P-PROJECTIVE GROUPS AND PRO-P TREES

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Abstract. Combining results of J-P.Serre and L.Ribes and P.Zalesskiĭ one concludes easiliy that a profinite group $\hat{G}$ acting without inversion of edges and with pro-p' stabilizers on a pro-p tree $\hat{\Gamma}$ must be projective (Thm. A). It is shown that a finitely generated p-projective virtual pro-p group $\hat{G}$ has such an action on a pro-p tree $\hat{\Gamma}$ (Thm.B). However, not every such profinite group $\hat{G}$ can act on a locally-finite pro-p tree co-compactly, without inversion of edges and with p' vertex stabilizers (Thm.D). This fact is deduced from the existence of p-irrational finitely generated p-projective virtual pro-p groups (Prop.C).

1. Introduction

From a theorem of A.Karrass, A.Pietrowski and D.Solitar (cf. [8]) and Bass-Serre theory (cf. [11, §5.1]) one obtains the following characterization of finitely generated virtually free groups: A discrete group $\tilde{F}$ is finitely generated and virtually free, if and only if $\tilde{F}$ has an action on a locally-finite tree Γ such that

(i) $\tilde{F}$ is acting without inversion of edges,
(ii) every vertex stabilizer $\tilde{F}_v$, $v \in \mathfrak{V}(\Gamma)$ is finite,
(iii) $\tilde{F}$ has finitely many orbits on the set of vertices $\mathfrak{V}(\Gamma)$.

A graph Γ is called *locally-finite*, if every vertex $v \in \mathfrak{V}(\Gamma)$ is the origin of only finitely many edges. Moreover, if the group $\tilde{F}$ is acting on the tree Γ such that (i), (ii) and (iii) are satisfied, one has also the following properties:

(iv) every edge stabilizer $\tilde{F}_{\mathbf{e}}$, $\mathbf{e} \in \mathfrak{E}(\hat{\Gamma})$, is finite,
(v) $\tilde{F}$ has finitely many orbits on the set of edges $\mathfrak{E}(\Gamma)$.

In this note we study p-projective groups and their action on pro-p trees. By a theorem of K.W.Gruenberg (cf. [7]), one knows that a profinite group $\hat{G}$ is p-projective, if and only if its cohomological p-dimension is less or equal to 1, i.e., $\mathrm{cd}_p(\hat{G}) \leq 1$. Thus $\hat{G}$ is p-projective, if and only if any pro-p Sylow subgroup $\hat{P} \in \mathrm{Syl}_p(\hat{G})$ is a free pro-p group (cf. [12, §I.3.3, Prop.14; §I.3.4]). From this fact and the characterization of pro-p groups acting regularly and without inversion of edges on pro-p trees obtained by L.Ribes and P.A.Zalesskiĭ (cf. [9, Cor.3.6]), one concludes the following:

Theorem A. *Let $\hat{G}$ be a profinite group acting on a pro-p tree $\hat{\Gamma}$ such that*

(i) *$\hat{G}$ is acting without inversion of edges,*
(ii) *every vertex stabilizer $\hat{G}_v$, $v \in \mathfrak{V}(\hat{\Gamma})$, is a pro-$p'$ group.*

Then $\hat{G}$ is p-projective.

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Theorem A raises the question whether it is true that every p-projective profinite group has an action on a pro-p tree satisfying (i) and (ii). We say that the action of the profinite group $\hat{G}$ on the profinite graph $\hat{\Gamma}$ has the *p-Bruhat-Tits property* (or for short the p-BT property), if

- (i) $\hat{G}$ acts without inversion of edges,
- (ii) every vertex stabilzer $\hat{G}_v$, $v \in \mathfrak{V}(\hat{\Gamma})$, is a pro-$p'$ group,
- (iii) every pro-p' subgroup $\hat{H}$ of $\hat{G}$ fixes a vertex.

In section 4 we establish the following theorem which answers this question for a particular type of p-projective profinite groups (cf. §4.8).

Theorem B. *Let $\hat{G}$ be a finitely generated p-projective virtual pro-p group. Then $\hat{G}$ has an action on a locally-finite pro-p tree $\hat{\Gamma}$ with the p-Bruhat-Tits property.*

Finitely generated p-projective virtual pro-p groups can be seen as the pro-p analogue of discrete finitely generated virtually free groups. Therefore, one might ask whether such a group $\hat{G}$ has also a *co-compact action* on a locally-finite pro-p tree $\hat{\Gamma}$ with the p-BT property, i.e., $\hat{G}$ has finitely many orbits on the vertex set $\mathfrak{V}(\hat{\Gamma})$. However, in general this is not possible.

In order to answer this question we develop in section 5 the theory of *Hattori-Stallings ranks* in analogy to the discrete case (cf. [2, Chap.IX.2]). As a consequence, for every virtual pro-p group $\hat{G}$ of type *p-FP* one can define a *p-Lefschetz number* $\mathcal{L}_p(\hat{G})$ which generalizes the p-Euler characteristic (cf. §5.5). Using the p-Lefschetz number, one can distinguish between *p-rational* and *p-irrational* virtual pro-p group of type *p-FP* (cf. (5.27)). Moreover, in case that $\hat{G}$ is a finitely generated p-projective virtual pro-p group, one can calculate the value of the p-Lefschetz number using the p-modular representation theory of the finite group $\hat{G}/\Phi_p(\hat{G})$, where $\Phi_p(\hat{G})$ denotes the pro-p Sylow subgroup of the Frattini subgroup of $\hat{G}$ (cf. Prop.5.5).

Let G be a finite group and let $\pi\colon Fr_p(G) \to G$ denote its universal p-Frattini cover. Then $Fr_p(G)$ is a finitely generated p-projective virtual pro-p group (cf. [5, §20.7]). E.g., for $G\colon = \mathrm{SL}_2(p)$ the description of the projective indecomposable $\mathbb{F}_p[G]$-modules given by R.Brauer and C.Nesbitt (cf. [1]) yields the following (cf. Prop.5.6).

Proposition C. *Let $p \geq 11$ and $p \neq 13$. Then the universal p-Frattini cover $Fr_p(G)$ of $G = \mathrm{SL}_2(p)$ is a finitely generated p-irrational p-projective virtual pro-p group.*

The p-irrationality of a finitely generated p-projective virtual pro-p group has the following consequence which gives a negative answer to the previously raised question (cf. Cor.5.4).

Theorem D. *Let $\hat{G}$ be a finitely generated p-projectve p-irrational virtual pro-p group. Assume that $\hat{G}$ is acting on the locally-finite pro-p tree $\hat{G}$ such that*

- (i) *$\hat{G}$ is acting without inversion of edges,*
- (ii) *every vertex stabiler $\hat{G}_v$, $v \in \mathfrak{V}(\hat{\Gamma})$ is a finite p'-group.*

Then $\hat{G}$ has infinitely many orbits on $\mathfrak{V}(\hat{\Gamma})$.

2. Pro-p trees

2.1. **Boolean spaces.** A *boolean (or profinite) space* X is a topological space which is Hausdorff, compact and totally disconnected. By **bool** we denote the category the objects of which are boolean spaces. For $X, Y \in ob(\mathbf{bool})$ the set of morphisms $\mathrm{mor}(X, Y)$ coincides with all continuous maps from X to Y.

Let $\mathbf{prf}_p$ denote the abelian category the objects of which are abelian pro-p groups. For $A, B \in ob(\mathbf{prf}_p)$ the set of morphisms $\mathbf{Hom}(A, B)$ is given by all continuous group homomorphisms from A to B. For our purpose the following elementary property will be useful.

Proposition 2.1. *For every boolean space X there exists a torsion free abelian pro-p group $\mathbb{Z}_p[\![X]\!]$ and a continuous injective map of boolean spaces $\iota_X \colon X \to \mathbb{Z}_p[\![X]\!]$ with the following property: For every continuous map $\phi \colon X \to A$ of X into the abelian pro-p group A there exists a unique continuous group homomorphism $\phi^\circ \colon \mathbb{Z}_p[\![X]\!] \to A$ making the diagram*

$$\begin{tikzcd}
X \arrow[r, "\iota_X"] \arrow[dr, "\phi"'] & \mathbb{Z}_p[\![X]\!] \arrow[d, "\phi^\circ"] \\
& A
\end{tikzcd} \tag{2.1}$$

*In other words, $\mathbb{Z}_p[\![\_]\!] \colon$ **bool** $\to \mathbf{prf}_p$ is the left adjoint of the forgetful functor* $\mathbf{for} \colon \mathbf{prf}_p \to \mathbf{bool}$.

Proof. Since X is a profinite space (cf. [10, Thm.1.1.12]), one has

$$X = \varprojlim_{i \in I} X_i \tag{2.2}$$

for some projective system of finite spaces $(X_i, \alpha_{i,j})_{i,j \in I}$. One checks easily that

$$\mathbb{Z}_p[\![X]\!] := \varprojlim_{i \in I} \mathbb{Z}_p[X_i], \tag{2.3}$$

where $\mathbb{Z}_p[X_i]$ is the free $\mathbb{Z}_p$-module generated bi X_i, has the required properties. □

From now on we will assume that the canonical map $\iota_X \colon X \to \mathbb{Z}_p[\![X]\!]$ is always given by inclusion. We list the fundamental properties of the functor $\mathbb{Z}_p[\![\_]\!]$ in the following proposition which easy proof is left to the reader.

Proposition 2.2. *Let $X, Y \in ob(\mathbf{bool})$ and let $\alpha \colon X \to Y$ be a continuous map.*

(a) $\mathbb{Z}_p[\![\emptyset]\!] = 0$.
(b) *If α is injective, $\mathbb{Z}_p[\![\alpha]\!] \colon \mathbb{Z}_p[\![X]\!] \to \mathbb{Z}_p[\![Y]\!]$ is injective.*
(c) *If α is surjective, $\mathbb{Z}_p[\![\alpha]\!] \colon \mathbb{Z}_p[\![X]\!] \to \mathbb{Z}_p[\![Y]\!]$ is surjective.*
(d)

$$\mathbb{Z}_p[\![X \sqcup Y]\!] = \mathbb{Z}_p[\![X]\!] \oplus \mathbb{Z}_p[\![Y]\!], \tag{2.4}$$

where $\sqcup$ denotes outer disjoint union.

(e) *Let $(X_i, \rho_{i,j})_{i \in I}$ be a directed system of boolean sets. Then*

$$\mathbb{Z}_p[\![\varprojlim_{i \in \mathbb{N}} X_i]\!] = \varprojlim_{i \in \mathbb{N}} \mathbb{Z}_p[\![X_i]\!] \tag{2.5}$$

2.2. **Profinite graphs.** A *profinite graph* $\hat{\Gamma}$ consists of a non-trivial boolean set of vertices $\mathfrak{V}(\hat{\Gamma})$, a boolean set of edges $\mathfrak{E}(\hat{\Gamma})$ and three continuous maps: A *terminus* $t\colon \mathfrak{E}(\hat{\Gamma}) \to \mathfrak{V}(\hat{\Gamma})$, an *origin* $o\colon \mathfrak{E}(\hat{\Gamma}) \to \mathfrak{V}(\hat{\Gamma})$ and an *inversion* $^{-}\colon \mathfrak{E}(\hat{\Gamma}) \to \mathfrak{E}(\hat{\Gamma})$. These maps have the following property: For every edge $\mathbf{e} \in \mathfrak{E}(\hat{\Gamma})$ one has

(i) $\bar{\mathbf{e}} \neq \mathbf{e}$ and $\bar{\bar{\mathbf{e}}} = \mathbf{e}$,
(ii) $t(\bar{\mathbf{e}}) = o(\mathbf{e})$, $o(\bar{\mathbf{e}}) = t(\mathbf{e})$.

If the map

$$(t, o)\colon \mathfrak{E}(\hat{\Gamma}) \to \mathfrak{V}(\hat{\Gamma}) \times \mathfrak{V}(\hat{\Gamma}) \tag{2.6}$$

is injective $\hat{\Gamma}$ is called *combinatorial.* If $t(\mathbf{e}) \neq o(\mathbf{e})$ for all $\mathbf{e} \in \mathfrak{E}(\hat{\Gamma})$, $\hat{\Gamma}$ will be called *loop-free.* All our definitions agree with the standard definitions translated in the category of boolean spaces (cf. [11, §2.1]).

2.3. **The $\mathbb{Z}_p$-vertex and $\mathbb{Z}_p$-edge group of a profinite graph.** We define the *$\mathbb{Z}_p$-vertex group* of the profinite graph $\hat{\Gamma}$ by

$$\mathbf{V}(\hat{\Gamma})\colon = \mathbb{Z}_p[\![\mathfrak{V}(\hat{\Gamma})]\!]. \tag{2.7}$$

Thus every vertex v can also be considered as an element in $\mathbf{V}(\hat{\Gamma})$. The *$\mathbb{Z}_p$-edge group* of the profinite graph $\hat{\Gamma}$ will be defined by

$$\mathbf{E}(\hat{\Gamma})\colon = \mathbb{Z}_p[\![\mathfrak{E}(\hat{\Gamma})]\!]/\langle\, \mathbf{e} + \bar{\mathbf{e}} \mid \mathbf{e} \in \mathfrak{E}(\hat{\Gamma})\,\rangle. \tag{2.8}$$

Here $\langle X\rangle$ denotes the closed subgroup generated by the set X. For $\mathbf{e} \in \mathfrak{E}(\hat{\Gamma})$ we denote by $\underline{\mathbf{e}}$ its canonical image in $\mathbf{E}(\hat{\Gamma})$. In particular, $\underline{\bar{\mathbf{e}}} = -\underline{\mathbf{e}}$.

The mappings $\partial\colon \mathbf{E}(\hat{\Gamma}) \to \mathbf{V}(\hat{\Gamma})$, $\partial(\underline{\mathbf{e}})\colon = t(\mathbf{e}) - o(\mathbf{e})$ for $\mathbf{e} \in \mathfrak{E}(\hat{\Gamma})$, and $\varepsilon\colon \mathbf{V}(\hat{\Gamma}) \to \mathbb{Z}_p$, $\varepsilon(v) = 1$ for $v \in \mathfrak{V}(\hat{\Gamma})$ define a chain complex $\mathbf{C}(\hat{\Gamma})$ of abelian pro-p groups

$$\mathbf{C}(\hat{\Gamma})\colon\quad 0 \longrightarrow \mathbf{E}(\hat{\Gamma}) \xrightarrow{\partial} \mathbf{V}(\hat{\Gamma}) \xrightarrow{\varepsilon} \mathbb{Z}_p \longrightarrow 0 \tag{2.9}$$

which we think is concentrated in degrees 1, 0 and -1.

One calls the profinite graph $\hat{\Gamma}$ *pro-p connected*, if $H_0(\mathbf{C}(\hat{\Gamma})) = 0$, and a *pro-$p$ tree*[1], if $H_0(\mathbf{C}(\hat{\Gamma})) = H_1(\mathbf{C}(\hat{\Gamma})) = 0$.

Remark 2.3. For a discrete graph Γ one can also associate a chain complex $C(|\Gamma|)$ of abelian groups, the singular chain complex of its geometric realization $|\Gamma|$. Hence, Γ is connected, if and only if $H_0(|\Gamma|, \mathbb{Z}) = H_0(C(|\Gamma|)) \simeq \mathbb{Z}$, and it is a tree, if and only if it is connected and

$$\pi_1(|\Gamma|, v)^{\mathrm{ab}} = H_1(|\Gamma|, \mathbb{Z}) = H_1(C(|\Gamma|)) = 0. \tag{2.10}$$

3. Profinite groups acting on profinite graphs

3.1. **Profinite $\hat{G}$-sets.** Let $\hat{G}$ be a profinite group. A boolean set X together with a continuous left $\hat{G}$-action will be called a *profinite $\hat{G}$-set.* Morphisms of profinite $\hat{G}$-sets are defined in the obvious way. It is an elementary exercise to verify that a profinite $\hat{G}$-set is indeed isomorphic to the inverse limit of finite discrete left $\hat{G}$-sets. By ${}_{\hat{G}}\mathbf{bool}$ we denote the category the objects of which are profinite left $\hat{G}$-sets. A profinite group $\hat{G}$ has an action on the profinite graph $\hat{\Gamma}$, if $\mathfrak{V}(\hat{\Gamma})$ and $\mathfrak{E}(\hat{\Gamma})$ are

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[1] Pro-p trees where introduced and studied by L.Ribes and P.A.Zalesskiĭ in [9]. However, the definition of a pro-p tree we are using in this paper is slightly more restrictive then their definition.

profinite $\hat{G}$-sets, and if $t\colon \mathfrak{E}(\hat{\Gamma}) \to \mathfrak{V}(\hat{\Gamma})$, $o\colon \mathfrak{E}(\hat{\Gamma}) \to \mathfrak{V}(\hat{\Gamma})$ and $^{-}\colon \mathfrak{E}(\hat{\Gamma}) \to \mathfrak{E}(\hat{\Gamma})$ are mappings of profinite $\hat{G}$-sets.

3.2. **Quotients of profinite graphs.** Let $\hat{N} \triangleleft \hat{G}$ be a closed normal subgroup and let X be a profinite left $\hat{G}$-set. The set of $\hat{N}$-orbits

$$X_{\hat{N}} \colon = \{\, \hat{N}.x \mid x \in X \,\} \tag{3.1}$$

carries canonically the structure of a profinite left $\hat{G}/\hat{N}$-set. Moreover,

$$\_|_{\hat{N}} \colon {}_{\hat{G}}\mathbf{bool} \longrightarrow {}_{\hat{G}/\hat{N}}\mathbf{bool} \tag{3.2}$$

is a covariant functor.

Let $\hat{G}$ be a profinite group acting on the profinite graph $\hat{\Gamma}$, and let $\hat{N}$ be a closed subgroup of $\hat{G}$ acting without inversion of edges on $\hat{\Gamma}$. Then $\hat{\Gamma}_{\hat{N}} \colon = (\mathfrak{V}(\hat{\Gamma})_{\hat{N}}, \mathfrak{E}(\hat{\Gamma})_{\hat{N}}, t, o, {}^{-})$, where t, o, $^{-}$ are the canonical maps, is a profinite graph. Moreover, in case that $\hat{N}$ is additionally a closed normal subgroup of $\hat{G}$, $\hat{G}/\hat{N}$ has a canonical left action on $\hat{\Gamma}_{\hat{N}}$.

3.3. **Profinite $\hat{G}$-sets and abelian pro-p groups with continuous left $\hat{G}$-action.** Let ${}_{\hat{G}}\mathbf{prf}_p$ denote the abelian category the objects of which are abelian pro-p groups with continuous left $\hat{G}$-action. Then ${}_{\hat{G}}\mathbf{prf}_p$ is an abelian category with enough projectives, and ${}_{\hat{G}}\mathbf{prf}_p$ is a subcategory of ${}_{\mathbb{Z}_p[\![\hat{G}]\!]}\,\mathrm{mod}$, the category of left $\mathbb{Z}_p[\![\hat{G}]\!]$-modules, where

$$\mathbb{Z}_p[\![\hat{G}]\!] \colon = \varprojlim_{\hat{U} \triangleleft_o \hat{G}} \mathbb{Z}_p[\hat{G}/\hat{U}] \tag{3.3}$$

denotes the *completed $\mathbb{Z}_p$-group algebra* of $\hat{G}$. Obviously, the inverse limit in (3.3) has to be taken over the directed system of all open normal subgroups of $\hat{G}$. For further details the reader may wish to consult [10, §5.1], [13, §3.2]. One has the following properties:

Proposition 3.1. *Let $\hat{G}$ be a profinite group.*

(a) $\mathbb{Z}_p[\![\_]\!] \colon {}_{\hat{G}}\mathbf{bool} \longrightarrow {}_{\hat{G}}\mathbf{prf}_p$ *is a covariant functor mapping finite disjoint unions to direct sums and commuting with inverse limits.*

(b) *If X is a profinite left $\hat{G}$-set such that*
   (i) *$\hat{G}$ has finitely many orbits on X,*
   (ii) *for every $x \in X$ the point stabilizer $\hat{G}_x$ is a pro-p' subgroup.*
   Then $\mathbb{Z}_p[\![X]\!] \in ob({}_{\hat{G}}\mathbf{prf}_p)$ is projective.

(c) *Let X be a profinite left $\hat{G}$-set such that $\hat{G}$ has finitely many orbits on X, and assume that $\hat{N}$ is a closed normal subgroup of $\hat{G}$. Then one has an isomorphism of topological left $\mathbb{Z}_p[\![\hat{G}/\hat{N}]\!]$-modules*

$$\mathbb{Z}_p[\![X|_{\hat{N}}]\!] \simeq \mathbb{Z}_p[\![\hat{G}/\hat{N}]\!] \hat{\otimes}_{\mathbb{Z}_p[\![\hat{G}]\!]} \mathbb{Z}_p[\![X]\!], \tag{3.4}$$

where $\hat{\otimes}_{\mathbb{Z}_p[\![\hat{G}]\!]}$ denotes the completed tensor product (cf. [3, §2.1]*).*

Proof. (a) is a direct consequence of Proposition 2.2. (b) Without loss of generality we may assume that $\hat{G}$ is transitive on X. For $x \in X$ one has isomorphisms

$$\mathbb{Z}_p[\![X]\!] \simeq \mathbb{Z}_p[\![\hat{G}/\hat{G}_x]\!] = \mathrm{ind}_{\hat{G}}^{\hat{G}_x}(\mathbb{Z}_p). \tag{3.5}$$

Since $\hat{G}_x$ is a profinite p'-group, $\mathbb{Z}_p \in ob({}_{\hat{G}_x}\mathbf{prf}_p)$ is projective. As $\mathrm{ind}_{\hat{G}}^{\hat{G}_x}(\_)$ is mapping projectives to projectives, this yields the claim. (c) follows by a similar argument. □

3.4. **P-perfect profinite groups.** Let p be a prime number. The profinite group $\hat{Y}$ is called *p-perfect*, if every finite homomorphic image of p-power order of $\hat{Y}$ is the trivial group. One has the following elementary property.

Proposition 3.2. *Every profinite group $\hat{G}$ has a maximal closed p-perfect characteristic subgroup*[2] *$O^p(\hat{G})$ which is given by*

$$O^p(\hat{G}) \colon = \mathrm{cl}(\langle\, g \in \hat{G} \mid p \nmid \mathrm{ord}(g)\,\rangle). \tag{3.6}$$

Proof. By definition, $O^p(O^p(\hat{G})) = O^p(\hat{G})$. Hence $O^p(\hat{G})$ is p-perfect. Since $\hat{G}/O^p(\hat{G})$ is a pro-p group, $O^p(\hat{G})$ is the maximal closed normal p-perfect subgroup of $\hat{G}$. By construction, it is characteristic. □

3.5. **Pro-p trees as quotients.** The following property will turn out to be useful.

Proposition 3.3. *Let $\hat{G}$ be a profinite group acting on the pro-p tree $\hat{\Gamma}$, and let $\hat{N} \lhd \hat{G}$ be a closed normal subgroup of $\hat{G}$ with the following properties:*

(i) *$\hat{\Gamma}$ is locally-finite,*
(ii) *$\hat{G}$ acts without inversion of edges,*
(iii) *every vertex stabilizer $\hat{G}_x$, $x \in \mathfrak{V}(\hat{\Gamma})$, is a pro-$p'$ group,*
(iv) *$\hat{G}$ has finitely many orbits on $\mathfrak{V}(\hat{\Gamma})$,*
(v) *$\hat{N}$ is p-perfect.*

Then

(I) *$\hat{\Gamma}_{\hat{N}}$ is a locally-finite pro-p tree,*
(II) *$\hat{G}/\hat{N}$ acts without inversion of edges,*
(III) *$\hat{G}/\hat{N}$ has finitely many orbits on $\mathfrak{V}(\hat{\Gamma}_{\hat{N}})$,*
(IV) *Let $\pi_{\mathfrak{V}} \colon \mathfrak{V}(\hat{\Gamma}) \to \mathfrak{V}(\hat{\Gamma}_{\hat{N}})$ denote the canonical projection of boolean sets. Let $y \in \mathfrak{V}(\hat{\Gamma}_{\hat{N}})$ and $x \in \mathfrak{V}(\hat{\Gamma})$ with $\pi_{\mathfrak{V}}(x) = y$. Then $(\hat{G}/\hat{N})_y = \hat{G}_x.\hat{N}$. In particular, $(\hat{G}/\hat{N})_y$ is a pro-p' group.*

Proof. The locally-finiteness of $\hat{\Gamma}_{\hat{N}}$, property (II) and (III) are immediate consequences of the hypothesis (i), (ii) and (iv). As $\hat{\Gamma}$ is a pro-p tree, Proposition 3.1(b) and hypothesis (iii) and (iv) imply that the chain complex $\mathbf{C}(\hat{\Gamma})$ (cf. (2.9)) is a projective resolution of the trivial left $\hat{G}$-module $\mathbb{Z}_p$ in ${}_{\hat{G}}\mathbf{prf}_p$. Since restriction maps projective objects in ${}_{\hat{G}}\mathbf{prf}_p$ to projective objects in ${}_{\hat{N}}\mathbf{prf}_p$, one concludes from Proposition 3.1(c) and the identity

$$\mathrm{res}_1^{\hat{G}/\hat{N}}(\mathbb{Z}_p[\![\hat{G}/\hat{N}]\!]\hat{\otimes}_{\mathbb{Z}_p[\![\hat{G}]\!]}\_) = \mathbb{Z}_p\hat{\otimes}_{\mathbb{Z}_p[\![\hat{N}]\!]}\,\mathrm{res}_{\hat{N}}^{\hat{G}}(\_), \tag{3.7}$$

that

(3.8)
$$\begin{aligned} H_0(\mathbf{C}(\hat{\Gamma}_{\hat{N}})) &= H_0(\mathbb{Z}_p[\![\hat{G}/\hat{N}]\!]\hat{\otimes}_{\mathbb{Z}_p[\![\hat{G}]\!]}\mathbf{C}(\hat{\Gamma})) &&= 0,\\ H_1(\mathbf{C}(\hat{\Gamma}_{\hat{N}})) &= H_1(\mathbb{Z}_p[\![\hat{G}/\hat{N}]\!]\hat{\otimes}_{\mathbb{Z}_p[\![\hat{G}]\!]}\mathbf{C}(\hat{\Gamma})) &&= \mathbf{H}_1(\hat{N},\mathbb{Z}_p) &&= (\hat{N}/O^p(\hat{N}))^{\mathrm{ab}}. \end{aligned}$$

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[2] This is the standard notation in the theory of finite groups (cf. [6, p.19]).

Since $\hat{N}$ is p-perfect, this implies that $\hat{\Gamma}_{\hat{N}}$ is a pro-p tree. The conclusion (IV) is obvious. $\square$

4. The universal p'-cover of a finite p-perfect group

In this section we assume that Y is a finite p-perfect group.

4.1. The p'-graph of groups of a finite p-perfect group. Let $\mathfrak{V}(\Xi)$ denote the set of all maximal p'-subgroups of the finite p-perfect group Y, and let Ξ denote the combinatorial loop-free complete graph over the set $\mathfrak{V}(\Xi)$, i.e., the combinatorial graph whose edge set is given by

$$\mathfrak{E}(\Xi)\colon = \big(\mathfrak{V}(\Xi)\times\mathfrak{V}(\Xi)\big)\setminus\Delta(\mathfrak{V}(\Xi)), \tag{4.1}$$

where $\Delta(\mathfrak{V}(\Xi))$ denotes the diagonal, and the origin, terminus and edge inversion mappings are the obvious ones. The assignment $\mathfrak{A}_A\colon = A$ for $A\in\mathfrak{V}(\Xi)$ and $\mathfrak{A}_{(A,B)}\colon = A\cap B$ for $(A,B)\in\mathfrak{E}(\Xi)$ defines a finite graph of finite groups $\mathfrak{A}$ on the graph Ξ, which will be called the *p'-graph of groups for Y*.

By construction, one has a canonical map

$$\beta\colon F(\mathfrak{A},\Xi)\longrightarrow Y, \tag{4.2}$$

where $F(\mathfrak{A},\Xi)$ is the group as defined in [11, §5.1], and $\beta(a)\colon = a$, $a\in\mathfrak{A}_A$, $A\in\mathfrak{V}(\Xi)$; $\beta(y_{\mathbf{e}})\colon = 1$ for $\mathbf{e}\in\mathfrak{E}(\Xi)$.

Let $T\subseteq\Xi$ be a maximal subtree of the finite graph Ξ, and let $\Pi\colon = \pi_1(\mathfrak{A},T)$ denote the fundamental group based at T, i.e., $\Pi = F(\mathfrak{A},\Xi)/N_T$ where N_T is the normal subgroup generated by $y_{\mathbf{e}}$, $\mathbf{e}\in\mathfrak{E}(T)$ (cf. [11, §5.1]). By construction, one has a canonical map $\beta_T\colon\Pi\to Y$. Moreover, since we assumed Y to be p-perfect, β_T is surjective.

4.2. The tree associated to $(\Xi,\mathfrak{A})$. Let $\mathfrak{E}^+(\Xi)\subseteq\mathfrak{E}(\Xi)$ be an orientation of Ξ. The universal cover Γ of $(\mathfrak{A},\Xi)$ with respect to $(T,\mathfrak{E}^+(\Xi))$ is a tree with a canonical Π-action (cf. [11, §I.5.3, Thm.12]). Since $(\Xi,\mathfrak{A})$ is a finite graph of groups, Γ is locally-finite. It comes equipped with two mappings

$$\begin{aligned}\sigma_{\mathfrak{V}}&\colon\mathfrak{V}(\Xi)\longrightarrow\mathfrak{V}(\Gamma),\\ \sigma_{\mathfrak{E}}&\colon\mathfrak{E}(\Xi)\longrightarrow\mathfrak{E}(\Gamma),\end{aligned} \tag{4.3}$$

whose restriction to the maximal subtree T is an immersion of graphs. The mapping $\sigma_{\mathfrak{E}}$ commutes with edge inversion, and for all $\mathbf{e}\in\mathfrak{E}^+(\Xi)\setminus\mathfrak{E}(T)$ one has $o(\sigma_{\mathfrak{E}}(\mathbf{e})) = \sigma_{\mathfrak{V}}(o(\mathbf{e}))$. Moreover, $\mathrm{im}(\sigma_{\mathfrak{V}})$ is a system of representatives of the orbits of Π on $\mathfrak{V}(\Gamma)$, and $\mathrm{im}(\sigma_{\mathfrak{E}})$ is a system of representatives of the orbits of Π on $\mathfrak{E}(\Gamma)$. In particular, for all $\mathbf{e}\in\mathfrak{E}^+(\Xi)\setminus\mathfrak{E}(T)$ there exists an element $\gamma_{\mathbf{e}}\in\Pi$ such that $t(\sigma_{\mathfrak{E}}(\mathbf{e})) = \gamma_{\mathbf{e}}.\sigma_{\mathfrak{V}}(t(\mathbf{e}))$. For $\mathbf{e}\in\mathfrak{E}^+(\Xi)\cap\mathfrak{E}(T)$ we put $\gamma_{\mathbf{e}}\colon = 1$. For the stabilizers one has the identities

$$\mathfrak{A}_v = \Pi_{\sigma_{\mathfrak{V}}(v)},\quad \mathfrak{A}_{\mathbf{e}} = \Pi_{\sigma_{\mathfrak{E}}(\mathbf{e})}, \tag{4.4}$$

$v\in\mathfrak{V}(\Xi)$, $\mathbf{e}\in\mathfrak{E}(\Xi)$.

4.3. **The $\mathcal{O}_p$-vertex and edge modules of Γ.** Let $\mathcal{O}_p\colon=(\mathbb{Z}\setminus p.\mathbb{Z})^{-1}.\mathbb{Z}$ denote the localization of $\mathbb{Z}$ at p, let

$$\mathbf{V}_{\mathcal{O}_p}(\Gamma)\colon=\mathcal{O}_p[\mathfrak{V}(\Gamma)]=\coprod_{v\in\mathfrak{V}(\Gamma)}\mathcal{O}_p.\langle v\rangle \tag{4.5}$$

denote the *$\mathcal{O}_p$-vertex module of* Γ, and let

$$\mathbf{E}_{\mathcal{O}_p}(\Gamma)\colon=\mathcal{O}_p[\mathfrak{E}(\Gamma)]/\langle\, \mathbf{e}+\bar{\mathbf{e}}\mid \mathbf{e}\in\mathfrak{V}(\Gamma)\,\rangle \tag{4.6}$$

denote the *$\mathcal{O}_p$-edge module of* Γ. Since Γ is a tree, one has a canonical exact sequence of left $\mathcal{O}_p[\Pi]$-modules

$$0\longrightarrow \mathbf{E}_{\mathcal{O}_p}(\Gamma)\xrightarrow{\ \partial\ }\mathbf{V}_{\mathcal{O}_p}(\Gamma)\xrightarrow{\ \varepsilon\ }\mathcal{O}_p\longrightarrow 0, \tag{4.7}$$

where $\partial(\mathbf{e})\colon=t(\mathbf{e})-o(\mathbf{e})$, $\mathbf{e}\in\mathfrak{E}(\Gamma)$, and $\varepsilon(v)=1$, $v\in\mathfrak{V}(\Gamma)$.

Let A be a finite p' subgroup of Π. The divisibility of $|A|$ in $\mathcal{O}_p$ implies that the $\mathcal{O}_p$-permutation module $\mathcal{O}_p[\Pi/A]$ is a projective $\mathcal{O}_p[\Pi]$-module. The exactness of (4.7) implies that

$$\coprod_{\mathbf{e}\in\mathfrak{E}^+(\Xi)}\mathcal{O}_p[\Pi/\Pi_{\sigma_{\mathfrak{E}}(\mathbf{e})}]\langle\mathbf{e}\rangle\xrightarrow{\ \partial\ }\coprod_{v\in\mathfrak{V}(\Xi)}\mathcal{O}_p[\Pi/\Pi_{\sigma_{\mathfrak{V}}(v)}]\langle v\rangle\xrightarrow{\ \varepsilon\ }\mathcal{O}_p \tag{4.8}$$

$\partial(\mathbf{e})\colon=\gamma_{\mathbf{e}}.\Pi_{\sigma_{\mathfrak{V}}(t(\mathbf{e}))}\langle t(\mathbf{e})\rangle-\Pi_{\sigma_{\mathfrak{V}}(o(\mathbf{e}))}\langle o(\mathbf{e})\rangle$, is a projective resolution of the trivial $\mathcal{O}_p[\Pi]$-module $\mathcal{O}_p$. In particular, $\mathrm{cd}_{\mathcal{O}_p}(\Pi)\le 1$.

4.4. **The profinite completion of Π.** As Π is residually finite, the canonical map $\iota\colon\Pi\to\hat{\Pi}$ is an injection. For $x\in\mathfrak{V}(\Gamma)\cup\mathfrak{E}(\Gamma)$ we put $\hat{\Pi}_x\colon=\iota(\Pi_x)$. Let

$$\begin{aligned}\mathfrak{V}(\hat{\Gamma})&\colon=\bigsqcup_{v\in\mathfrak{V}(\Xi)}\hat{\Pi}/\hat{\Pi}_{\sigma_{\mathfrak{V}}(v)}.v\qquad\text{and}\\ \mathfrak{E}(\hat{\Gamma})&\colon=\bigsqcup_{\mathbf{e}\in\mathfrak{E}^+(\Xi)}\hat{\Pi}/\hat{\Pi}_{\sigma_{\mathfrak{E}}(\mathbf{e})}.\mathbf{e}\ \sqcup\ \bigsqcup_{\mathbf{e}\in\mathfrak{E}^+(\Xi)}\hat{\Pi}/\hat{\Pi}_{\sigma_{\mathfrak{E}}(\mathbf{e})}.\bar{\mathbf{e}},\end{aligned} \tag{4.9}$$

where $\sqcup$ denotes "outer disjoint union". Since $\mathfrak{V}(\Xi)$ and $\mathfrak{E}^+(\Xi)$ are finite sets, $\mathfrak{V}(\hat{\Gamma})$ and $\mathfrak{E}(\hat{\Gamma})$ are boolean sets. There is an edge inversion map $\bar{\ }\colon\mathfrak{E}(\hat{\Gamma})\to\mathfrak{E}(\hat{\Gamma})$, which is defined in the obvious way. Moreover, the assignment

$$\begin{aligned}o(g.\hat{\Pi}_{\sigma_{\mathfrak{E}}(\mathbf{e})}.\mathbf{e})&\colon=g.\hat{\Pi}_{\sigma_{\mathfrak{V}}(o(\mathbf{e}))}.o(\mathbf{e}),\\ t(g.\hat{\Pi}_{\sigma_{\mathfrak{E}}(\mathbf{e})}.\mathbf{e})&\colon=\begin{cases}g.\hat{\Pi}_{\sigma_{\mathfrak{V}}(t(\mathbf{e}))}.t(\mathbf{e}) & \text{for } \mathbf{e}\in\mathfrak{E}(T),\\ g\gamma_{\mathbf{e}}.\hat{\Pi}_{\sigma_{\mathfrak{V}}(t(\mathbf{e}))}.t(\mathbf{e}) & \text{for } \mathbf{e}\notin\mathfrak{E}(T),\end{cases}\end{aligned} \tag{4.10}$$

$\mathbf{e}\in\mathfrak{E}^+(\Xi)$, $g\in\hat{\Pi}$, makes $\hat{\Gamma}$ a profinite graph with a continuous left $\hat{\Pi}$-action.

4.5. **Good groups of type FP_∞.** The group Π is virtually free, and therefore "p-good" (cf. [12, §I.2.6, Ex.2]), i.e., for every finite $\mathcal{O}_p[\Pi]$-module M one has natural isomorphisms

$$\iota^k\colon H^k(\hat{\Pi},M)\longrightarrow H^k(\Pi,M), \tag{4.11}$$

for all $k\in\mathbb{N}_0$. The natural isomorphisms $\iota^\bullet$ are induced by the mapping of $\mathcal{O}_p$-algebras

$$\iota\colon\mathcal{O}_p[\Pi]\longrightarrow\mathbb{Z}_p[\![\hat{\Pi}]\!]. \tag{4.12}$$

One has the following.

Proposition 4.1. *Let G be a discrete group which is p-good, and let $(P_\bullet, \partial_\bullet, \varepsilon)$ be a projective resolution of the trivial left $\mathcal{O}_p[G]$-module $\mathcal{O}_p$ such that P_k is a finitely generated left $\mathcal{O}_p[G]$-module for all $k \geq 0$. Then $(\mathbb{Z}_p[\![\hat{G}]\!] \otimes_{\mathcal{O}_p[G]} P_\bullet, \partial_\bullet^\otimes, \varepsilon^\otimes)$ is a projective resolution of $\mathbb{Z}_p$ in ${}_{\hat{G}}\mathbf{prf}_p$.*

Proof. By hypothesis, the augmented $\mathcal{O}_p$-algebra $\mathcal{O}_p[G]$ is of type FP_∞. Hence $H^k(G, \_)$ commutes with direct limits for all k and for all direct systems of $\mathcal{O}_p[G]$-modules. The functors $H^k(\hat{G}, \_)$ commutes with direct limits for all $k \geq 0$ (cf. [12, §I.2]).

Let $\mathbb{I}_p \colon = \mathbb{Q}_p/\mathbb{Z}_p$ and let $J \colon = \operatorname{coind}^1_{\hat{G}}(\mathbb{I}_p)$ denote the discrete coinduced Galois module[3]. Then J is an injective $\hat{G}$-Galois module and a discrete topological $\mathbb{Z}_p[\![\hat{G}]\!]$-module. In particular, $H^k(\hat{G}, J) = 0$ for all $k > 0$. Thus the previously mentioned argument implies that $H^k(G, J) = 0$.

The functor $\mathfrak{F} \colon = \mathbb{Z}_p[\![\hat{G}]\!] \otimes_{\mathcal{O}[G]} \_$ is mapping finitely generated projective left $\mathcal{O}[G]$-modules to finitely generated projective left $\mathbb{Z}_p[\![\hat{G}]\!]$-modules. Moreover, as mappings of finitely generated left $\mathbb{Z}_p[\![\hat{G}]\!]$-modules are continuous,

$$Q_k \colon = H_k(\mathbb{Z}_p[\![\hat{G}]\!] \otimes_{\mathcal{O}_p[G]} P_\bullet, \partial_\bullet^\otimes) \in ob({}_{\hat{G}}\mathbf{prf}_p). \tag{4.13}$$

The functor $\mathfrak{J} \colon = \mathbf{Hom}_{\hat{G}}(\_, J) \colon {}_{\hat{G}}\mathbf{prf}_p^{\mathrm{op}} \to \mathbf{ab}$, where $\mathbf{ab}$ denotes the abelian category of abelian groups and

$$\mathbf{Hom}_{\hat{G}}(\_{}_1, \_{}_2) \colon {}_{\hat{G}}\mathbf{prf}_p^{\mathrm{op}} \times {}_{\hat{G}}\mathbf{dis}_p \longrightarrow \mathbf{ab} \tag{4.14}$$

denotes the Hom-functor defined by A.Brumer (cf. [3, §2.1]), is an additive contravariant exact and faithful functor, i.e., $\mathfrak{J}(M) = 0$ implies $M = 0$. Hence for $k \geq 1$

$$0 = H^k(G, J) = \mathbf{Hom}_{\mathbb{Z}_p[\![\hat{G}]\!]}(Q_k, J) \tag{4.15}$$

and this yields the claim. □

Proposition 4.1 has the following consequence.

Proposition 4.2. *The profinite graph $\hat{\Gamma}$ as defined in* (4.9) *and* (4.10) *is a pro-p tree.*

Proof. For every finite p'-subgroup A of Π, one has

$$\mathbb{Z}_p[\![\hat{\Pi}]\!] \otimes_{\mathcal{O}_p[\Pi]} \mathcal{O}_p[\hat{\Gamma}/A] = \mathbb{Z}_p[\![\hat{\Pi}/\iota(A)]\!]. \tag{4.16}$$

Moreover, Π is p-good and of type FP_∞. Hence by Proposition 4.1, $\mathbb{Z}_p[\![\hat{\Pi}]\!] \otimes_{\mathcal{O}_p[\Pi]} \_$ applied to (4.7) is also exact. This shows that $\hat{\Gamma}$ is a pro-p tree (cf. §2.3). □

4.6. **The universal p'-cover of a finite p-perfect group.** In the previous subsection we have seen that the profinite group $\hat{\Pi}$ has a canonical action on a pro-p tree $\hat{\Gamma}$ with the following properties:

(i) $\hat{\Gamma}$ is locally-finite,
(ii) $\hat{\Pi}$ is acting without inversion of edges,
(iii) every vertex stabilizer $\hat{\Pi}_x$, $x \in \mathfrak{V}(\hat{\Gamma})$, is a finite p'-group.
(iv) $\hat{\Pi}$ has finitely many orbits on $\mathfrak{V}(\hat{\Gamma})$.

---

[3] In [12, §I.2.5] these modules are called induced modules.

In section 4.1 we have also seen that one has a canonical surjective map

$$\hat{\beta}_T \colon \hat{\Pi} \longrightarrow Y. \tag{4.17}$$

We define the *universal p'-cover* $\hat{Y}^p$ of the p-perfect finite group Y to be

$$\hat{Y}^p \colon = \hat{\Pi}/O^p(\ker(\hat{\beta}_T)). \tag{4.18}$$

From Proposition 3.3 one concludes the following:

Theorem 4.3. *Let Y be a finite p-perfect group, and let $\hat{\beta}_Y \colon \hat{Y}^p \to Y$ be its universal p'-cover. Then $\hat{Y}^p$ has a canonical action on a locally-finite pro-p tree $\hat{\Gamma}_Y$ satisfying the p-Bruhat-Tits property.*

Proof. By Proposition 3.3, the profinite graph $\hat{\Gamma}_Y \colon = \hat{\Gamma}_{O^p(\ker(\hat{\beta}_T))}$ is a locally-finite pro-p tree, and the action of $\hat{Y}^p$ on $\hat{\Gamma}_Y$ satisfies (i) and (ii) of the definition of the p-Bruhat-Tits property (cf. §1). Let $\pi \colon \Pi \to \hat{Y}^p$ denote the canonical map. By construction and the Schur-Zassenhaus theorem, every pro-p' subgroup of $\hat{Y}^p$ is finite and $\hat{Y}^p$-conjugate to a subgroup of $\pi(\mathfrak{A}_v)$ for some $v \in \mathfrak{V}(\Gamma)$. Thus the action of $\hat{Y}^p$ on $\hat{\Gamma}_Y$ also satisfies (iii). □

4.7. **The universal p'-cover and universal p-Frattini covers.** In order to complete the proof of Theorem B, one has to show that every finitely generated p-projective virtual pro-p group $\hat{G}$ can be embedded in the universal p'-cover $\hat{Y}^p$ of a finite p-perfect group Y. This goal will be achieved in the following proposition.

Proposition 4.4. (a) *Let $\hat{G}$ be a finitely generated p-projective virtual pro-p group. Then the pro-p Sylow subgroup $\Phi_p(\hat{G})$ of the Frattini subgroup of $\hat{G}$ is an open normal subgroup of $\hat{G}$. Moreover, $\hat{G}$ is (non-canonically) isomorphic to the universal p-Frattini cover $Fr_p(\hat{G}/\Phi_p(\hat{G}))$ of $\hat{G}/\Phi_p(\hat{G})$.*

(b) *Let H be a subgroup of the finite group G. Then the universal p-Frattini cover $Fr_p(H)$ of H is (non-canonically) isomorphic to a closed subgroup of the universal p-Frattini cover $Fr_p(G)$ of G.*

(c) *Let Y be a finite p-perfect group. Then the universal p-Frattini cover $Fr_p(Y)$ is (non-canonically) isomorphic to a closed subgroup of the universal p'-cover $\hat{Y}^p$.*

Proof. (a) By hypothesis, $O_p(\hat{G})$, the largest closed normal pro-p subgroup of $\hat{G}$ is open in $\hat{G}$, and thus finitely generated. As $\Phi_p(O_p(\hat{G})) \leq \Phi_p(\hat{G})$, $\Phi_p(\hat{G})$ is open. The second part follows from [5, Cor.20.32].

(b) is a direct consequence of the fact that the universal p-Frattini cover coincides with the minimal p-projective cover (cf. [5, Prop.20.33]).

(c) Since $\ker(\hat{\beta}_Y)$ is a pro-p group, the embedding problem

$$\begin{tikzcd} & Fr_p(Y) \arrow[dl, dashed, "\tilde{\pi}"'] \arrow[d, "\pi"] \\ \hat{Y}^p \arrow[r, "\hat{\beta}_Y"] & Y \end{tikzcd} \tag{4.19}$$

has a solution $\tilde{\pi} \colon Fr_p(Y) \to \hat{Y}^p$. Hence $\hat{X} \colon = \operatorname{im}(\tilde{\pi}) \leq \hat{Y}^p$ is p-projective, and $\hat{\beta}_Y|_{\hat{X}} \colon \hat{X} \to Y$ is a p-projective cover of Y. As $\pi \colon Fr_p(Y) \to Y$ is the minimal p-projective cover, $\tilde{\pi}$ must be injective, and this yields the claim. □

4.8. **Proof of Theorem B.**

Proof of Theorem B. Let $\hat{G}$ be a finitely generated p-projective virtual pro-p group. Then $\hat{G}/\Phi_p(\hat{G})$ is finite (cf. Prop.4.4(a)) and thus - by Cayley's theorem - isomorphic to a subgroup of an alternating group A_n, $n \geq 5$. In particular, $\hat{G}$ is isomorphic to a closed subgroup of $Fr_p(A_n)$ (cf. Prop.4.4(b)), and also to a closed subgroup of $\hat{A}_n^p$ (cf. Prop.4.4(c)). Hence Theorem 4.3 yields the claim. □

Remark 4.5. Not every action of a finitely generated p-projective virtual pro-p group $\hat{G}$ on a locally-finite pro-p tree $\hat{\Gamma}$ satisfies neessarily the p-Bruhat-Tits property. Let $\hat{G} \colon = C_r \times \mathbb{Z}_p$ where C_r denotes a cyclic group of order r co-prime to p, $r \geq 3$. Then $\hat{G}$ acts without inversion of edges and vertex transitively on a circuit $\mathfrak{C}_{rp^k}$, $\mathfrak{V}(\mathfrak{C}_{rp^k}) = rp^k$, $k \geq 1$, and thus also on the profinite graph $\mathfrak{C} \colon = \varprojlim_{k \in \mathbb{N}_0} \mathfrak{C}_{rp^k}$. An easy calculation shows that $\mathfrak{C}$ is a pro-p tree, and the vertex stabilizers $\hat{G}_x$, $x \in \mathfrak{V}(\mathfrak{C})$, are all trivial. Hence the action of $\hat{G}$ on $\mathfrak{C}$ does not satisfy the p-Bruhat-Tits property.

5. The p-Lefschetz number for virtual pro-p groups of type p-FP

In this section we assume that R is a commutative ring with 1.

5.1. **Traces.** Let A be an associative R-algebra, and let $Q \in ob({}_A\,\mathrm{mod})$ be a finitely generated projective left A-module. Then $Q^* \colon = \mathrm{Hom}_A(Q, A)$ is a finitely generated projective right A-module. Let $T(A) \colon = A/[A, A]$, where $[A, A]$ is the R-module spanned by all commutators $ab - ba$, $a, b \in A$. One has an evaluation map

$$\begin{gathered} \mathrm{ev}_Q \colon Q^* \otimes_A Q \longrightarrow T(A), \\ \mathrm{ev}_Q(q^* \otimes q) \colon = q^*(q) + [A, A], \end{gathered} \tag{5.1}$$

where $q^* \in Q^*$, $q \in Q$. Let $c_Q \colon Q^* \otimes_A Q \to \mathrm{Hom}_A(Q, Q)$ denote the canonical isomorphism (cf. [2, §I.8 Prop.3]). The *trace on* Q is the map

$$\mathrm{tr}_Q \colon = \mathrm{ev}_Q \circ c_Q^{-1} \colon \mathrm{Hom}_A(Q, Q) \longrightarrow T(A) \tag{5.2}$$

(cf. [2, §IX.2, Ex.1]). The *Hattori-Stallings rank* $R_A(Q)$ of the projective left A-module Q is defined by

$$R_A(Q) \colon = \mathrm{tr}_Q(\mathrm{id}_Q). \tag{5.3}$$

Hence, by definition R_A is additive, i.e., one has

$$R_A(Q \oplus Q') = R_A(Q) + R_A(Q'), \tag{5.4}$$

and $R_A(A) = 1_A + [A, A]$ (cf. [2, §IX]).

5.2. **Hattori-Stallings ranks for projective modules of finite groups.** Let G be a finite group, and let $A \colon = \mathbb{Z}_p[G]$ denote its $\mathbb{Z}_p$-group algebra. Then one can identify $T(\mathbb{Z}_p[G])$ with the free $\mathbb{Z}_p$-module generated by the set $\{\, [g] \mid g \in G \,\}$ of G-conjugacy classes, i.e., if $\mathfrak{R} \subseteq G$ is a system of representatives of the G-conjugacy classes, one has

$$T(\mathbb{Z}_p[G]) = \coprod_{\gamma \in \mathfrak{R}} \mathbb{Z}_p.[\gamma]. \tag{5.5}$$

Tensoring by an extension field K of $\mathbb{Q}_p$ has no effect on the Hattori-Stallings rank, i.e., one has

$$R_G(Q) \colon = R_{K[G]}(K \otimes_{\mathbb{Z}_p} Q) = R_{\mathbb{Z}_p[G]}(Q). \tag{5.6}$$

In particular, if $\chi_Q \colon G^G \to \mathbb{Z}_p$ denotes the character associated to Q, one has

$$R_G(Q) = \frac{1}{|G|} \sum_{g \in G} \chi_Q(g^{-1}) \cdot [g] = \sum_{\gamma \in \mathfrak{R}} \frac{\chi_Q(\gamma^{-1})}{|C_G(\gamma)|} \cdot [\gamma] \tag{5.7}$$

(cf. [2, §IX.4, Ex.3]). Moreover, $\chi_Q(g)$ vanishes for every group element g which order is divisible by p (cf. [4, Thm.18.26]). Let $\mathfrak{R}_{p'}$ be a system of representatives for the p-regular[4] G-conjugacy classes of G. Hence (5.7) implies that

$$R_G(Q) \in T_{p'}(G) \colon = \coprod_{\gamma \in \mathfrak{R}_{p'}} \mathbb{Z}_p.[\gamma]. \tag{5.8}$$

For short we define $t_{[\gamma]}(Q) \in \mathbb{Z}_p$, $\gamma \in \mathfrak{R}_{p'}$, by

$$R_G(Q) = \sum_{\gamma \in \mathfrak{R}_{p'}} t_{[\gamma]}(Q).[\gamma]. \tag{5.9}$$

Since every character value is an algebraic integer, one has

$$t_{[\gamma]}(Q) \in \mathrm{int}_{\mathbb{Q}_p}(\mathbb{Z}), \tag{5.10}$$

where $\mathrm{int}_{\mathbb{Q}_p}(\mathbb{Z})$ denotes the integral closure of $\mathbb{Z}$ in $\mathbb{Q}_p$. Let $\mathfrak{S}_p(G)$ denote the set of isomorphism types of irreducible left $\mathbb{F}_p[G]$-modules. For every $[S] \in \mathfrak{S}_p(G)$ there exists a unique projective indecomposable left $\mathbb{Z}_p[G]$-module Q_S which head is isomorphic to S. Moreover,

$$\dim_{\mathbb{Q}_p}(\mathbb{Q}_p \otimes_{\mathbb{Z}_p} Q_S) = c_S \cdot |P|, \tag{5.11}$$

where $P \in \mathrm{Syl}_p(G)$ is a p-Sylow subgroup of G and c_S is a positive integer (cf. [14]). In particular, if we put $|G|_{p'} \colon = |G|/|P|$, we obtain from (5.7) the formula

$$t_{[1]}(Q_S) = \frac{c_S}{|G|_{p'}}. \tag{5.12}$$

Let $A \leq G$ be a subgroup of G of p'-order. Then the permutation module $\mathbb{Z}_p[G/A] = \mathrm{ind}_G^A(\mathbb{Z}_p)$ is a projective left $\mathbb{Z}[G]$-module. Moreover, from an elementary calculation using (5.7) or from an alternative description of the Hattori-Stallings rank, one concludes easily that

$$R_G(\mathbb{Z}_p[G/A]) = \frac{1}{|A|} \sum_{a \in A} [a]. \tag{5.13}$$

5.3. **Deflation.** Let $\pi \colon A \to B$ be a surjective morphism of associative R-algebras, and let $\bar{\pi} \colon T(A) \to T(B)$ denote the induced surjective map of R-modules. Then

$$\pi\_ \colon = B \otimes_A \_ \colon {}_A\,\mathrm{mod} \longrightarrow {}_B\,\mathrm{mod}, \tag{5.14}$$

is called the *deflation functor*. It is an additive right-exact functor mapping projectives to projectives, and it is left-adjoint to the inflation functor

$$\mathrm{inf}_A^B \colon {}_B\,\mathrm{mod} \to {}_A\,\mathrm{mod}\,. \tag{5.15}$$

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[4] An element $g \in G$ is called *p-regular*, if its order is co-prime to p.

By $\_{}_{\pi}\colon = \_ \otimes_A B\colon \mathrm{mod}_A \to \mathrm{mod}_B$ we denote the deflation functor on right modules. For a finitely generated projective left A-module $Q \in ob({}_A\mathrm{mod})$ one has canonical isomorphisms

$$(Q^*)_\pi = \mathrm{Hom}_A(Q,A) \otimes_A B \simeq \mathrm{Hom}_A(Q, \inf_A^B(B)) \simeq \mathrm{Hom}_B({}_\pi Q, B) = ({}_\pi Q)^*. \tag{5.16}$$

In particular, one has a commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}_A(Q,Q) \arrow[r, "c^{-1}"] \arrow[d, "\pi_*"'] & Q^* \otimes_A Q \arrow[r, "\mathrm{ev}"] \arrow[d] & T(A) \arrow[d, "\bar{\pi}"] \\
\mathrm{Hom}_B({}_\pi Q, {}_\pi Q) \arrow[r, "c^{-1}"] & Q_\pi \otimes_B {}_\pi Q \arrow[r, "\mathrm{ev}"] & T(B)
\end{tikzcd} \tag{5.17}$$

and $\pi_*(\mathrm{id}_Q) = \mathrm{id}_{{}_\pi Q}$. This shows that $R_B({}_\pi Q) = \bar{\pi}(R_A(Q))$.

Let $\pi_\circ\colon \tilde{G} \to G$ be a surjective homomorphism of finite groups such that $\ker(\pi_\circ)$ is a p-group, and let $\pi\colon \mathbb{Z}_p[\tilde{G}] \to \mathbb{Z}_p[G]$ denote the induced surjective homomorphism of $\mathbb{Z}_p$-algebras. Then $\bar{\pi}$ induces an isomorphism $\bar{\pi}_*\colon T_{p'}(\tilde{G}) \to T_{p'}(G)$. In particular, from our previous discussion one concludes the following fact.

Lemma 5.1. *Let $\pi_\circ\colon \tilde{G} \to G$ be a surjective homomorphism of finite groups such that* $\ker(\pi_\circ)$ *is a p-group, and let $\pi\colon \mathbb{Z}_p[\tilde{G}] \to \mathbb{Z}_p[G]$ denote the induced surjective homomorphism of $\mathbb{Z}_p$-algebras. Let Q be a finitely generated projective left $\mathbb{Z}_p[\tilde{G}]$-module. Then for the associated characters χ_Q and $\chi_{{}_\pi Q}$ one has*

$$\chi_Q(g) = \frac{|C_{\tilde{G}}(g)|}{|C_G(\pi_\circ(g))|} \cdot \chi_{{}_\pi Q}(\pi_\circ(g)) \tag{5.18}$$

for all p-regular elements $g \in \tilde{G}$.

5.4. **Virtual pro-p groups.** Let $\hat{G}$ be a virtual pro-p group, and let $\mathbb{Z}_p[\![\hat{G}]\!]$ denote its completed $\mathbb{Z}_p$-group algebra. We put

$$\mathbf{T}(\mathbb{Z}_p[\![\hat{G}]\!])\colon = \mathbb{Z}_p[\![\hat{G}]\!] / \mathrm{cl}([\mathbb{Z}_p[\![\hat{G}]\!], \mathbb{Z}_p[\![\hat{G}]\!]]), \tag{5.19}$$

where $\mathrm{cl}(\_)$ denotes the topological closure operation. In particular, one has a canonical map $\tau\colon T(\mathbb{Z}_p[\![\hat{G}]\!]) \to \mathbf{T}(\mathbb{Z}_p[\![\hat{G}]\!])$, and a canonical isomorphism

$$\mathbf{T}(\mathbb{Z}_p[\![\hat{G}]\!]) = \varprojlim_{U \triangleleft \hat{G}} T(\mathbb{Z}_p[\hat{G}/U]) \tag{5.20}$$

Let Q be a finitely generated left $\mathbb{Z}_p[\![\hat{G}]\!]$-module. We define the *profinite Hattori-Stallings rank* of Q by

$$R_{\hat{G}}(Q) = \tau(\mathrm{tr}_Q(\mathrm{id}_Q)) \in \mathbf{T}(\mathbb{Z}_p[\![\hat{G}]\!]) \tag{5.21}$$

where tr_Q is the trace as described in subsection §5.1. One has the following:

Proposition 5.2. *Let $\hat{G}$ be a finitely generated virtual pro-p group, and let Q be a finitely generated projective left $\mathbb{Z}_p[\![\hat{G}]\!]$-module. Let $\pi_\circ\colon \hat{G} \to G$ be a surjective continuous morphism on the finite group G with* $\ker(\pi)$ *a pro-p group. Then*

$$\bar{\pi}(R_{\hat{G}}(Q)) = R_G({}_\pi Q), \tag{5.22}$$

where $\pi\colon \mathbb{Z}_p[\![\hat{G}]\!] \to \mathbb{Z}_p[G]$ *and* $\bar{\pi}\colon \mathbf{T}(\hat{G}) \to T(G)$ *denote the canonical maps. In particular,*

$$R_{\hat{G}}(Q) \in \mathbf{T}_{p'}(\hat{G})\colon = \coprod_{\gamma\in\mathfrak{R}_{p'}} \mathbb{Z}_p.[\gamma], \tag{5.23}$$

where $\mathfrak{R}_{p'} \subseteq \hat{G}$ *is a system of representative of the* $\hat{G}$*-conjugacy classes of elements of* p'*-order.*

Proof. Let $\mathfrak{U}$ be a base of neighbourhoods of $1 \in \hat{G}$, such that every $U \in \mathfrak{U}$ is an open normal pro-p subgroup of $\hat{G}$, and let $\pi_\circ(U)\colon \hat{G} \to \hat{G}/U$ denote the canonical map. By (5.20) and (5.17), one has $\bar{\pi}(R_{\hat{G}}(Q)) = R_{\hat{G}/U}({}_{\pi_\circ(U)}Q)$. Hence, (5.8) yields (5.23). The identity (5.22) follows from the fact that for $U, V \in \mathfrak{U}$, $V \leq U$,

$$\bar{\pi}_*(V,U)\colon T_{p'}(\hat{G}/V) \longrightarrow T_{p'}(\hat{G}/U) \tag{5.24}$$

is an isomorphism (cf. §5.3). □

5.5. **The p-Lefschetz number for virtual pro-p group of type p-FP.** A profinite group $\hat{G}$ is called of *type p-FP*, if the trivial $\hat{G}$-module $\mathbb{Z}_p$ has a finite and finitely generated projective resolution $(Q_\bullet, \partial^Q_\bullet, \varepsilon)$ in ${}_{\hat{G}}\mathbf{prf}_p$ (cf. [13, §4.2]). For such a group we define the *p-Lefschetz number* by

$$\mathcal{L}_p(\hat{G})\colon = \sum_{k\in\mathbb{N}_0} (-1)^k.R_{\hat{G}}(Q_k) \in \mathbf{T}_{p'}(\hat{G}). \tag{5.25}$$

Since $(Q_\bullet, \partial^Q_\bullet, \varepsilon)$ is supposed to be concentrated in finitely many degrees, almost all coefficients $R_{\hat{G}}(Q_k)$ are 0. The additivity of $R_{\hat{G}}$ implies that the p-Lefschetz number $\mathcal{L}_p(\hat{G})$ is independent of the choice of projective resolution $(Q_\bullet, \partial^Q_\bullet, \varepsilon)$.

A p-projective virtual pro-p group is of type p-FP, if and only if it is finitely generated (cf. [12, §I.4.2]). Let

$$\mathcal{L}_p(\hat{G}) = \sum_{\gamma\in\mathfrak{R}} t_{[\gamma]}(\mathbb{Z}_p).[\gamma] \in \mathbf{T}_{p'}(\hat{G}) = \sum_{\gamma\in\mathfrak{R}} \mathbb{Z}_p.[\gamma], \tag{5.26}$$

where $\mathfrak{R}_{p'} \subseteq \hat{G}$ is a system of representatives of the $\hat{G}$-conjugacy classes of elements of p'-order. Then the coefficient $t_{[1]}(\mathbb{Z}_p)$ coincides the *p-Euler characteristic* of $\hat{G}$ (cf. [12, §I.4.1]).

As before let $\mathcal{O}_p = (\mathbb{Z} \setminus p.\mathbb{Z})^{-1}\mathbb{Z} \leq \mathbb{Q}$ denote the localization of $\mathbb{Z}$ at the prime ideal $p.\mathbb{Z}$. Then $\mathcal{O}_p = \mathbb{Q} \cap \mathbb{Z}_p \leq \mathbb{Q}_p$. We call the virtual pro-p group $\hat{G}$ of type p-FP *p-rational*, if

$$\mathcal{L}_p(\hat{G}) \in \sum_{\gamma\in\mathfrak{R}_{p'}} \mathcal{O}_p.[\gamma]. \tag{5.27}$$

If $\hat{G}$ is not p-rational, it will be called *p-irrational.* From (5.10) and Proposition 5.2 one concludes that for all virtual pro-p groups $\hat{G}$ of type p-FP one has

$$\mathcal{L}_p(\hat{G}) \in \sum_{\gamma\in\mathfrak{R}_{p'}} \mathrm{int}_{\mathbb{Q}_p}(\mathbb{Z}).[\gamma]. \tag{5.28}$$

Moreover, $t_{[1]}(\mathbb{Z}_p) \in \mathcal{O}_p$ (cf. §5.2). Hence p-rationality/irrationality can be detected only on non-trivial conjugacy classes of elements of p'-order. For our purpose the following property will be useful.

Proposition 5.3. *Let $\hat{G}$ be a finitely generated p-projective virtual pro-p group, which acts on a locally-finite pro-p tree $\hat{\Gamma}$ such that*

(i) *$\hat{G}$ acts without inversion of edges,*
(ii) *every vertex stabilizer $\hat{G}_v$, $v \in \mathfrak{V}(\hat{\Gamma})$, is a finite p'-group,*
(iii) *$\hat{G}$ has finitely many orbits on $\mathfrak{V}(\hat{\Gamma})$.*

Then $\hat{G}$ is a p-rational virtual pro-p group.

Proof. The hypothesis imply that $\hat{G}$ has also finitely many orbits on $\mathfrak{E}(\hat{\Gamma})$, and that every edge stabilizer $\hat{G}_{\mathbf{e}}$, $\mathbf{e} \in \mathfrak{E}(\hat{\Gamma})$, is also a finite p'-group. Let $\mathfrak{E}^+(\hat{\Gamma}) \subset \mathfrak{E}(\hat{\Gamma})$ be a $\hat{G}$-invariant orientation of the pro-p tree $\hat{\Gamma}$. Let $V \subset \mathfrak{V}(\hat{\Gamma})$ be a system of representatives of the $\hat{G}$-orbits on $\mathfrak{V}(\hat{\Gamma})$, and let $E^+ \subset \mathfrak{E}^+(\hat{\Gamma})$ be a system of representatives of the $\hat{G}$-orbits on $\mathfrak{E}^+(\hat{\Gamma})$. The chain complex $\mathbf{C}(\hat{\Gamma})$ is a projective resolution of $\mathbb{Z}_p$ in ${}_{\hat{G}}\mathbf{prf}_p$, and this yields

$$\begin{aligned}\mathcal{L}_p(\hat{G}) &= R_{\hat{G}}(\mathbf{V}(\hat{\Gamma})) - R_{\hat{G}}(\mathbf{E}(\hat{\Gamma})) \\ &= \sum_{x \in V} \frac{1}{|\hat{G}_x|} \sum_{g \in \hat{G}_x} [g] - \sum_{\mathbf{e} \in E} \frac{1}{|\hat{G}_{\mathbf{e}}|} \sum_{g \in \hat{G}_{\mathbf{e}}} [g]\end{aligned} \tag{5.29}$$

(cf. (5.13), Prop.5.2). Hence $\hat{G}$ is p-rational. □

An immediate consequence of Proposition 5.3 is the following.

Corollary 5.4. *Let $\hat{G}$ be a finitely generated p-projectve p-irrational virtual pro-p group. Assume that $\hat{G}$ is acting on the locally-finite pro-p tree $\hat{G}$ such that*

(i) *$\hat{G}$ is acting without inversion of edges,*
(ii) *every vertex stabiler $\hat{G}_v$, $v \in \mathfrak{V}(\hat{\Gamma})$ is a finite p'-group.*

Then $\hat{G}$ has infinitely many orbits on $\mathfrak{V}(\hat{\Gamma})$.

5.6. **The p-Lefschetz number of universal p-Frattini covers of finite groups.** Let $\pi\colon Fr_p(G) \to G$ be the universal p-Frattini cover of the finite group G. Thus

$$\ker(\pi)^{\mathrm{ab}} \simeq \Omega_2(G, \mathbb{Z}_p), \tag{5.30}$$

where $\Omega_2(G, \mathbb{Z}_p)$ denotes the second Heller translate of the trivial left $\mathbb{Z}[G]$-module $\mathbb{Z}_p$. Hence using the notation of §5.2 one obtains the following:

Proposition 5.5. *Let $\pi\colon Fr_p(G) \to G$ be the universal p-Frattini cover of the finite group G. Then*

$$\mathcal{L}_p(Fr_p(G)) = \frac{1}{|G|} \sum_{g \in G_{p'}} (1 - \chi_{\Omega_2(G,\mathbb{Z}_p)}(g^{-1}))[\tilde{g}], \tag{5.31}$$

where $\chi_{\Omega_2(G,\mathbb{Z}_p)}$ denotes the character on $\Omega_2(G, \mathbb{Z})$, $G_{p'}$ denotes the set of elements of p'-order in G, and $\tilde{g} \in Fr_p(G)$ denotes the unique element of order $\mathrm{ord}(g)$ *satisfying $\pi(\tilde{g}) = g$, $g \in G_{p'}$.*

Proof. Let $P_1 \xrightarrow{\partial_1} P_0 \xrightarrow{\varepsilon} \mathbb{Z}_p$ be a minimal projective resolution of the trivial left $Fr_p(G)$-module in ${}_{Fr_p(G)}\mathbf{prf}_p$. Applying the deflation functor $\mathrm{def}^{\pi}(\_)$ one obtains a partial minimal projective resolution ${}_{\pi}P_1 \to {}_{\pi}P_0 \to \mathbb{Z}_p$ (cf. [16, Prop.1], [15, §3.4]). In particular,

$$0 \longrightarrow \Omega_2(G, \mathbb{Z}_p) \longrightarrow {}_{\pi}P_1 \xrightarrow{{}_{\pi}\partial_1} {}_{\pi}P_0 \longrightarrow \mathbb{Z}_p \longrightarrow 0 \tag{5.32}$$

is exact. By Proposition 5.2, one has

$$\mathcal{L}_p(Fr_p(G)) = R_G({}_\pi P_0) - R_G({}_\pi P_1). \tag{5.33}$$

Hence, (5.7), the additvity of $\chi\_$ and (5.32) yield the claim. □

Proposition 5.5 shows, that in case that one knows the Brauer character of $\Omega_2(G, \mathbb{F}_p)$, one can also compute the p-Lefschetz number of $Fr_p(G)$. In the subsequent proposition we illustrate this for the finite group $G = \mathrm{SL}_2(p)$.

Proposition 5.6. *Let $p \geq 5$, let $G \colon = \mathrm{SL}_2(p)$ and let $\pi \colon Fr_p(G) \to G$ denote its universal p-Frattini cover. Let*

$$T \colon = \{\, \mathfrak{t}_\eta \colon = \mathrm{diag}(\eta, \eta^{-1}) \mid \eta \in \mathbb{F}_p^* \,\} \leq G \tag{5.34}$$

be a maximally split torus, and let $\tau \colon \mathbb{F}_p^* \to \mathbb{Z}_p^*$ denote the Teichmüller-section[5]. *Let $\mathfrak{R}_{p'}$ be a system of representatives of the $Fr_p(G)$-conjugacy classes of elements of p'-order, and let*

$$\mathcal{L}_p(Fr_p(G)) = \sum_{\gamma \in \mathfrak{R}_{p'}} t_{[\gamma]}.[\gamma] \tag{5.35}$$

Then - using the notation of Proposition 5.5 - one has

$$t_{[\tilde{\mathfrak{t}}_\eta]} = \begin{cases} -\frac{\tau(\eta)^2 + \tau(\eta)^{-2}}{p-1} & \text{if } t_\eta \notin Z(G), \\ -\frac{1}{p^2-1} & \text{if } t_\eta \in Z(G). \end{cases} \tag{5.36}$$

In particular, if $p \geq 11$ and $p \neq 13$, then $Fr_p(G)$ is p-irrational.

Proof. For short we put $\Omega \colon = \Omega_2(G, \mathbb{Z}_p)$. One has a short exact sequence

$$0 \longrightarrow M_{p-3} \longrightarrow \Omega \otimes_{\mathbb{Z}_p} \mathbb{F}_p \longrightarrow M_2 \longrightarrow 0 \tag{5.37}$$

of left $\mathbb{F}_p[G]$-modules, where M_k, $k = 0, \dots, p-1$, denotes the irreducible $\mathbb{F}_p[G]$-module of heighest weight k (cf. [14]). Hence

$$\chi_\Omega(\mathfrak{t}_\eta) = 1 + \tau(\eta)^2 + \tau(\eta)^{-2} + \sum_{i=0}^{p-3} \tau(\eta)^{p-3-2i}. \tag{5.38}$$

Let $\xi \colon = \tau(\eta)^2 \neq 1$ and $k \colon = p - 3$. Then one verifies easily that

$$\sum_{i=0}^{p-3} \tau(\eta)^{p-3-2i} = \xi^{k/2} \sum_{i=0}^{k} \xi^{-k} = \xi^{k/2} \frac{\xi^{-(k+1} - 1}{\xi^{-1} - 1} = 0. \tag{5.39}$$

If $\tau(\eta)^2 = 1$, one has $\chi_\Omega(\mathfrak{t}_\eta) = p + 1$. Hence (5.31) yields (5.36). Let ζ be a primitive root of unity of order $(p-1)/2$. Then $|\mathbb{Q}(\zeta + \zeta^{-1}) \colon \mathbb{Q}| = \frac{1}{2}\varphi(\frac{p-1}{2})$, where φ denotes the Euler function. Moreover, $\varphi(n) = 2$ implies $n \in \{3, 4, 6\}$. Thus, if $p \geq 11$ and $p \neq 13$, $Fr_p(G)$ is p-irrational. □

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[5] This is the unique map such that $\tau(\eta)$ is a unit of p'-order satisfying $\tau(\eta) + p.\mathbb{Z}_p = \eta$.

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