

ON PROFINITE GROUPS WITH FINITE ABELIANIZATIONS

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ABSTRACT. Profinite groups with finite p -abelianizations arise in various contexts: group theory, number theory and geometry. Using Ph.Furtwängler's transfer vanishing theorem it will be proved that a finitely generated profinite group \hat{G} with this property satisfies $H^1(\hat{G}, \mathbb{F}_p[[\hat{G}]]) = 0$ (Thm.A). As a consequence one obtains that a hereditary just-infinite non-virtually cyclic pro- p group has only 1 end (Cor.B). Applied to 3-dimensional Poincaré duality groups one obtains a generalization of A.Reznikov's theorem on 3-dimensional co-compact hyperbolic lattices violating W.Thurston's conjecture (Thm.C).

1. INTRODUCTION

Let \hat{G} be a profinite group and let p be a prime number. Then \hat{G} is said to have *finite p -abelianizations*, if the largest abelian pro- p quotient U_p^{ab} of every open subgroup U of \hat{G} is finite.

Examples of profinite groups with finite p -abelianizations arise in group theory and number theory, and moreover, this property is also related to a famous conjecture in hyperbolic geometry: (a) If \hat{G} is a hereditary just-infinite non-virtually cyclic pro- p group, \hat{G} has finite p -abelianizations. (b) Let K be a number field, and let K^{un}/K denote the maximal algebraic extension of K which is unramified above all finite places. Then, by a famous theorem of E.S.Golod and I.R.Šafarevič (cf. [4]), its Galois group $G_K^{fin} := Gal(K^{un}/K)$ is not necessarily finite. However, the finiteness of the ideal class group implies that it has finite p -abelianizations for all prime numbers p . (c) Let $\Gamma \leq O(n, 1)$, $n \geq 2$, be a hyperbolic lattice for which W.Thurston's conjecture fails. Then its profinite completion $\hat{\Gamma}$ has finite p -abelianizations for all prime numbers p .

The purpose of this short note is to establish the following theorem (cf. Thm.2.2) and to discuss its consequences.

Theorem A. *Let G be a finitely generated profinite group with finite p -abelianizations. Then*

$$(1.1) \quad H^1(\hat{G}, \mathbb{F}_p[[\hat{G}]]) = 0.$$

Here $H^\bullet(\hat{G}, _)$ denotes *continuous cochain cohomology* as defined by J.Tate (cf. [11]), and $\mathbb{F}_p[[\hat{G}]]$ denotes the completed \mathbb{F}_p -group algebra of \hat{G} . In [7] A. A. Korenev established a theory of ends for pro- p groups. In this context Theorem A implies the following (cf. §2.3):

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Corollary B. *Let \hat{G} be a finitely generated infinite pro- p group with finite p -abelianizations. Then \hat{G} has 1 end. In particular, if \hat{G} is a hereditary just-infinite non-virtually cyclic pro- p group, then \hat{G} has 1 end.*

In [9] A. Reznikov considered the pro- p completion of torsionfree co-compact lattices in $SL_2(\mathbb{C})$ under the hypothesis that W.Thurston's conjecture does not hold for them. Although the author mentioned the analogy between the theory of 3-manifolds and number theory, he does make use of the "transfer vanishing theorem" established by Ph. Furtwängler (cf. [3]), which implies Theorem A, and which was also the basic tool for proving the "principle ideal theorem" conjectured by D.Hilbert more that thirty years earlier (cf. [5]). From Theorem A one concludes the following theorem (cf. Prop.3.2, Thm.3.3) which can be seen as a generalization of [9, Thm.10.2].

Theorem C. *Let Γ be a discrete orientable 3-dimensional Poincaré duality group, and let p be a prime number with the following properties:*

- (i) *All subgroups U of Γ of finite index have trivial Schur multiplier, i.e., $H_2(U, \mathbb{Z}) = H^1(U, \mathbb{Z}) = 0$.*
- (ii) *The pro- p completion $\hat{\Gamma}(p)$ of Γ is infinite.*

Then the profinite completion $\hat{\Gamma}$ of Γ as well as the pro- p completion $\hat{\Gamma}(p)$ of Γ are orientable p -Poincaré duality groups of dimension 3.

2. THE TRANSFER VANISHING THEOREM

2.1. The transfer. Let G be a discrete group, and let $U \leq G$ be a subgroup of finite index. Then the transfer $tr_{G,U}: G^{ab} \rightarrow U^{ab}$ is defined as follows: Let \mathcal{R} be a set of representatives of G/U . For $g \in G$ and $r \in \mathcal{R}$ one has $g.r = r'.u(g,r)$ for some unique element $u(g,r) \in U$. The transfer from G to U is given by

$$(2.1) \quad tr_{G,U}(g.[G,G]) = \prod_{r \in \mathcal{R}} u(g,r).[U,U].$$

Ph. Furtwängler's transfer vanishing theorem can be stated as follows (cf. [8, Thm.VI.7.6]):

Theorem 2.1. *Let G be a finitely generated group, such that $G' := [G,G]$ and $G'' := [G',G']$ are subgroups of finite index. Then $tr_{G,G'} = 0$.*

2.2. Computing $H^1(\hat{G}, \mathbb{F}_p[[\hat{G}]])$. Let \hat{G} be a profinite group, and let $\mathcal{U}_{\hat{G}}$ denote the directed set of open subgroups of \hat{G} . The completed \mathbb{F}_p -algebra of \hat{G} is given by

$$(2.2) \quad \mathbb{F}_p[[\hat{G}]] = \varprojlim_{U \in \mathcal{U}_{\hat{G}}} \mathbb{F}_p[\hat{G}/U].$$

Then, in case \hat{G} is a finitely generated profinite group,

$$(2.3) \quad H^1(\hat{G}, \mathbb{F}_p[[\hat{G}]]) = Z^1(\hat{G}, \mathbb{F}_p[[\hat{G}]]) / B^1(\hat{G}, \mathbb{F}_p[[\hat{G}]]),$$

is the abelian pro- p group of crossed homomorphisms $Z^1(\hat{G}, \mathbb{F}_p[[\hat{G}]])$ of \hat{G} into $\mathbb{F}_p[[\hat{G}]]$ modulo inner crossed homomorphisms $B^1(\hat{G}, \mathbb{F}_p[[\hat{G}]])$. In particular, by the Faddeev-Shapiro lemma and Pontryagin duality

$$(2.4) \quad \begin{aligned} H^1(\hat{G}, \mathbb{F}_p[[\hat{G}]]) &= \varprojlim_{U \in \mathcal{U}_{\hat{G}}} H^1(G, \mathbb{F}_p[\hat{G}/U]) \\ &= \varprojlim_{U \in \mathcal{U}_{\hat{G}}} H^1(U, \mathbb{F}_p) \\ &= \varprojlim_{U \in \mathcal{U}_{\hat{G}}} (U^{ab}/p)^* = \left(\varinjlim_{U \in \mathcal{U}_{\hat{G}}} U^{ab}/p \right)^*, \end{aligned}$$

where $U_{/p}^{ab}$ is the largest elementary p -abelian quotient of U , and $_*$ denotes the Pontryagin dual. Note that for $U, V \in \mathcal{U}_{\hat{G}}$, $V \leq U$, the map in the direct limit in (2.4) is just $tr_{U,V} \otimes \mathbb{F}_p$. This yields:

Theorem 2.2. *Let G be a finitely generated profinite group with finite p -abelianizations. Then $H^1(\hat{G}, \mathbb{F}_p[[\hat{G}]]) = 0$.*

Proof. By (2.4) it suffices to show that for $U \in \mathcal{U}_{\hat{G}}$ there exist $V \in \mathcal{U}_{\hat{G}}$ such that

$$(2.5) \quad tr_{U,V} \otimes \mathbb{F}_p: U_{/p}^{ab} \longrightarrow V_{/p}^{ab}$$

is the trivial map. Let $V := \ker(U \rightarrow U_{/p}^{ab})$. Then by hypothesis, V is open. Moreover, for $W := \ker(V \rightarrow V_{/p}^{ab})$, U/W is a metabelian p -group with commutator subgroup equal to V/W . Hence by Theorem 2.1, $tr_{U,V} = 0$ and this yields the claim. \square

2.3. Just-infinite pro- p groups. A profinite group \hat{G} is called *just-infinite*, if \hat{G} is not finite and every closed normal subgroup of \hat{G} is open. It is called *hereditary just-infinite*, if every open subgroup is just-infinite.

Hence by definition, a hereditary just-infinite pro- p group \hat{G} is finitely generated. Furthermore, it is either virtually cyclic, or the closure of the commutator subgroup $cl([U, U])$ is open for all open subgroups $U \in \mathcal{U}_{\hat{G}}$. Thus a hereditary just-infinite non-virtually cyclic pro- p group has finite p -abelianizations.

If \hat{G} is a finitely generated infinite pro- p group, the number of ends $e(\hat{G})$ equals 1 plus the \mathbb{F}_p -dimension of $H^1(\hat{G}, \mathbb{F}_p[[\hat{G}]])$ (cf. [7]). Thus Corollary B is a direct consequence of Theorem A.

3. POINCARÉ DUALITY GROUPS

3.1. Goodness of discrete groups. Let Γ be a discrete group, and let $\hat{\Gamma}$ denote its profinite completion. The canonical morphism $\iota: \Gamma \rightarrow \hat{\Gamma}$ induces canonical maps

$$(3.1) \quad \iota^k(M): H^k(\hat{\Gamma}, M) \longrightarrow H^k(\Gamma, M)$$

for all $k \geq 0$ and for all left $\mathbb{Z}[\Gamma]$ -modules M of finite order. Furthermore, the sequence of natural transformations ι^\bullet commutes with cup products.

Let p be a prime number. Following [10], one calls the discrete group Γ *p -good*, if $\iota^k(M)$ are isomorphisms for all $k \geq 0$ and for all finite left $\mathbb{Z}[\Gamma]$ -modules M of p -power order.

Let $\hat{\Gamma}(p)$ denote the pro- p completion of Γ , and let $\iota_p: \Gamma \rightarrow \hat{\Gamma}(p)$ denote the canonical map. This map induces canonical morphisms

$$(3.2) \quad \iota_p^k(M): H^k(\hat{\Gamma}(p), M) \longrightarrow H^k(\Gamma, M)$$

for all $k \geq 0$ and for all finite left $\mathbb{Z}[\Gamma]$ -modules M of p -power order for which all composition factors are trivial Γ -modules. Again the sequence of natural transformations ι_p^\bullet commutes with cup-product.

The discrete group Γ will be called *pro- p good*, if $\iota_p^k(\mathbb{F}_p)$ are isomorphisms for all $k \geq 0$. Since ι_p^\bullet is a natural transformation of cohomological functors, this implies that for a pro- p good group Γ , $\iota_p^k(M)$ are isomorphisms for all $k \geq 0$ and for all finite left $\mathbb{Z}[\Gamma]$ -modules M of p -power order for which all composition factors are trivial Γ -modules.

These two notions of goodness are not unrelated. One has:

Proposition 3.1. *Let Γ be a discrete group. Assume further that Γ and all its subgroups of finite index are pro- p good. Then Γ is p -good.*

Proof. The hypothesis implies that for every subgroup of finite index U of Γ , for every finite trivial U -module of p -power order M and for every $\eta \in H^k(U, M)$, $k \geq 0$, there exists a subgroup V of U of p -power index such that $\text{res}_{U,V}(\eta) = 0$ (cf. [10, §I.2.6, Ex.1]). Hence for every finite Γ -module N of p -power order, and $\omega \in H^k(\Gamma, N)$, $k \geq 0$, there exists a subgroup of finite index W in Γ such that $\text{res}_{\Gamma,W}(\omega) = 0$. This yields the claim. \square

3.2. Poincaré duality. Let \hat{G} be a profinite group of cohomological p -dimension d , $d \in \mathbb{N}$. Then \hat{G} is called a p -Poincaré duality group of dimension d , if

- (i) for every finite discrete left \hat{G} -module of p -power order M and for all $k \geq 0$ one has $|H^k(\hat{G}, M)| < \infty$,
- (ii) the p -dualizing module $I_{\hat{G},p}$ of \hat{G} (cf. [10, §I.3.5]) is isomorphic to $\mathbb{I}_p := \mathbb{Q}_p/\mathbb{Z}_p$ as abelian group,
- (iii) for every finite discrete left \hat{G} -module M of p -power order, cup-product induces a non-degenerate pairing

$$(3.3) \quad H^k(\hat{G}, M') \times H^{d-k}(\hat{G}, M) \xrightarrow{H^d(\text{ev}_M) \circ (\cdot \cup \cdot)} H^d(\hat{G}, I_{\hat{G},p}) \xrightarrow{i} \mathbb{I}_p,$$

where $M' := \text{Hom}(M, I_{\hat{G},p})$, $\text{ev}_M: M' \times M \rightarrow I_{\hat{G},p}$ is the evaluation map and i is given as in [10, §I.3.5].

Furthermore, one calls the p -Poincaré duality group \hat{G} *orientable*, if $I_{\hat{G},p}$ is a trivial \hat{G} -module.

The definition of Poincaré duality for profinite groups looks quite different to the definition given by R.Bieri and B.Eckmann for discrete groups (cf. [1]). Nevertheless, one has the following:

Proposition 3.2. *Let Γ be an orientable discrete Poincaré duality group of dimension d .*

- (a) *If Γ is p -good, $\hat{\Gamma}$ is an orientable p -Poincaré duality group of dimension d .*
- (b) *If Γ is pro- p good, $\hat{\Gamma}(p)$ is an orientable p -Poincaré duality group of dimension d .*

Proof. Cap product with the fundamental class $z \in H^d(\Gamma, \mathbb{Z})$ (cf. [2, Chap.VIII, Thm.10.1, 2nd proof]) induces a canonical isomorphism $i: H^d(\Gamma, \mathbb{I}_p) \rightarrow \mathbb{I}_p$. For a finite left Γ -module M of p -power order let $M^* := \text{Hom}(M, \mathbb{I}_p)$. The canonical pairing

$$(3.4) \quad H^k(\Gamma, M^*) \otimes H^{d-k}(\Gamma, M) \xrightarrow{\cdot \cup \cdot} H^d(\Gamma, M^* \otimes M) \xrightarrow{i \circ H^d(\text{ev}_M)} \mathbb{I}_p$$

induced by cup-product is non-degenerate (cf. [12, 2.8]). Hence the claim follows from the commutativity of ι^\bullet and ι_p^\bullet , respectively, with cup-product. \square

3.3. 3-dimensional Poincaré duality groups. For an orientable 3-dimensional Poincaré duality group Theorem 2.2 implies the following:

Theorem 3.3. *Let Γ be an orientable 3-dimensional discrete Poincaré duality group.*

- (a) *Γ is pro- p good, if and only if the pro- p completion $\hat{\Gamma}(p)$ of Γ is an infinite pro- p group with 1 end.*

- (b) Assume that Γ and all its subgroups of finite index have trivial Schur multiplier, i.e., for every subgroup U of finite index in Γ one has $H^1(U, \mathbb{Z}) = H_2(U, \mathbb{Z}) = 0$. Then the following are equivalent:
- (i) The pro- p completion $\hat{\Gamma}(p)$ of Γ is infinite.
 - (ii) Γ is pro- p good.
- (c) Assume that Γ and all its subgroups of finite index have trivial Schur multiplier, and that the pro- p completion $\hat{\Gamma}(p)$ of Γ is infinite. Then Γ is p -good.

Proof. (a) If Γ is pro- p good, $\hat{\Gamma}(p)$ is a profinite Poincaré duality pro- p group of dimension 3 (cf. Prop.3.2), and thus it is infinite and has 1 end (cf. [7]). If $\hat{\Gamma}(p)$ is infinite and has 1 end, Poincaré duality for Γ implies

$$(3.5) \quad \begin{aligned} 0 &= (H^1(\hat{\Gamma}(p), \mathbb{F}_p[\llbracket \hat{\Gamma}(p) \rrbracket]))^* = \varinjlim_{U \in \mathcal{U}(p)} U_{/p}^{ab} \\ &= \varinjlim_{U \in \mathcal{U}(p)} H^2(U, \mathbb{F}_p) \end{aligned}$$

where $\mathcal{U}(p)$ denotes the directed set of subgroups of Γ of p -power index, and the maps in the direct limits are given by restriction. This shows the following: Let M be a finite Γ -module which is a finite-dimensional \mathbb{F}_p -vector space and which composition factors are all trivial modules. Then for every element $\eta \in H^2(\Gamma, M)$ there exists a subgroup U of Γ of p -power index such that $res_{\Gamma, U}(\eta) = 0$.

For such a Γ -module M Poincaré duality of Γ and the non-finiteness of $\hat{\Gamma}(p)$ imply that for every element $\omega \in H^3(\Gamma, M)$ there exists a subgroup of Γ of p -power index U such that $res_{\Gamma, U}(\omega) = 0$ (cf. [12, proof of Thm.2.4]). The argument used to show the equivalences between A_n , B_n , C_n and D_n in [10, I.2.6, Ex.1] then yields that $\iota_p^k(M)$ are isomorphisms for all finite Γ -modules M which are finite-dimensional \mathbb{F}_p -vector space with trivial composition factors.

(b) From the hypothesis one concludes that $\hat{\Gamma}(p)$ has finite p -abelianizations, and thus by Theorem 2.2, $H^1(\hat{\Gamma}(p), \mathbb{F}_p[\llbracket \hat{\Gamma}(p) \rrbracket]) = 0$. Thus (a) yields the claim.

(c) Note that the hypothesis implies that the pro- p completion $\hat{U}(p)$ is infinite for every subgroup U of Γ of finite index. Hence (b) and Proposition 3.1 yields the claim. \square

Remark 3.4. The author just learnt that part (a) of Theorem 3.3 was also independently proved by D.H.Kochloukova and P.Zalesskii (cf. [6, Thm.C]).

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