Uniform Hölder bounds and regularity properties of the limiting profile for highly competing nonlinear systems of Schrödinger equations

Susanna Terracini
Dipartimento di Matematica e Applicazioni
Università di Milano Bicocca

joint works with
Monica Conti and Gianmaria Verzini *Politecnico di Milano*
Benedetta Noris and Hugo Tavares, *Università di Milano Bicocca*

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1 Gross Pitaevskii systems

We are interested in systems of nonlinear Schrödinger equations

$$\boxed{\begin{cases} -\Delta u_i + (V_i(x) + \lambda_i) u_i = \mu_i u_i^3 - u_i \sum_{j \neq i} \beta_{i,j} u_j^2, \\ u_i \in H^1(\mathbb{R}^N), \quad u_i > 0, \qquad i = 1, \dots, k \end{cases}}$$

with $N = 2, 3$, $k \geq 3$, and β. Such type of system arise in the study of solitary wave solutions of systems of $k \geq 3$ coupled nonlinear Schrödinger equations, known in the literature as Gross–Pitaevskii equations:

$$\boxed{\begin{cases} -\imath \partial_t(\phi_i) = \Delta \phi_i - V_i(x) \phi_i + \mu_i |\phi_i|^2 \phi_i - \sum_{j \neq i} \beta_{ij} |\phi_j|^2 \phi_i, \qquad i = 1, \dots, k \\ \phi_i \in H^1(\mathbb{R}^N; \mathbb{C}), \qquad N = 1, 2, 3. \end{cases}}$$

2 Condensation of Bose gas in multiple states

This system has been proposed as a mathematical model for multispecies Bose–Einstein condensation in k different hyperfine spin states; it was the Indian physicist S.N. Bose who realized in 1924 that the statistics governing photons is determined by restricting the physical Hilbert space to be the symmetric tensor product of single photon states. Shortly afterwards, Einstein applied this idea to massive particles, such as a gas of atoms, and described the phenomenon that we now call Bose-Einstein condensation.

The first empirical evidence for its existence was only obtained in 1995, in experiments performed by groups led by Cornell and Wieman at the University of Colorado at Boulder and by Ketterle at MIT. In these experiments, atomic gases were initially trapped by magnetic fields and cooled down at very low temperatures. Then the magnetic traps were switched off and the consequent time evolution of the gas was observed; for sufficiently small temperatures, the whole gas moves as a single particle.

A rigorous derivation of the model has not been completely settled. Lieb, Yngvason, and Seiringer considered a trapped Bose gas consisting of N three- dimensional particles described by the Hamiltonian

$$\boxed{H_N = \sum_{1\le j\le N} (\Delta_j + V_{ext}(x_j)) + \sum_{1\le i<j\le N} V_N(x_i - x_j).}$$

where e V_{ext} is an external confining potential and $V_N(x) = N^2V(Nx)$ is a spherically symmetric repulsive interaction potential. Letting $N \to \infty$, they showed that the ground state energy $E(N)$ divided by the number of particle N converges to the ground state energy of the Gross-Pitaevskii energy functional

$$\boxed{\mathcal{E}_{GP}(u) = \int |\nabla u|^2 + V(x)u^2 + 4\pi a_0 u^4}.$$

Lieb and Seiringer also proved that the ground state of the Hamiltonian exhibits complete Bose-Einstein condensation into the minimizer of the Gross-Pitaevskii energy functional. The convergence of the whole dynamics of Bose-Einstein Condensates has been recently derived rigorously in the de–focusing case by Adami, Golse and Teta (2007) in one space dimension and by Erdös, Schlein and Yau (2008) in three dimensions.

3 Condensation in multiple states

Bose–Einstein condensation has been experimentally observed also in double and triple states. In this case the complex valued functions ϕ_i's are the wave functions of the i–th condensate, the functions V_i's represent the trapping magnetic potentials, and the positive constants μ_i's and β_{ij}'s are the intraspecies and the interspecies scattering lengths, respectively. The interactions between like particles are can be attractive (the focusing case) or repulsive (the de–focusing case), while the interactions between the unlike ones are repulsive; we shall assume that $\beta_{ij} = \beta_{ji}$, which gives the system a gradient structure. To obtain solitary wave solutions we set

$$\boxed{\phi_i(t,x) = e^{-\imath\lambda_i t} u_i(x),}$$

obtaining that the real functions u_i's satisfy

$$\boxed{\begin{cases} -\Delta u_i + \left[V_i(x) + \lambda_i\right] u_i = \mu_i u_i^3 - \sum_{j\neq i} \beta_{ij} u_j^2 u_i, \qquad i = 1,\dots,k \\ u_i \in H^1(\mathbb{R}^N). \end{cases} \qquad \text{(Sys)}}$$

For the sake of simplicity we assume $V_i(x) \equiv 0$, $\lambda_i = \mu_i = \pm 1$ and $\beta_{ij} = \beta$, for every i and j, and $N = 2, 3$, even though our method works also in more general cases.

4 Ground states

In the de-focusing case, i.e. ($\mu_i < 0$) the energy associated with the system is

$$\mathcal{E}_\beta(u_1,\dots,u_k) = \sum_{i=1}^{k}\left[\int_\Omega \frac{1}{2}|\nabla u_i|^2 + \frac{1}{2}V_i(x)u^2 + \frac{|\mu_i|}{4}\int_\Omega u_i^4\,dx\right] + \frac{\beta}{4}\sum_{\substack{i,j=1\\ i\neq j}}^{k}\int_\Omega u_i^2u_j^2\,dx$$

Ground state solutions are minimizers of the energy with the mass constraint:

$$\int_\Omega u_i^2 = 1 \qquad i = 1,\dots,k.$$

We define

$$c_\beta = \min\{\mathcal{E}_\beta(u_1,\dots,u_k) \ : \ \int_\Omega u_i^2 = 1 \qquad i = 1,\dots,k\} \,.$$

The functional associated with the focusing case ($\mu_i > 0$)is

$$J_\beta(u_1,\dots,u_k) = \sum_{i=1}^{k}\left[\int_{\mathbb{R}^N} \frac{1}{2}|\nabla u_i|^2 + \frac{1}{2}V_i(x)u^2 - \frac{\mu_i}{4}\int_{\mathbb{R}^N} u_i^4\,dx\right] + \frac{\beta}{4}\sum_{\substack{i,j=1\\ i\neq j}}^{k}\int_{\mathbb{R}^N} u_i^2u_j^2\,dx$$

in this case, solutions having minimal solutions are mountain–pass solutions or, equivalently, minimizers of the functional on the Nearin's manifold. Again, we define

$$c_\beta = \min_{(u_1,\dots,u_k)}\max_{r_i>0} J_\beta(r_1u_1,\dots,r_ku_k) \,.$$

5 Segregation for ground states

When $\beta \to \infty$ as limiting variational problems we find, in the de–focusing case

$$c_\infty = \min_{\substack{\int_\Omega u_i^2 = 1 \,, i,j=1,\dots,k \\ u_i(x)u_j(x)\equiv 0 \text{ for } i\neq j}} \mathcal{E}_\beta(u_1, \dots, u_k) \, . \tag{LVP}$$

and in the focusing one

$$c_\infty = \min_{\substack{(u_1,\dots,u_k) \\ u_i(x)u_j(x)\equiv 0 \text{ for } i\neq j}} \max_{r_i>0} J_\beta(r_1u_1, \dots, r_ku_k) \, . \tag{LVP}$$

It is not difficult to prove that,

Theorem 1 *As $\beta \to +\infty$, there holds*

- $c_\beta \nearrow c_\infty$;
- *there is strong H^1–convergence of the minimizers to a minimizer of (LVP).*

For a fixed k, as the interspecific competition goes to infinity, the wave amplitudes u_i's segregate, that is, their supports tend to be disjoint. This phenomenon, called phase separation, has been studied, in the case of $\mu_i > 0$ (focusing), starting from

M. Conti, S. Terracini, G. Verzini, An optimal partition problem related to non linear eigenvalues, *J. Funct. Anal.* **198** (2003), no. 1, 160-196;

M. Conti, S. Terracini, G. Verzini, On a class of optimal partition problems related to the Fučík spectrum and to the monoticity formulae, *Calc. Var.* **22** (2005), 45-72.

and, in the case $\mu_i < 0$, for least energy solutions in bounded domains:

S.M. Chang, C.S. Lin, T.C. Lin, and W.W. Lin: Segregated nodal domains of two-dimensional multispecies Bose-Einstein condensates. *Phys. D* **196**, 341–361 (2004)

Optimal partition problems

In the limit, we find both a limiting profile, a minimizer of (LVP), and a partition of the original domain, which is optimal with respect to the sum of the ground state energies of the subdomains. Indeed, let us define

$$\varphi_i(\omega) = \min_{\substack{u\in H_0^1(\omega)\\ \int u^2=1}} \left[\int_\omega \frac{1}{2}|\nabla u|^2 + \frac{1}{2}V_i(x)u^2 + \frac{|\mu_i|}{4}\int_\omega u^4\,dx\right]$$

or, in the focusing case,

$$\varphi_i(\omega) = \min_{u\in H_0^1(\omega)} \max_{r>0} \left[\int_\omega \frac{r^2}{2}|\nabla u|^2 + \frac{1}{2}V_i(x)u^2 - \frac{r^4\mu_i}{4}\int_\omega u^4\,dx\right]$$

Given such a ground state energy as a function of the domain, the unknown is the union of the interfaces: in other words we are lead to solve the optimal partition problem:

$$\boxed{\inf_{(\omega_i)\in\mathcal{P}} \sum_{i=1}^{k} \varphi_i(\omega_i) \qquad\qquad \text{(OPP)}}$$

where

$$\mathcal{P} := \left\{(\omega_1,\ldots,\omega_k) : \bigcup_{i=1}^{k}\overline{\omega}_i = \overline{\Omega},\ \omega_i\cap\omega_j = \emptyset \text{ if } i\neq j\right\}.$$

As the real unknown is the nodal set determined by the optimal partition. Hence, we can regard the phase segregation as a free boundary problem.

Problems:

- definition of a class of the admissible partitions;
- regularity of the free boundary;
- connectedness of the subdomains.

Of course, the number of connected domains of segregation is at least the number of different phases surviving in the limit. For the minimal solutions, the limiting states have *connected* supports Indeed, in the case $\mu_i > 0$, it is easy to prove that the supports of the limiting segregated states solve an optimal partition problem, where the total cost is additive (and strictly positive) with respect to the disjoint union; this penalizes non connected supports. On the other hand, when $\mu_i < 0$, it results from numerical evidence.

7 Extremality conditions for the limiting problem

Introduce the notation:

$$f_i(x,u) = -\lambda_i u_i + \mu_i u_i^3$$

$$\widehat{u}_i = u_i - \sum_{j\neq i} u_j, \qquad \widehat{f}(x,\widehat{u}_i) = f_i(x,u_i) - \sum_{j\neq i} f_j(x,u_j).$$

Let us define the class $\mathcal{S}$ as

$$\boxed{\mathcal{S} = \left\{ U = (u_1,\dots,u_k) \in \left(H_0^1(\Omega)\right)^k : \begin{array}{l} u_i \geq 0,\ u_i\cdot u_j = 0 \text{ if } i\neq j, \ \text{ in } \Omega \\ -\Delta u_i \leq f_i(x,u_i(x)), \\ -\Delta \widehat{u}_i \geq \widehat{f}_i(x,\widehat{u}_i(x)) \end{array} \right\}.}$$

In fact, we have

Theorem 2 *Let $\bar{U} = (\bar{u}_1,\dots,\bar{u}_k)$ a minimizer of the minimization problem (LVP) then*

$$\boxed{(\bar{u}_1,\dots,\bar{u}_k) \in \mathcal{S}.}$$

8 Regularity in the class $\mathcal{S}$

Thus the study of $\mathcal{S}$ provides the information on the segregated states induced by strong competition. In particular:

- the elements in the class $\mathcal{S}$ are Lipschitz continuous;
- their nodal set has some regularity properties: up to a set of Hausdorff dimension $N-2$ consists of a collection of $\mathcal{C}^{1,\alpha}$ $N-1$–dimensional manifolds.

Conti M., Terracini S., Verzini G., *A variational problem for the spatial segregation of reaction–diffusion systems*, Indiana Univ. Math. J. 54 (2005), no. 3, 779–815.

L. Caffarelli, F.-H- Lin, Singularly perturbed elliptic systems and multi-valued harmonic functions with free boundaries, *J. Amer. Math. Soc.* **21** (2008), 847-862;

9 Sign changing solutions and the case of two competing densities

In the focusing case, let us assume $\lambda_i = \mu_i = 1$. Then, the limiting profiles are associated with the changing–sign solutions of the scalar equation

$$\boxed{-\Delta w + w = w^3. \qquad \text{(SE)}}$$

Indeed, the system of inequalities in the definition of the class $\mathcal{S}$ read

$$-\Delta(u_i - u_j) \geq -(u_i - u_j) + (u_i - u_j)^3 \,, \qquad i, j = 1, 2 \,, i \neq j$$

and hence, as u_1 and u_2 have disjoint supports:

$$u_1 = w^+ \qquad u_2 = w^- \,.$$

Of course, as its nodal partition is optimal with respect to the (OOP), this solution minimizes the energy among all sign–changing solutions and has exactly two nodal regions.

Conti M., Terracini S., Verzini G., *Nehari method for PDE's and competing species systems*,Ann. Inst. H. Poincar Anal. Non Linaire 19 (2002), no. 6, 871–888

10 Segregation for excited states

We see that there are two (intertwined) possible points of view: we can seek solutions of (Sys), for large β, emanating from a give sign-changing solution of (ES), or we can prove convergence of solutions of (Sys) corresponding to excited states to limiting profiles (possibly in connections with (SE). We know that (SE) possesses an infinity of sign changing solutions (radially symmetric). The nodal solutions of the radially symmetric case are not ground states and their nodal partition needs not to be optimal. Hence the results exposed so far do not apply. Existence of multiple positive radially symmetric solutions for a system of two equal equations has been proved in:

J.C. Wei and T. Weth: Radial solutions and phase separation in a system of two coupled Schrödinger equations. *Arch. Rational Mech. Anal.*, to appear.

and in the general case in:

E.N. Dancer, J.C. Wei, and T. Weth: A priori bounds versus multiple existence of positive solutions for a nonlinear Schrödinger system. *preprint* (2007)

However, the problem of phase separation in the general case remains open. In particular one wonders

- do families of bounded solutions converge in spaces of Hölder continuous (or Lipschitz) functions;
- what are the properties of their limiting profiles?
- Is there a class $\mathcal{S}$?

11 Phase separation in the radially symmetric case

It is well known that the single equation (SE) admits infinitely many nodal solutions. More precisely, following Bartsch and Willem, for any $h \in \mathbb{N}$ equation (SE) possesses radial solutions with exactly $h-1$ changes of sign, that is h nodal components ("bumps"), with a variational characterization. In a recent paper Wei and Weth have shown that, in the case of $k=2$ components, there are solutions (u_1, u_2) such that the difference $u_1 - u_2$, for large values of β, approaches some sign–changing solution w of (SE).

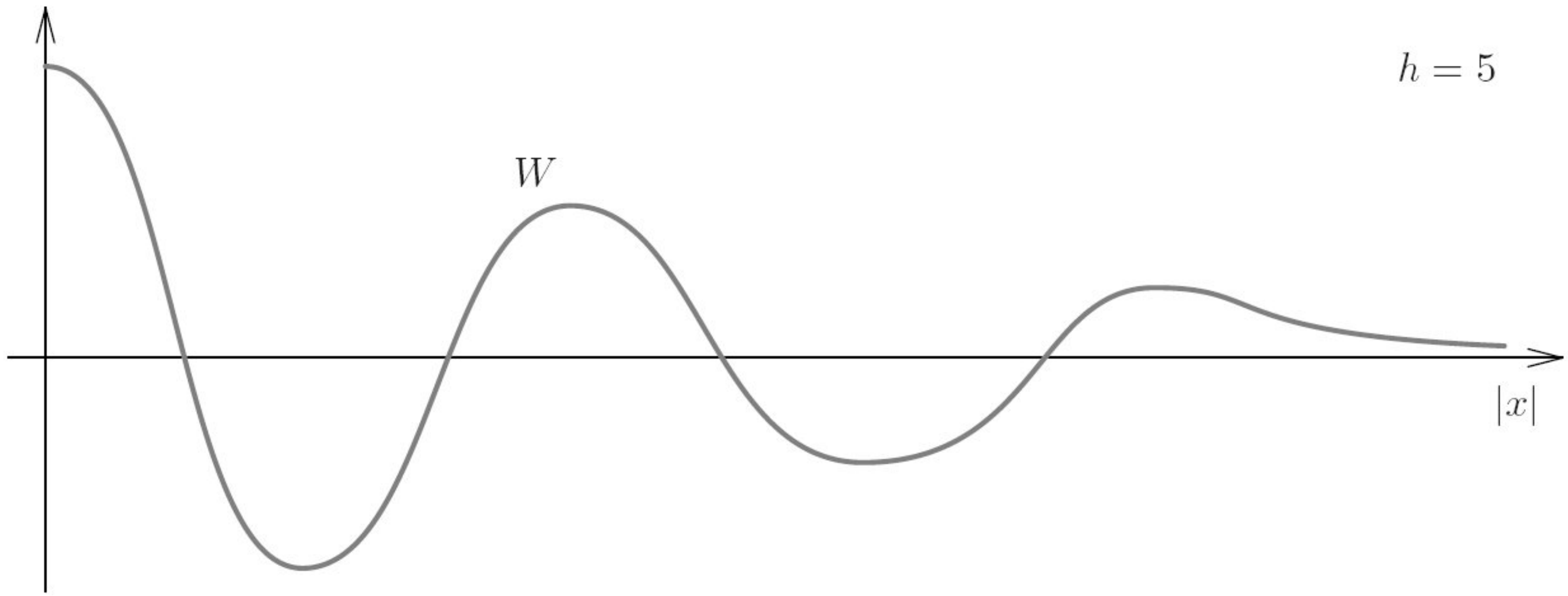

Figure 1: Graph of a solution of (SE) with four nodes

Hence, one can prescribe the limit shape of u_1 and u_2 as w^+ and w^-: this means that each

u_i can be seen as the sum of pulses, each converging to one of the bumps of $|w|$.

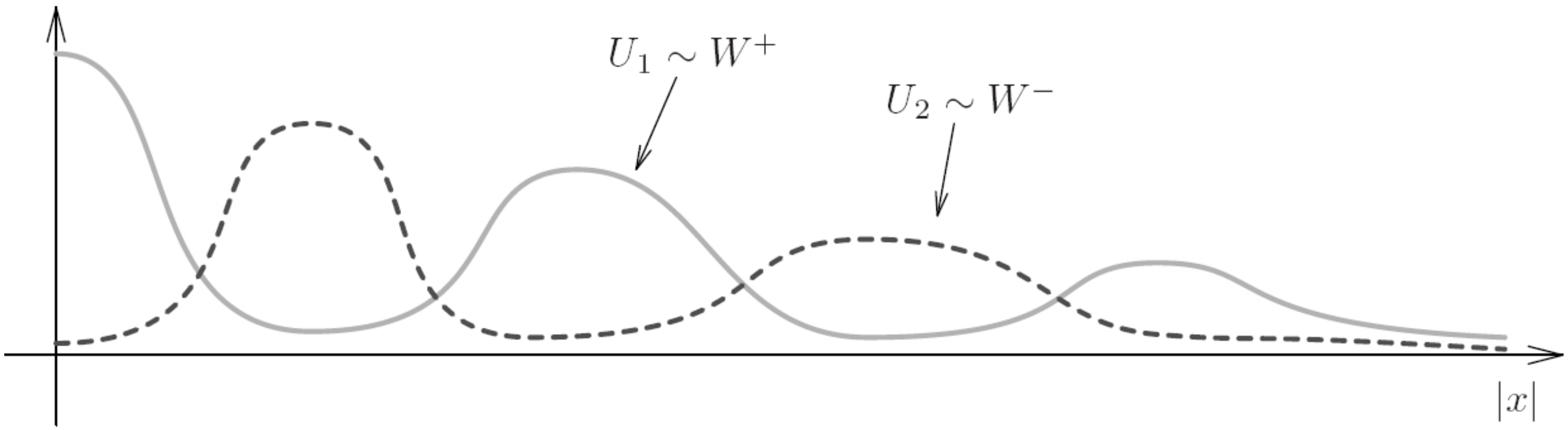

Figure 2: The corresponding solution of a two component (Sys), according to Wei–Weth

In a recent joint paper with G. Verzini, we extend this result to the case of an arbitrary number of components $k \geq 3$, proving the existence of solutions to (SE) with the property that, for β large, each component u_i is near the sum of some non–consecutive bumps of $|W|$.

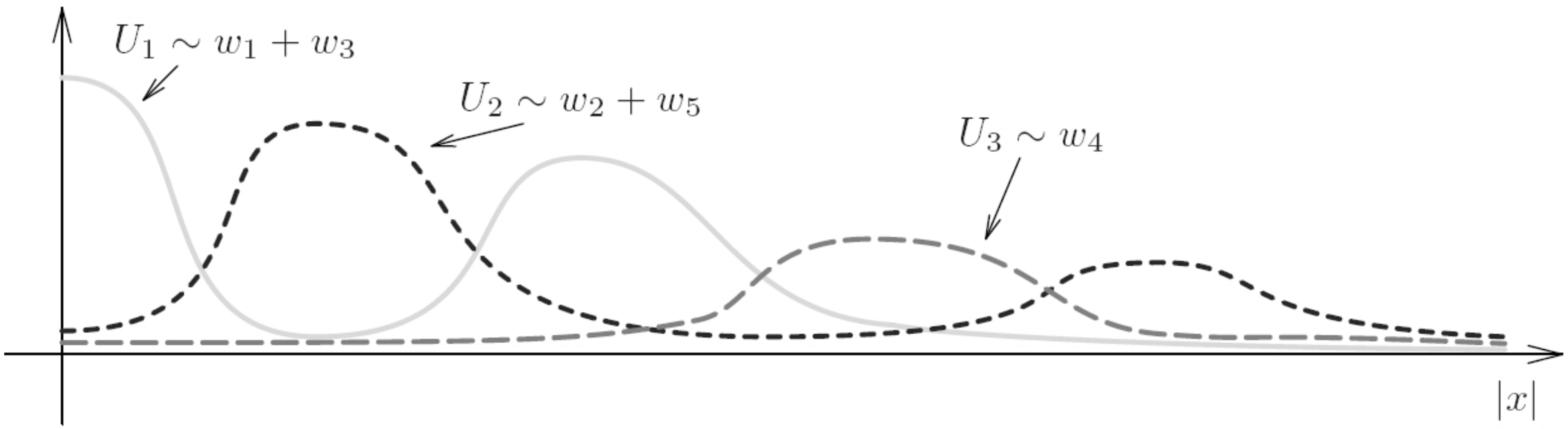

Figure 3: The corresponding solution of a three component (Sys), according to T.–Verzini

Furthermore, we can prescribe the correspondence between such bumps of $|W|$ and the index i of the component u_i. This, compared with the case $k = 2$, provides a much richer structure of the solution set for (Sys). This goal will be achieved by a suitable construction inspired by the extended Nehari's method.

12 Bounds in Hölder spaces and regularity of the limiting profile

Let $\Omega \subset \mathbb{R}^N$ be a regular bounded domain. For every fixed $\beta > 0$, let us consider the system:

$$\boxed{\begin{cases} -\Delta u_\beta + \lambda_\beta u_\beta = \mu_1 u_\beta^3 - \beta u_\beta v_\beta^2 & \text{in } \Omega \\ -\Delta v_\beta + \mu_\beta v_\beta = \mu_2 v_\beta^3 - \beta u_\beta^2 v_\beta & \text{in } \Omega \\ u_\beta,\ v_\beta \in H_0^1(\Omega), \quad u_\beta, v_\beta \geq 0 & \text{in } \Omega \end{cases} \qquad \text{(Sys)}}$$

where $\lambda_\beta, \mu_\beta \in \mathbb{R}$ is a bounded sequence, $\mu_1, \mu_2 \in \mathbb{R}$ are fixed constants.

First we prove uniforma Hölder bounds:

Theorem 3 *Let u_β, v_β be solutions of (Sys) uniformly bounded in $L^\infty(\Omega)$. Then for every $\alpha \in (0,1)$ there exists $C > 0$ (independent of β) such that*

$$\boxed{\max_{x,y\in\overline{\Omega}} \frac{|u_\beta(x) - u_\beta(y)|}{|x-y|^\alpha}, \ \max_{x,y\in\overline{\Omega}} \frac{|v_\beta(x) - v_\beta(y)|}{|x-y|^\alpha} \leq C,}$$

for all $\beta > 0$.

In addition we have:

Theorem 4 *Let u_β, v_β be solutions of (Sys) uniformly bounded in $L^\infty(\Omega)$. Then there exists $(u, v) \in C^{0,\alpha}$, $\forall\ \alpha \in (0,1)$, such that up to a subsequence there holds* *(i)* $u_\beta \to u, v_\beta \to v$ *in* $C^{0,\alpha}(\Omega) \cap H^1(\Omega)$, $\forall\ \alpha \in (0, \alpha^*)$*;*

(ii) $u \cdot v \equiv 0$ *in* Ω *and* $\displaystyle\int_\Omega \beta u_\beta^2 v_\beta^2 \to 0$ *as* $\beta \to +\infty$*;*

(iii) the limiting functions u, v *satisfy the following equations:*

$$\boxed{\begin{cases} -\Delta u + \lambda u &= \mu_1 u^3 \quad \text{in } \{u > 0\}, \\ -\Delta v + \mu v &= \mu_2 v^3 \quad \text{in } \{v > 0\}, \end{cases}}$$

where $\lambda := \lim_{\beta\to+\infty} \lambda_\beta$, $\mu := \lim_{\beta\to+\infty} \mu_\beta$, *and* h, k *denote the* L^2*-weak limits of* h_β, k_β *as* $\beta \to +\infty$.

We can improve the regularity result for the limiting profile.

Theorem 5 *The limiting profiles* (u, v) *is Lipschitz continuous in the interior of* Ω.

This result generalizes:

J.C. Wei and T. Weth Asymptotic behavior of solutions of planar systems with strong competition. *Nonlinearity* **21**, 305–317 (2008)

13 Monotonicity formulæ

Theorem 6 (Alt-Caffarelli-Friedman Monotonicity formula) *Let* $u, v \in H^1_{\rm loc}(\mathbb{R}^N) \cap C(\mathbb{R}^N)$ *be nonnegative functions such that* $u \cdot v \equiv 0$*. Assume moreover that* $-\Delta u \leq 0, -\Delta v \leq 0$ *in* $\mathbb{R}^N$ *and* $u(x_0) = v(x_0) = 0$*. Then the function*

$$\boxed{J(r) := \frac{1}{r^2}\int_{B_r(x_0)} \frac{|\nabla u|^2}{|x-x_0|^{N-2}} \cdot \frac{1}{r^2}\int_{B_r(x_0)} \frac{|\nabla v|^2}{|x-x_0|^{N-2}}}$$

is non decreasing for $r \in (0, +\infty)$.

We shall need a similar result for functions u, v which do not have disjoint supports of the following system:

$$\boxed{\begin{cases} -\Delta u = -uv^2 & \text{in } \mathbb{R}^N \\ -\Delta v = -u^2 v & \text{in } \mathbb{R}^N. \end{cases} \qquad \text{(LSys)}}$$

With this, we can prove a perturbed version of the monotonicity lemma. We introduce a C^1 auxiliary function

$$f(r) = \begin{cases} \frac{2-N}{2} r^2 + \frac{N}{2} & r \leq 1 \\ \frac{1}{r^{N-2}} & r > 1 \end{cases}$$

and denote $m(|x|) := -\frac{\Delta f(|x|)}{2}$. Notice that $m(|x|)$ is bounded on $\mathbb{R}^N$, vanishes in $\mathbb{R}^N \setminus B_1$ and $m(|x|) \geq 0$ a.e. x.

Theorem 7 *Let u, v be positive solutions of (LSys) and let $\varepsilon > 0$ be fixed. Then there exists $\bar{r} > 1$ such that for every $x_0 \in \mathbb{R}^N$ the function*

$$\boxed{\begin{aligned} J(r) := \frac{1}{r^{4-\varepsilon}} \int_{B_r(x_0)} & \left[f(|x|)\left(|\nabla u|^2 + u^2v^2\right) + m(|x|)u^2\right] \\ & \int_{B_r(x_0)} \left[f(|x|)\left(|\nabla v|^2 + u^2v^2\right) + m(|x|)v^2\right] \end{aligned}}$$

is increasing for $r \in (\bar{r}, +\infty)$.

Then we have the following Liouville–type result.

Theorem 8 *Let u, v be non negative entire solutions of (Sys), uniformly Hölder continuous on $\mathbb{R}^N$ for some $\alpha \in (0, 1)$. Then one of the functions is identically zero and the other is a constant.*

$$\boxed{\begin{aligned} E(r) &= \frac{1}{r^{N-2}} \int_{B_r} \left(|\nabla u_\beta|^2 + |\nabla v_\beta|^2 + \lambda_\beta u_\beta^2 + \mu_\beta v_\beta^2 - \mu_1 u_\beta^4 - \mu_2 v_\beta^4 + 2\beta u_\beta^2 v_\beta^2 \right), \\ H(r) &= \frac{1}{r^{N-1}} \int_{\partial B_r} (u_\beta^2 + v_\beta^2), \end{aligned}}$$

We define the (modified) Almgren's quotient as follows:

$$\boxed{N(r) = \frac{E(r) + H(r)}{H(r)}.}$$

Theorem 9 *There exist $\bar{r}, C > 0$, such that for $0 < r < \bar{r}$ we have $H(r) \neq 0$ and also $E(r) + H(r) \neq 0$,*

$$N'(r) \geq -CrN(r)$$

(in particular the function $\tilde{N} = e^{\frac{C}{2}r^2} N(r)$ is non decreasing) and moreover

$$\frac{d}{dr} \log(H(r)) = \frac{2}{r}(N(r) - 1).$$