trivial $\hat{G}$-module and $\mathbf{H}^\bullet$ denotes continuous cochain cohomology.

Proof. The implication $(i) \Rightarrow (ii)$ is implicitly already contained in a letter from J.Tate to J-P.Serre (cf. [12, App.1]) Here one should only note that the second property of a Poincaré duality group ensures that $\mathbf{H}^k(\tilde{G}, \mathbb{F}_p[\![\tilde{G}]\!])^* = E_k(\mathbb{F}_p)$.

Note that property (4.5) already implies that (4.4) holds for all finite $\mathbb{F}_p$-vector spaces which are discrete $\hat{G}$-modules. Then the same argument used in the proof of [12, Prop.I.32]) shows that (4.4) holds for all finite discrete $\hat{G}$-modules of p power order. □

4.3. **Cohomological Mackey functors for p-Frattini extensions.** Let $\mathbf{X}$ be a cohomological $\mathcal{F}(\tilde{G})$-Mackey functor, such that $\mathbf{X}_U$ are finitely generated $\mathbb{F}_p[\tilde{G}/U]$-modules for all $U \in \mathcal{F}(\tilde{G})$. Then applying $\mathrm{Hom}_{\tilde{G}}(\_, \mathbb{F}_p)$ and changing the role of i and N defines a new cohomological $\mathcal{F}(\tilde{G})$-Mackey functor which we denote by $\mathbf{X}^*$. The functor * is obviously contravariant and exact.

For short put $\mathbf{S}(\mathbb{F}_p) := \mathfrak{X}(\mathbb{F}_p)$, $\mathbf{T}(\mathbb{F}_p) := \mathbf{S}(\mathbb{F}_p)^*$. Then $\mathbf{S}(\mathbb{F}_p)$ is a cohomological $\mathcal{F}(\tilde{G})$-Mackey functor with all mapping $N^{\mathbf{S}(\mathbb{F}_p)}_{V,U}$ bijective, and $\mathbf{T}(\mathbb{F}_p)$ is a $\mathcal{F}(\tilde{G})$-Mackey functor with all mapping $i^{\mathbf{T}(\mathbb{F}_p)}_{U,V}$ bijective, $U, V \in \mathcal{F}(\tilde{G})$, $V \leq U$.

Thus one has an exact sequence of cohomological $\mathcal{F}(\tilde{G})$-Macke functors

$$0 \longrightarrow \mathbf{T}(\mathbb{F}_p) \overset{\mathfrak{X}(\varepsilon^{/p})^*}{\longrightarrow} \mathfrak{X}(Q_0)^* \overset{\mathfrak{X}(\delta^{/p})^*}{\longrightarrow} \mathfrak{X}(Q_1)^*. \tag{4.6}$$

We put

$$\begin{aligned} \Omega^1(\tilde{G}/\_, \mathbb{F}_p) &:= ker(\mathfrak{X}(\delta^{/p})^*), \\ \Omega^2(\tilde{G}/\_, \mathbb{F}_p) &:= coker(\mathfrak{X}(\delta^{/p})^*). \end{aligned} \tag{4.7}$$

It is an easy exercise to show that $\Omega^1(\tilde{G}/\_, \mathbb{F}_p)$ is i-injective and N-surjective, and that $\Omega^2(\tilde{G}/\_, \mathbb{F}_p)$ is of type H_0.

4.4. **Extending injective maps $\Omega^1(G, \mathbb{F}_p) \to \Omega_2(G, \mathbb{F}_p)$.** The first step in proving Theorem B is establishing the following proposition:

Proposition 4.3. *Let G be a finite group, and let $\alpha\colon \Omega^1(G, \mathbb{F}_p) \to \Omega^2(G, \mathbb{F}_p)$ be a mapping of $\mathbb{F}_p[G]$-modules. Then there exists a closed normal subgroup N, $N \leq ker(\pi_p)$ of the universal p-Frattini extension $\tilde{G}_p$, $\tilde{G}\colon = \tilde{G}_p/N$, and a map of cohomological $\mathcal{F}(\tilde{G})$-Mackey functors*

$$\boldsymbol{\alpha}\colon \boldsymbol{\Omega}^1(\tilde{G}/\_, \mathbb{F}_p) \longrightarrow \mathbf{C}^{/p}, \tag{4.8}$$

satisfying $im(\boldsymbol{\alpha}) = \mathfrak{N}^{/p}$ and $\boldsymbol{\alpha}_{ker(\pi_p)} = \iota_{ker(\pi_p)}\colon \alpha$, where $\iota\colon \mathfrak{N}^{/p} \to \mathbf{C}^{/p}$ denotes the canonical map.

Moreover, if α is injective, $\boldsymbol{\alpha}$ is injective.

Proof. Put $V_0\colon = ker(\pi_p)$ and $\boldsymbol{\alpha}_0\colon = \alpha\colon \Omega^1(G, \mathbb{F}_p) \to \Omega_2(G, \mathbb{F}_p)$. Assume we have constructed open normal subgroups $V_0, .., V_{k-1}$ and injective morphisms

$$\boldsymbol{\alpha}_{V_i}\colon \Omega^1(\tilde{G}_p/V_i) \longrightarrow \Omega_2(\tilde{G}_p/V_i, \mathbb{F}_p), \tag{4.9}$$

$i = 0, ..., k-1$, such that the diagrams

$$\begin{array}{ccc} \Omega^1(\tilde{G}_p/V_{i-1}, \mathbb{F}_p) & \xrightarrow{\boldsymbol{\alpha}_{V_{i-1}}} & \Omega_2(\tilde{G}_p/V_{i-1}, \mathbb{F}_p) \\ {\scriptstyle i^{\Omega^1}_{V_{i-1},V_i}}\downarrow & & \downarrow{\scriptstyle i^{\Omega_2}_{V_{i-1},V_i}} \\ \Omega^1(\tilde{G}_p/V_i, \mathbb{F}_p) & \xrightarrow{\boldsymbol{\alpha}_{V_i}} & \Omega_2(\tilde{G}_p/V_i, \mathbb{F}_p) \end{array} \tag{4.10}$$

$$\begin{array}{ccc} \Omega^1(\tilde{G}_p/V_{i-1}, \mathbb{F}_p) & \xrightarrow{\boldsymbol{\alpha}_{V_{i-1}}} & \Omega_2(\tilde{G}_p/V_{i-1}, \mathbb{F}_p) \\ {\scriptstyle N^{\Omega^1}_{V_i,V_{i-1}}}\uparrow & & \uparrow{\scriptstyle N^{\Omega_2}_{V_i,V_{i-1}}} \\ \Omega^1(\tilde{G}_p/V_i, \mathbb{F}_p) & \xrightarrow{\boldsymbol{\alpha}_{V_i}} & \Omega_2(\tilde{G}_p/V_i, \mathbb{F}_p) \end{array} \tag{4.11}$$

commute, $i = 1, .., k-1$. In the first step we construct V_k and a mapping

$$\boldsymbol{\alpha}_{V_k}\colon \Omega^1(\tilde{G}_p/V_k, \mathbb{F}_p) \to \Omega_2(\tilde{G}_p/V_k, \mathbb{F}_p) \tag{4.12}$$

such the diagrams (4.10) and (4.11) commute for $(k-1, k)$.

Let $V_k \leq ker(\pi_p)$ be the unique open normal subgroup such that V_{k-1}/V_k is elementary p-abelian, and $im(\boldsymbol{\alpha}_{V_{k-1}}) = im(N^{\Omega_2}_{V_k,V_{k-1}})$. The uniqueness is guaranteed by axiom (iii) of a $/p$-class field theory. Since $(Q_0)^*_{V_k}$ is a projective $\mathbb{F}_p[\tilde{G}_p/V_k]$-module, there exists a mapping $\alpha'\colon (Q_0)^*_{V_k} \to \Omega_2(\tilde{G}_p/V_k, \mathbb{F}_p)$ making the diagram

$$\begin{array}{ccc} \Omega^1(\tilde{G}_p/V_{k-1}, \mathbb{F}_p) & \xrightarrow{\boldsymbol{\alpha}_{V_{k-1}}} & \Omega_2(\tilde{G}_p/V_{k-1}, \mathbb{F}_p) \\ {\scriptstyle N}\uparrow & & \uparrow{\scriptstyle N^{\Omega_2}_{V_k,V_{k-1}}} \\ (Q_0)^*_{V_k} & \xrightarrow{\alpha'} & \Omega_2(\tilde{G}_p/V_k, \mathbb{F}_p) \end{array} \tag{4.13}$$

commute, where $N\colon (Q_0)^*_{V_k} \to \Omega^1(\tilde{G}_p/V_{k-1}, \mathbb{F}_p)$ is the canonical map. Since the $\mathbb{F}_p[\tilde{G}_p/V_k]$-module $\Omega_2(\tilde{G}_p/V_k, \mathbb{F}_p)$ is directly indecomposable, and as $(Q_0)^*_{V_k}$ is also injective, α' cannot be injective. Hence α' factors through a mapping

$$\boldsymbol{\alpha}_{V_k}\colon \Omega^1(\tilde{G}_p/V_k, \mathbb{F}_p) \to \Omega_2(\tilde{G}_p/V_k, \mathbb{F}_p). \tag{4.14}$$

for which diagram (4.11) commutes for $(k-1,k)$.

Let $x \in \Omega^1(\tilde{G}_p/V_{k-1},\mathbb{F}_p)$. As $\Omega^1(\tilde{G}_p/\_,\mathbb{F}_p)$ is N-surjective, there exists $y \in \Omega^1(\tilde{G}_p/V_k,\mathbb{F}_p)$ such that $N^{\Omega^1}_{V_k,V_{k-1}}(y) = x$. Thus

$$\begin{aligned} i^{\Omega_2}_{V_{k-1},V_k}(\boldsymbol{\alpha}_{V_{k-1}}(x)) &= i^{\Omega_2}_{V_{k-1},V_k}(\boldsymbol{\alpha}_{V_{k-1}}(N^{\Omega^1}_{V_k,V_{k-1}}(y))), \\ &= i^{\Omega_2}_{V_{k-1},V_k}(N^{\Omega_2}_{V_k,V_{k-1}}(\boldsymbol{\alpha}_{V_k}(y))) = N_{V_{k-1}/V_k}(\boldsymbol{\alpha}_{V_k}(y)), \end{aligned} \tag{4.15}$$

where $N_{V_{k-1}/V_k} \colon = \sum_{g\in V_{k-1}/V_k} g$. On the other hand

$$\begin{aligned} \boldsymbol{\alpha}_{V_k}(i^{\Omega^1}_{V_{k-1},V_k}(x)) &= \boldsymbol{\alpha}_{V_k}(i^{\Omega^1}_{V_{k-1},V_k}(N^{\Omega^1}_{V_k,V_{k-1}}(y))) \\ &= \boldsymbol{\alpha}_{V_k}(N_{V_{k-1}/V_k}(y)) = N_{V_{k-1}/V_k}(\boldsymbol{\alpha}_{V_k}(y)), \end{aligned} \tag{4.16}$$

i.e., the diagram (4.10) commutes for $(k-1,k)$ aswell.

Since $i^{\Omega^1}_{V_{k-1},V_k} \colon \operatorname{soc}(\Omega^1(\tilde{G}_p/V_{k-1},\mathbb{F}_p) \to \operatorname{soc}(\Omega^1(\tilde{G}_p/V_k,\mathbb{F}_p)$ is bijective, and as $\mathbf{C}^{/p}$ is of type H^0, $\boldsymbol{\alpha}_{V_k}$ is injective provided $\boldsymbol{\alpha}_{V_{k-1}}$ is injective.

Let $N \colon = \bigcap_{k\in\mathbb{N}_0} V_k$. Then $\{V_k/N\}_{k\in\mathbb{N}_0}$ is a basis of open neighbourhoods of $1 \in \tilde{G}_p/N$.

Let $V \in \mathcal{F}_N \colon = \{U \in \mathcal{F} \mid N \leq U\}$. Then there exist $k \in \mathbb{N}_0$ such that $V_k \leq V$. Since $\Omega^1(\tilde{G}_p/\_,\mathbb{F}_p)$ and $\Omega_2(\tilde{G}_p/\_,\mathbb{F}_p)$ are i-injective cohomological $\mathcal{F}$-Mackey functors, there exists a unique mapping

$$\boldsymbol{\alpha}_V \colon \Omega^1(\tilde{G}_p/V,\mathbb{F}_p) \longrightarrow \Omega_2(\tilde{G}_p/V,\mathbb{F}_p) \tag{4.17}$$

making the diagram

$$\begin{array}{ccc} \Omega^1(\tilde{G}_p/V,\mathbb{F}_p) & \xrightarrow{\boldsymbol{\alpha}_V} & \Omega_2(\tilde{G}_p/V,\mathbb{F}_p) \\ {\scriptstyle i^{\Omega^1}_{V,V_k}}\big\downarrow & & \big\downarrow{\scriptstyle i^{\Omega_2}_{V,V_k}} \\ \Omega^1(\tilde{G}_p/V_k,\mathbb{F}_p) & \xrightarrow{\boldsymbol{\alpha}_{V_k}} & \Omega_2(\tilde{G}_p/V_k,\mathbb{F}_p) \end{array} \tag{4.18}$$

commute. It is easy to check that for all $U, V \in \mathcal{F}_N$, $V \leq U$, the diagram

$$\begin{array}{ccc} \Omega^1(\tilde{G}_p/U,\mathbb{F}_p) & \xrightarrow{\boldsymbol{\alpha}_U} & \Omega_2(\tilde{G}_p/U,\mathbb{F}_p) \\ {\scriptstyle i^{\Omega^1}_{U,V}}\big\downarrow & & \big\downarrow{\scriptstyle i^{\Omega_2}_{U,V}} \\ \Omega^1(\tilde{G}_p/V,\mathbb{F}_p) & \xrightarrow{\boldsymbol{\alpha}_V} & \Omega_2(\tilde{G}_p/V,\mathbb{F}_p) \end{array} \tag{4.19}$$

commutes. Note that $\Omega_2(\tilde{G}/\_,\mathbb{F}_p)$ is i-injective, and that for $x \in \Omega^1(\tilde{G}_p/V,\mathbb{F}_p)$

$$i^{\Omega_2}_{U,V}(\boldsymbol{\alpha}_U(N^{\Omega^1}_{V,U}(x))) = \boldsymbol{\alpha}_V(i^{\Omega^1}_{U,V}(N^{\Omega^1}_{V,U}(x))) = \boldsymbol{\alpha}_V(N_{U/V}(x)), \tag{4.20}$$

$$i^{\Omega_2}_{U,V}(N^{\Omega_2}_{V,U}(\boldsymbol{\alpha}_V(x))) = N_{V/U}(\boldsymbol{\alpha}_V(x)) = \boldsymbol{\alpha}_V(N_{U/V}(x)). \tag{4.21}$$

Hence the diagram

$$\begin{array}{ccc} \Omega^1(\tilde{G}_p/U,\mathbb{F}_p) & \xrightarrow{\boldsymbol{\alpha}_U} & \Omega_2(\tilde{G}_p/U,\mathbb{F}_p) \\ {\scriptstyle N^{\Omega^1}_{V,U}}\big\uparrow & & \big\uparrow{\scriptstyle N^{\Omega_2}_{V,U}} \\ \Omega^1(\tilde{G}_p/V,\mathbb{F}_p) & \xrightarrow{\boldsymbol{\alpha}_V} & \Omega_2(\tilde{G}_p/V,\mathbb{F}_p) \end{array} \tag{4.22}$$

commutes as well showing that

$$\boldsymbol{\alpha} \colon \Omega^1(\tilde{G}_p/\_,\mathbb{F}_p) \longrightarrow \Omega_2(\tilde{G}_p/\_,\mathbb{F}_p) \tag{4.23}$$

is a morphism of cohomological $\mathcal{F}(\tilde{G}_p/N)$-Mackey functors. By construction, one has $im(\boldsymbol{\alpha}) = \mathfrak{N}^{/p}$. Moreover, if α is injective, then the construction shows that $\boldsymbol{\alpha}$ is also injective. This yields the claim. □

4.5. **Ω^1-relator p-Frattini extensions.** Let $\pi\colon \tilde{G} \to G$ be a p-Frattini extension of G, and let $(\mathbf{C}^{/p}, \gamma^{/p})$ denote its $/p$-Frattini class field theory. We call π an *Ω^1-relator p-Frattini extension*, if there exists a map

$$\boldsymbol{\alpha}\colon \Omega^1(\tilde{G}/\_, \mathbb{F}_p) \to \mathbf{C}^{/p} \tag{4.24}$$

of cohomological $\mathcal{F}(\tilde{G})$-Mackey functors with $im(\alpha) = \mathfrak{N}^{/p}$. If necessary we include the mapping $\boldsymbol{\alpha}$ in the notation, i.e., we write (π, α) for a Ω^1-relator p-Frattini extension.

For the universal p-Frattini extension $\pi_p\colon \tilde{G}_p \to G$ one has $\mathfrak{N}^{/p} = 0$, and thus π_p is a Ω^1-relator p-Frattini extension.

From Proposition 4.3 one concludes that one can also construct such a p-Frattini extenion starting from a map $\alpha\colon \Omega^1(G, \mathbb{F}_p) \to \Omega_2(G, \mathbb{F}_p)$.

Another source of examples arises in the context of modular towers. The starting point in the study of modular towers is a fixed surjective morphism $\phi\colon \hat{G} \to G$ where $\hat{G}$ is a certain profinite orientable p-Poincaré duality group of dimension 2 onto a finite group G. A modular tower consists of all open normal subgroups U in $\hat{G}$ contained in $ker(\phi)$ such that the induced map $\phi_U\colon \tilde{G}/U \to G$ is a p-Frattini extension (cf. [1]). The 'limit groups' of a modular tower correspond to a closed normal subgroup $A \le ker(\phi)$ such that $\phi_A\colon \hat{G}/A \to G$ is a maximal p-Frattini extension ϕ can factor through. In particular, (ϕ_A, π_A), $\pi_A\colon \tilde{G} \to \hat{G}/A$ the canonical projection, is a *maximal p-Frattini quotient* of ϕ (cf. [16]). These p-Frattini extension have the following property.

Proposition 4.4. *Let $\phi\colon \hat{G} \to G$ be a surjective map of the profinite weakly-orientable p-Poincaré duality group $\hat{G}$ of dimension 2 onto the finite group G. Then for every maximal p-Frattini quotient (π, β), $\pi\colon im(\beta) \to G$ is a Ω^1-relator p-Frattini extension of G.*

Proof. Let $B\colon = im(\beta)$, and let

$$Q_1 \xrightarrow{\delta^{/p}} Q_0 \longrightarrow \mathbb{F}_p \tag{4.25}$$

be a partial minimal projective resolution in ${}_B\mathbf{prf}_{/p}$. Put $M\colon = ker(\delta)$. By [16, Prop.3.4], one has a surjective map $\alpha\colon Q_0 \to M$. Since $\mathfrak{N}^{/p}$ is norm surjective (cf. Prop.4.1(b)), one has a surjective map of cohomological $\mathcal{F}(B)$-Mackey functors

$$\rho\colon \mathfrak{X}(Q_0) \longrightarrow \mathfrak{X}(M) \longrightarrow \mathfrak{N}^{/p}. \tag{4.26}$$

Since $\mathfrak{N}^{/p}$ is a $\mathcal{F}(B)$-sub Mackey functor of $\mathbf{C}$, and as $(Q_0)_U$ is an injective $\mathbb{F}_p[B/U]$-module, $\rho_U\colon (Q_0)_U \to \mathfrak{N}_U^{/p} \le \Omega_2(B/U, \mathbb{F}_p)$ cannot be injective, i.e, $\mathrm{soc}((Q_0)_U) \le ker(\rho_U)$. Hence ρ induces a surjective mapping

$$\rho_*\colon \Omega^1(B/\_, \mathbb{F}_p) \longrightarrow \mathfrak{N}^{/p} \tag{4.27}$$

of cohomological $\mathcal{F}(B)$-Mackey functors and this yields the claim. □

In order to finish the proof of Theorem B, we establish the following theorem:

Theorem 4.5. *Let $(\pi, \boldsymbol{\alpha})$, $\pi\colon \tilde{G} \to G$, be a Ω^1-relator p-Frattini extension. Assume further that $\boldsymbol{\alpha}$ is injective, and that $\boldsymbol{\alpha}_{ker(\pi)}$ is not an isomorphism. Then $\tilde{G}$ is a weakly-orientable p-Poincaré duality group of dimension* 2.

Proof. Note that $\dim_{\mathbb{F}_p}(\Omega_2(G,\mathbb{F}_p)) > \dim_{\mathbb{F}_p}(\Omega^1(G,\mathbb{F}_p))$ implies that $\tilde{G}$ is infinite (cf. [16, Prop.3.5]). It suffices to prove that $\mathbf{H}^k(\tilde{G},\mathbb{F}_p[\![\tilde{G}]\!]) = 0$ for $k \neq 2$, and $\mathbf{H}^2(\tilde{G},\mathbb{F}_p[\![\tilde{G}]\!]) \simeq \mathbb{F}_p$. As beforre $\mathbf{H}^\bullet$ denotes continuous cochain cohomology.

By definition, one has exact sequences of cohomological $\mathcal{F}(\tilde{G})$-Mackey functors

$$0 \longrightarrow \mathbf{T}(\mathbb{F}_p) \longrightarrow \mathfrak{X}(Q_0) \longrightarrow \Omega^1(\tilde{G}/\_,\mathbb{F}_p) \longrightarrow 0, \tag{4.28}$$

$$0 \longrightarrow \Omega^1(\tilde{G}/\_,\mathbb{F}_p) \longrightarrow \Omega_2(\tilde{G}/\_,\mathbb{F}_p) \longrightarrow \mathbf{Ab}^{/p} \longrightarrow 0, \tag{4.29}$$

$$0 \longrightarrow \Omega_2(\tilde{G}/\_,\mathbb{F}_p) \longrightarrow \mathfrak{X}(Q_1) \longrightarrow \mathfrak{X}(Q_0) \longrightarrow \mathbf{S}(\mathbb{F}_p) \longrightarrow 0. \tag{4.30}$$

As $\tilde{G}$ is infinite $m(\mathbf{T}(\mathbb{F}_p)) = m(\mathbf{Ab}^{/p}) = 0$. Thus applying the functor m yields that one has a minimal projective resolution

$$0 \longrightarrow Q_0 \longrightarrow Q_1 \longrightarrow Q_0 \longrightarrow \mathbb{F}_p \longrightarrow 0 \tag{4.31}$$

of $\mathbb{F}_p$ in ${}_{\tilde{G}}\mathbf{prf}_{/p}$. Hence $\tilde{G}$ is of cohomological p-dimension 2.

In his letter to J-P.Serre (cf. [12, App.1]), J.Tate described how one can compute the Pontryagin dual of the cohomology groups $\mathbf{H}^k(\tilde{G},\mathbb{F}_p[\![\tilde{G}]\!])$. Translated to our situation we obtain

$$\begin{aligned} \mathbf{H}^2(\tilde{G},\mathbb{F}_p[\![\tilde{G}]\!])^* &= \varinjlim_U \mathbf{H}_2(U,\mathbb{F}_p), \\ \mathbf{H}^1(\tilde{G},\mathbb{F}_p[\![\tilde{G}]\!])^* &= \varinjlim_U \mathbf{H}_1(U,\mathbb{F}_p). \end{aligned} \tag{4.32}$$

Since $\tilde{G}$ is infinite, $\mathbf{H}^0(\tilde{G},\mathbb{F}_p[\![\tilde{G}]\!]) = 0$. From the exact sequences (4.28) it follows that one has an isomorphism of $\mathcal{F}(\tilde{G})$-Mackey functors $\mathbf{H}_2(\_,\mathbb{F}_p) \simeq \mathbf{T}(\mathbb{F}_q)$. This yields $\mathbf{H}^2(\tilde{G},\mathbb{F}_p[\![\tilde{G}]\!]) \simeq \mathbb{F}_p$.

Let $\boldsymbol{\alpha}^*\colon \Omega^2(\tilde{G}/\_,\mathbb{F}_p) \longrightarrow \Omega_1(\tilde{G}/\_,\mathbb{F}_p)$ be the Pontryagin dual of $\boldsymbol{\alpha}$. Then by (4.32), $\mathbf{H}^1(\tilde{G},\mathbb{F}_p[\![\tilde{G}]\!]) \simeq m(ker(\boldsymbol{\alpha}^*))$. Moreover, $\boldsymbol{\alpha}^*$ is surjective. Since for all $U \in \mathcal{F}(\tilde{G})$, one has an isomorphism

$$hd(\boldsymbol{\alpha}^*_U)\colon hd(\Omega_2(\tilde{G}/U,\mathbb{F}_p)) \longrightarrow hd(\Omega_1(\tilde{G}/U,\mathbb{F}_p)), \tag{4.33}$$

where $hd(\_)$ denotes the head of a module, one obtains a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Omega^1(\tilde{G}/\_,\mathbb{F}_p) & \longrightarrow & \mathfrak{X}(Q_1)^* & \longrightarrow & \Omega^2(\tilde{G}/\_,\mathbb{F}_p) & \longrightarrow & 0 \\ & & \Big\downarrow{\scriptstyle\rho} & & \Big\downarrow{\scriptstyle\sigma} & & \Big\downarrow{\scriptstyle\boldsymbol{\alpha}^*} & & \\ 0 & \longrightarrow & \Omega_2(\tilde{G}/\_,\mathbb{F}_p) & \longrightarrow & \mathfrak{X}(Q_1)^* & \longrightarrow & \Omega_1(\tilde{G}/\_,\mathbb{F}_p) & \longrightarrow & 0. \end{array} \tag{4.34}$$

By (4.33), σ is an isomorphism. So by the snake lemma, ρ is injective, and one has an isomorphism $coker(\rho) = ker(\boldsymbol{\alpha}^*)$. Since $\Omega^1(\tilde{G}/\_,\mathbb{F}_p)$ is N-surjective, all elements in $im(\sigma)$ are universal norms. Hence by dimension arguments, $im(\rho) = im(\boldsymbol{\alpha})$ and this yields

$$m(ker(\boldsymbol{\alpha}^*)) \simeq m(coker(\rho)) \simeq m(\mathbf{Ab}^{/p}) = 0. \tag{4.35}$$

This yields the claim. □

Corollary 4.6. *Let G be a finite group and let p be a prime number. Then the following are equivalent:*

(i) *There exists a p-Frattini extension* $\pi\colon \tilde{G} \to G$ *with* $\tilde{G}$ *a profinite weakly-orientable p-Poincaré duality group of dimension* 2.
(ii) *There exists an injection* $\alpha\colon \Omega^1(G, \mathbb{F}_p) \to \Omega_2(\mathbb{F}_p)$ *which is not an isomorphism.*

Proof. This is a direct consequence of [16, Thm.4.1] and Theorem 4.5. □

Remark 4.7. (a) Let $p = 2$ and let $G = PSl_2(q)$, $q \equiv 3 \mod 4$. The explicit description of the projective indecomposable $\mathbb{F}_2[G]$-modules obtained by K.Erdmann [5] shows that in this case one has an injection $\alpha\colon \Omega^1(G, \mathbb{F}_p) \to \Omega_2(G, \mathbb{F}_p)$.
(b) If G is p-perfect, i.e., $G_p^{ab} = 0$, $\tilde{G}$ is p-perfect too. Thus every $\tilde{G}$-module $M \in ob({}_{\tilde{G}}\mathbf{prf}_p)$, which underlying abelian pro-p group is isomorphic to $\mathbb{Z}_p$ and which reduction mod p $M/p.M$ is a trivial $\tilde{G}$-module, must be trivial. Hence in this case one can conclude that $\tilde{G}$ is indeed a orientable p-Poincaré duality group of dimension 2.
(c) In [16, Ex.1.4] an example was given were for any maximal p-Frattini quotient (π, β) of a morphism $\phi\colon \hat{G} \to PSl_2(7)$, the p-Frattini extension π is of the type described in Theorem 4.5.
(d) One question which has been untouched completely is to describe all isomorphism types of extensions $\pi\colon \tilde{G} \to G$ satisfying (i) of Corollary 4.6. The construction we used does not give any evidence how one can achieve this goal.

5. Δ-Frattini extensions

Throughout this section we fix a prime number p. For a given finite group G we denote by $\mathfrak{S}_p(G)$ the set of isomorphism types of irreducible (left) $\mathbb{F}_p[G]$-modules. For an irreducible $\mathbb{F}_p[G]$-module S we use the symbol $[S] \in \mathfrak{S}_p(G)$ to denote its isomorphism type.

5.1. **The Δ-head of an $\mathbb{F}_p[G]$-module.** Let $\Delta \subseteq \mathfrak{S}_p(G)$ be a set of isomorphism types of irreducible $\mathbb{F}_p[G]$-modules. For short we call an $\mathbb{F}_p[G]$-module $M \in ob({}_G\operatorname{mod}_p)$ of finite $\mathbb{F}_p$-dimension a Δ-*module*, if M has a composition series $(M_k)_{0 \le k \le m}$, $0 = M_0 < M_1 < \cdots < M_m = M$, with each composition factor being contained in Δ, i.e., $[M_k/M_{k-1}] \in \Delta$ for all $k = 1, .., m$. We also assume that $0 \in ob({}_G\operatorname{mod}_p)$ is a Δ-module.

Let M be an $\mathbb{F}_p[G]$-module of finite $\mathbb{F}_p$-dimension. We call an $\mathbb{F}_p[G]$-submodule $N \le M$ a Δ-*kernel*, if M/N is a Δ-module. Obviously, the intersection of any set of Δ-kernels $N_i \le M$, $i \in I$, is again a Δ-kernel. Hence there exists a minimal Δ-kernel $M_\Delta \le M$. For short we call

$$hd_\Delta(M) \colon = M/M_\Delta \tag{5.1}$$

The Δ-*head* of M.

5.2. **The universal Δ-Frattini extension.** Let

$$1 \longrightarrow \Omega_2(G, \mathbb{F}_p) \xrightarrow{\ \iota\ } \tilde{G}_{/p} \xrightarrow{\ \pi_{/p}\ } G \longrightarrow 1 \tag{5.2}$$

be the universal elementary p-abelian Frattini extension of G, where ι is considered to be given by inclusion. Factoring by the minimal Δ-kernel $\Omega_2(G, \mathbb{F}_p)_\Delta$ of $\Omega_2(G, \mathbb{F}_p)$ yields a Δ-Frattini extension

$$1 \longrightarrow hd(\Omega_2(G, \mathbb{F}_p)) \xrightarrow{\ \iota\ } \tilde{G}_{/\Delta} \xrightarrow{\ \pi_{/\Delta}\ } G \longrightarrow 1 \tag{5.3}$$

which is easily seen to be universal with respect to all elementary p-abelian Δ-Frattini extensions of G. Thus for $G_0 \colon = G$, and $\pi_{i+1,i} \colon G_{i+1} \to G_i$ the universal elementary p-abelian Δ-Frattini extension of G_i, we obtain an inverse system whose inverse limit

$$\tilde{G}_\Delta \colon = \varprojlim_{i \in \mathbb{N}_0} G_i \tag{5.4}$$

together with the canonical map $\pi_\Delta \colon \tilde{G}_\Delta \to G$ is a Δ-Frattini extension of G. The universality aswell as the uniqueness up to isomorphism follows by the same arguments which were used to prove these statements for the universal p-Frattini extension (cf. [6]).

At this point we have to deal with the question how one characterize the universal Δ-Frattini extension among all Δ-Frattini extensions. This is the subject of the following proposition.

Proposition 5.1. *Let $\pi \colon \tilde{G} \to G$ be a Δ-Frattini extension of G, $\Delta \subseteq \mathfrak{S}_p(G)$. Then the following are equivalent:*

(i) *π coincides with the universal Δ-Frattini extension of G.*
(ii) *$H^2(\tilde{G}, S) = 0$ for all irreducible $\mathbb{F}_p[G]$-modules S, $[S] \in \Delta$.*

Proof. Assume that $\pi \colon \tilde{G} \to G$ is the universal Δ-Frattini extension of G, and that there exists an irreducible $\mathbb{F}_p[G]$-module S, $[S] \in \Delta$, with $H^2(\tilde{G}, S) \neq 0$. For $\eta \in H^2(\tilde{G}, S)$, $\eta \neq 0$, the associated extension of profinite groups

$$\mathbf{s}(\eta) \colon \quad 1 \longrightarrow S \longrightarrow X \xrightarrow{\ \tau\ } \tilde{G} \longrightarrow 1 \tag{5.5}$$

is non-split and thus $\tau \circ \pi \colon X \to G$ is a Δ-Frattini extension. The universality of π implies that τ has a section $\sigma \colon \tilde{G} \to X$ contradicting the fact that $\mathbf{s}(\eta)$ is non-split. Thus (i) implies (ii).

Assume that $H^2(\tilde{G}, S) = 0$ for all $[S] \in \Delta$, and let $\pi_\Delta \colon \tilde{G}_\Delta \to G$ be the universal Δ-Frattini extension of G. Then one has a surjective map $\beta \colon \tilde{G}_\Delta \to \tilde{G}$, and thus an isomorphism

$$\tilde{\beta}^{-1} \colon \tilde{G} \longrightarrow \tilde{G}_\Delta / ker(\beta). \tag{5.6}$$

Assume that $ker(\beta) \neq 1$ is non-trivial, and let $U \leq ker(\beta)$ be a maximal open subgroup of $ker(\beta)$ which is normal in $\tilde{G}_\Delta$. Since $[ker(\beta)/U] \in \Delta$, one has $H^2(\tilde{G}, ker(\beta)/U) = 0$. Hence the embedding problem

$$\begin{array}{ccccccccc} & & & & & & \tilde{G} & & \\ & & & & & & \big\downarrow{\scriptstyle \tilde{\beta}^{-1}} & & \\ \mathbf{s} \colon & 1 & \longrightarrow & ker(\beta)/U & \longrightarrow & \tilde{G}_\Delta/U & \longrightarrow & \tilde{G}_\Delta/ker(\beta) & \longrightarrow & 1 \end{array} \tag{5.7}$$

has a weak solution (cf. [16, Prop.3.2]). This implies that $\mathbf{s}$ is split exact, which contradicts the fact that $\mathbf{s}$ is also a p-Frattini extension. Thus $ker(\beta) = 1$, and this yields the claim. □

5.3. **Chevalley groups over $\mathbb{Z}_p$.** For a given Dynkin diagram D let X_D be the simple simply-connected $\mathbb{Z}$-Chevalley group scheme associated to D, i.e., if D is of type A_n, one has $X_D = Sl_{n+1}$. It has been proved in [18, Thm.B] that

$$\pi_D \colon X_D(\mathbb{Z}_p) \longrightarrow X_D(\mathbb{F}_p) \tag{5.8}$$

is a p-Frattini extension apart from possibly 11 explicitly known values of (D,p). It was also shown that in 8 of these 11 cases (5.8) fails to be a p-Frattini extension.

In case π_D is a p-Frattini extension, then it is also a Δ_D-Frattini extension, where Δ_D consists of all the $\mathbb{F}_p[X(F_p)]$-composition factors of the $\mathbb{F}_p$-Chevalley Lie algebra $\mathfrak{L}_D \otimes \mathbb{F}_p$ (cf. [18, (2.5)]). If one has additionally

$$(5.9)\qquad \begin{aligned}(D,p) \notin \{\, &(A_n,p), p|(n+1), (B_n,2), (C_n,2), (D_n,2), \dots \\ &\dots, (E_6,3), (E_7,2), (F_4,2), (G_2,2), (G_2,3)\,\},\end{aligned}$$

then $\mathfrak{L}_D \otimes \mathbb{F}_p$ is an irreducible $\mathbb{F}_p[X_D(\mathbb{F}_p)]$-module (cf. [18, Lemma 2.10]), and thus $\Delta_D = \{[\mathfrak{L}_D \otimes \mathbb{F}_p]\}$.

The question raised in [6, Prob.20.40] can now be restated in the following way.

Question 5.2. *Assume that p is large with respect to the Coxeter number of D. Is it true that the p-Frattini extension $\pi_D\colon X(\mathbb{Z}_p) \to X(\mathbb{F}_p)$ coincide with the universal Δ_D-Frattini extension?*

From Proposition 5.1 one concludes that the problem of Question 5.2 is equivalent to the following vanishing problem.

Question 5.3. *Assume that p is large with respect to the Coxeter number of D. Is it true that*

$$(5.10)\qquad H^2(X_D(\mathbb{Z}_p), \mathfrak{L}_D \otimes \mathbb{F}_p) = 0?$$

As we see in the following theorem both questions have an affirmative answer for $X_D = Sl_2$.

Theorem 5.4. *Let p be a prime number different from 2, 3 or 5. Then*

$$(5.11)\qquad \pi_{A_1}\colon Sl_2(\mathbb{Z}_p) \to Sl_2(\mathbb{F}_p)$$

coincides with the universal Δ-Frattini extension for all $\Delta \subseteq \mathfrak{S}_p(Sl_2(\mathbb{F}_p))$ satisfying $[M_2] \in \Delta$, $[M_{p-3}] \notin \Delta$, where M_k, $k = 0,..,p-1$ denotes the irreducible $\mathbb{F}_p[Sl_2(\mathbb{F}_p)]$-module of heighest weight k and $\mathbb{F}_p$-dimension $k+1$.

Proof. By the previously mentioned remark and Proposition 5.1 it suffices to show that $H^2(Sl_2(\mathbb{Z}_p), M_k) = 0$ for all $k \neq p-3$.

As $p \neq 2,3$, $\tilde{G}\colon = Sl_2(\mathbb{Z}_p)$ is p-torsionfree, and thus a p-Poincaré duality group of dimension d (cf. [13, Prop.4.4.1]). As we assumed $p \neq 2,3$, $\tilde{G}$ is perfect (cf. [18, Prop.3.2]). Thus its p-dualizing module $\mathbb{I}_{\tilde{G},p}$ is a trivial $\tilde{G}$-module. Hence by Poincaré duality and the Universal Coefficiant Theorem one has

$$(5.12)\qquad H^2(Sl_2(\mathbb{Z}_p), M_k) \simeq H_1(Sl_2(\mathbb{Z}_p), M_k) \simeq H^1(Sl_2(Z_p), M_k)^*,$$

where * denotes the Pontryagin dual. Moreover, from [16, Prop.3.1] and [17] one concludes that

$$(5.13)\qquad H^1(Sl_2(\mathbb{Z}_p), M_k) \simeq H^1(Sl_2(\mathbb{F}_p), M_k) = 0$$

for $k \neq p-3$. This yields the claim. □

Remark 5.5. Theorem 5.4 does not hold for $p = 2$, 3 or 5, but in each case for a different reason.

For $p = 2$ or 3, π_{A_1} is not a 2-Frattini extension (cf. [18, Thm.B]). For $p = 3$, π_{A_1} is even a split extension, since in this case $\mathfrak{L}_{A_1} \otimes \mathbb{F}_3$ is isomorphic to the Steinberg module for $Sl_2(\mathbb{F}_3)$.

For $p = 5$, $\Omega_2(Sl_2(\mathbb{F}_5), \mathbb{F}_5)$ is a Δ_{A_1}-module (cf. [17]). Hence the universal elementary p-abelian Δ_{A_1}-extension coincides with the universal elementary p-abelian Frattini extension $\pi_{/p}$. However,

$$\dim_{\mathbb{F}_5}(\Omega_2(Sl_2(\mathbb{F}_5), \mathbb{F}_5) = 6, \quad \dim_{\mathbb{F}_5}(ker(\pi_{A_1})^{ab}) = 3. \tag{5.14}$$

This phenomenon can also be explained by analyzing cohomology groups. Since $p - 3 = 2$, Poincaré duality and [16, Prop.3.1] implies that

(5.15)

$$H^2(Sl_2(\mathbb{Z}_5), \mathfrak{L}_{A_1} \otimes \mathbb{F}_5)^* \simeq H^1(Sl_2(\mathbb{Z}_5), \mathfrak{L}_{A_1} \otimes \mathbb{F}_5) \simeq H^1(Sl_2(\mathbb{F}_5), \mathfrak{L}_{A_1} \otimes \mathbb{F}_5) \simeq \mathbb{F}_5.$$

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Th. Weigel, Università di Milano-Bicocca, U5-3067, Via R.Cozzi, 53, 20125 Milano, Italy

E-mail address: `thomas.weigel@unimib.it`