Singularities and collisions of generalized solutions to the N–body problem

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1 Papers

Ferrario, D. L., and Terracini, S. *On the existence of collisionless equivariant minimizers for the classical n-body problem* Inv. Math, (2004)

Barutello V., Ferrario D.L., Terracini S. , *Symmetry groups of the planar three-body problems and action-minimizing trajectories*, Arch. Rat. Mech. Anal., to appear

Barutello V., Ferrario D.L., Terracini S. , *On the singularities of generalized solutions to n–body type problems*, (2007), submitted

Related papers adopting a variational approach by: Ambrosetti, Bahri, Barutello, Bessi, Chenciner, Chen, Coti Zelati, Degiovanni, Desolneux, Giannoni, Gordon, Majer, Marchal, Marino, Montgomery, Rabinowitz, Riahi, Sbano, Serra, Tanaka, Terracini, Venturelli.

2 Singular systems

Many systems of interacting bodies of interest in Celestial and other areas of classical Mechanics have the form

$$m_i \ddot{x}_i = \frac{\partial U}{\partial x_i}(t, x), \qquad i = 1, \ldots, n$$

where the forces $\frac{\partial U}{\partial x_i}$ are undefined on a singular set Δ.

➡ Example: the set of collisions between two or more particles in the n–body problem:

$$U(x) = \sum_{i<j} \frac{m_i m_j}{|x_i - x_j|^\alpha}$$
$$\Delta = \bigcup_{i \neq j} \{(x_1, \ldots, x_n) \ : \ x_i = x_j\}$$

Such singularities play a fundamental role in the phase portrait and strongly influence the global orbit structure, as they can be held responsible, among others, of the presence of chaotic motions and of motions becoming unbounded in a finite time (Diacu, Devaney, Gerver, Gutzwiller, Mather, Saari, Simò, Xia).

3 Key points in the analysis of the impact of the bounded singularities in the n–body problem

- the asymptotic analysis along a single collision trajectory (total or partial); this analysis goes back, in the classical case, to the works by Sundman, Wintner and, in more recent years by Sperling, Pollard, Saari, Diacu and other authors;
- blowing–up the singularity by a suitable change of coordinates (nowadays named after McGehee) and replacing it by an invariant boundary –the collision manifold– where the flow can be extended in a smooth manner;
- it turns out that, in many interesting applications, the flow on the collision manifold has a simple structure: it is a gradient–like, Morse–Smale flow featuring a few stationary points and heteroclinic connections;
- the analysis of the extended flow allows us to obtain a full picture of the behavior of solutions near the singularity, despite the flow fails to be fully regularizable (except in a few cases).

4 Extension of the asymptotic estimates near collisions

- We wish to take into account of a very general notion of solution for the dynamical system, which fits particularly well to solutions found by variational techniques.

  - our notion of solution includes, besides all classical noncollision trajectories, all the locally minimal solutions (with respect to compactly supported variations) which are often termed minimal the sense of Morse;

  - furthermore, we include in the set of generalized solutions all the limits of classical and locally minimal solutions.

- We extend our analysis to a wide class of potentials including not only homogeneous and quasi-homogeneous potentials, but also those with weaker singularities of logarithmic type.

- We allow potentials to strongly depend on time (we only require its time derivative to be controlled by the potential itself). In this way, for instance, we can take into account models where masses vary in time.

As a consequence of the asymptotic estimates, the presence of a total collision prevents the occurrence of partial ones for neighboring times.

This observation plays a central role when extending the asymptotic estimates to the full n–body problem since it allows us to reduce from partial (even simultaneous) collisions to total ones by decomposing the system in colliding clusters.

5 The variational approach to the periodic N–body problem

- Settings: n point particles with masses $m_1, m_2, \ldots m_n$ and positions $x_1, x_2, \ldots, x_n \in \mathbb{R}^d$, with $d \geq 2$.
- Homogeneous (Newton) potential of degree $-\alpha < 0$ on the configuration space $\mathcal{X}$: $U(x) \simeq \sum_{i<j} \frac{m_i m_j}{|x_i - x_j|^\alpha}$.
- Many results can be extended to logarithmic potentials:almost parallel N–vortex problem;
- On collisions ($x_i = x_j$ for some $i \neq j$) potential $U = +\infty$.
- T-periodic orbits: solutions of the Newton equations (such that $\forall t : x(t+T) = x(t) \in \mathcal{X}$).

$$m_i \ddot{x}_i = \frac{\partial U}{\partial x_i}.$$

- Lagrangian: $L(x, \dot{x}) = L = K + U = \sum_i \frac{1}{2} m_i |\dot{x}_i|^2 + \sum_{i<j} \frac{m_i m_j}{|x_i - x_j|^\alpha}.$
- Action functional: $\mathcal{A}(x) = \int_0^T L(x(t), \dot{x}(t)) dt.$

6 Critical points of the action functional

- Sobolev space of T-periodic trajectories: $\Lambda = H^1(\mathbb{T}, \mathcal{X})$.
- Find critical points of action functional

$$\mathcal{A} \colon \Lambda \to \mathbb{R} \cup \infty,$$

constrained on suitable linear subspaces $\Lambda_0 \subset \Lambda$ (natural constraints for $\mathcal{A}$).

Main problems:

- The action functional $\mathcal{A}$ is not coercive on Λ. The minimum needs not to be achieved.
- We can seek critical point others than minimizers: e.g.
  - Local minimizers
  - Constrained minimizers
  - Other type of critical points (mountain pass).
- The action functional $\mathcal{A}$ does not satisfy the Palais Smale condition on Λ: sequences of almost–critical points may diverge.
- The potential U is singular on collisions, and thus miminizers or other critical points can *a priori* be collision trajectories.

7 A plethora of periodic trajectories

Our first aim is to prove the existence of a multiplicity of periodic trajectories by a systematic use of equivariant variational methods. This involves:

- The classification of all the admissible symmetry groups [Barutello, Ferrario, Terracini].
- The analysis of possible collisions for equivariant minimizers and the determination of those groups whose minimizers are free of collisions.
- A further study of qualitative properties of equivariant minimizers to understand whether different classes of symmetric loops may share the same minimizers.
- The (numerical) analysis of the linear stability of solutions.

Further developments

(1) Develop an equivariant Morse Theory specific for the N–body problem, taking into account of all possible collisions.

(2) Fully understand the impact of collisions on the variational characterization (Morse index) of periodic trajectories.

8 LOcally minimal solutions

In general, the outcome of our variational method (minimization or other) is a trajectory that can interact with the singularities of the potential. The very same notion of solutions has to be changed to take into account of this possibility. The first notion of generalized solutions is that of local minimizer with respect to compactly supported variations by Morse (or Morrey):

> **Definition:** A path $x : (a, b) \to \mathcal{X}$ is called a locally minimizing solution if, for every $t_0 \in (a, b)$, there exists $\delta > 0$ such that the restriction of x to $[t_0 - \delta, t_0 + \delta]$ is a local minimizer for the action with the same boundary conditions. .

Remarks:

- The definition can be modified in a obvious manner to include symmetries.
- Equivariant minimizers are locally minimizing solutions.
- If the potential is of class $\mathcal{C}^2$ outside collisions then every which is free of collisions is a locally minimizing solution.
- The locally minimal solutions possess an index: the minimal number of intervals I_j needed to cover (a, b) such that the restriction of x to I_j is a local minimizer for the action.
- There is a natural notion of maximal existence interval for generalized solutions, even without the unique extension property.

9 Generalized solutions

Definition: A path $x : (a, b) \to \mathcal{X}$ is called a generalized solution if there exists a sequence x_n of locally minimal solutions such that

(1) $x_n \to x$ uniformly on compact subsets of (a, b);

(2) for almost all $t \in (a, b)$ the associated total energy $h_n(t) := K(\dot{x}_n(t)) - U(t, x_n(t))$ converges.

To avoid trivialities we shall assume that the collision set $x^{-1}(\Delta)$ is not the full interval (a, b).

Remarks:

- Locally minimizing solutions are generalized solutions.
- This class includes all the limits of families solutions obtained as limits either by adding a strong force penalization sequence, or by expliting a sequence of smoothed potentials.
- There is a natural notion of maximal existence interval for generalized solutions, even without the unique extension property.
- Generalized solutions do not solve neither a differential equation nor a minimization problem. However, being the limit of functions having these properties, they inherit some of the fetures of solutions of such problems.

10 Singularities and collisions

Generally speaking we are dealing with systems of the form

$$M\ddot{x} = \nabla U(t,x), \qquad t \in (a,b), \qquad M_{ij} = m_i \delta_{ij}$$

where the potential U possesses a singular set $(a,b) \times \Delta$, in the sense that

$$\lim_{x \to \Delta} U(t,x) = +\infty \ , \qquad \text{uniformly in } t.$$

The set Δ is called the collision set. We assume Δ to be a cone in $\mathbb{R}^{nd}$:

$$x \in \Delta \implies \lambda x \in \Delta.$$

We say that a generalized solution x on the interval (a,b), has a singularity at $t^* < +\infty$ if

$$\limsup_{t \to t^*} U(t, x(t)) = +\infty.$$

When $t^* \in (a,b)$ we will say that x has an interior singularity at $t = t^*$, while when $t^* = a$ or $t^* = b$ (when finite) we will talk about a boundary singularity.

11 The theorems of Painlevé and Von Zeipel

We say that a classical solution x on the interval (a, b), has a singularity at $t^* < +\infty$ if it is not possible to extend x as a (classical, locally minimal, generalized) solution to a larger interval $(a, t^* + \delta)$.

Painlevé's Theorem: *Let $\bar{x}$ be a classical solution for the n-body dynamical system on the interval $[0, t^*)$. If $\bar{x}$ has a singularity at $t^* < +\infty$, then*

$$\lim_{t \to t^*} U(\bar{x}(t)) = +\infty.$$

Painlevé's Theorem does not necessarily imply that a collision (i.e. that is a singularity such the configuration has a definite limit) occurs when there is a singularity at a finite time (on this subject we refer to Pollard and Saari, McGehee). The next result has been stated by Von Zeipel in 1908 and definitely proved by Sperling in 1970:

Von Zeipel's Theorem: *If $\bar{x}$ is a classical solution for the n-body dynamical system on the interval (a, t^*) with a singularity at $t^* < +\infty$ and $\lim_{t \to t^*} \|\bar{x}(t)\| < +\infty$, then $\bar{x}(t)$ has a definite limit configuration x^* as t tends to t^*.*

12 Von Zeipel's Theorem and the structure of the collision set

To the aim of extending the Von Zeipel's Theorem we need to introduce some assumptions on the potential U and its singular set Δ.

$$\Delta = \bigcup_{\mu \in \mathcal{M}} V_\mu,$$

where the V_μ's are distinct linear subspaces of $\mathbb{R}^k$ and $\mathcal{M}$ is a finite set. We endow the family of the V_μ's with the inclusion partial ordering and we assume the family to be closed with respect to intersection. With each $\xi \in \Delta$ we associate

$$\mu(\xi) = \min\{\mu \ : \ \xi \in V_\mu\} \qquad \text{i.e.,} \qquad V_{\mu(\xi)} = \bigcap_{\xi \in V_\mu} V_\mu.$$

Fixed $\mu \in \mathcal{M}$ we define the set of collision configurations satisfying

$$\Delta_\mu = \{\xi \in \Delta : \mu(\xi) = \mu\}$$

and we observe that this is an open subset of V_μ and its closure $\overline{\Delta_\mu}$ is V_μ. We also notice that the map $\xi \to \dim(V_{\mu(\xi)})$ is lower semi continuous.

We denote by p_μ the orthogonal projection onto V_μ and we write

$$x = p_\mu(x) + w_\mu(x),$$

where, of course, $w_\mu = \mathbb{I} - p_\mu$.

We assume that, near the collision set, the potential depends, roughly, only on the projection orthogonal to the collision set: more precisely we assume

[U5] For every $\xi \in \Delta$, there is $\varepsilon > 0$ such that

$$U(t,x) - U(t, w_{\mu(\xi)}(x)) = W(t,x) \in \mathcal{C}^1\left((a,b) \times B_\varepsilon(\xi)\right),$$

where $B_\varepsilon(\xi) = \{x \ : \ |x-\xi| < \varepsilon\}$.

We can extend then Von Zeipel's Theorem to generalized solutions:

Theorem. *Let $\bar{x}$ be a generalized solution on the bounded interval (a,b). If $\bar{x}$ is bounded on the whole interval (a,b) then the singularities of $\bar{x}$ are collisions.*

13 One side conditions on the potential and its radial derivative

Our potential U satisfies some one–side homogeneity conditions:

[U1] there exists $C_1 \geq 0$ such that for every $(t, x) \in \big((a, b) \times \mathbb{R}^{nd} \setminus \Delta\big)$

$$\left|\frac{\partial U}{\partial t}(t, x)\right| \leq C_1 U(t, x).$$

[U2] there exist $\alpha \in (0, 2)$, $\gamma > 0$ and $C_2 \geq 0$ such that

$$\nabla U(t, x) \cdot x + \alpha U(t, x) \geq -C_2 |x|^\gamma U(t, x),$$

whenever $|x|$ is small.

We observe that when U is homogeneous functions of degree $-\alpha$, the equality in condition (U2) is attained with $C_2 = 0$; this assumption is satisfied also when we take into account potentials of the form $U(t, x) = U_\alpha(x) + U_\beta(x)$ where U_α is homogeneous of degree $-\alpha$, U_β is homogeneous of degree $-\beta$, $0 < \beta < \alpha$, and U_α is positive (extends previous results by F. Diacu, Journ. Differential Equations, 128, 58-77, 1996.).

14 Limiting behaviour of the potential

We introduce polar coordinates:

$$r = \sqrt{Mx \cdot x} = \sqrt{I}, \qquad s = \frac{x}{r},$$

where $s \in \mathcal{E} = \{x \; : \; I^2(x) = Mx \cdot x = 1\}$ belongs the ellipsoid of all the configurations having unitary moment of inertia.

[U3$_h$] there exists a function $\tilde{U}$ defined on $(a,b) \times (\mathcal{E} \setminus \Delta)$, such that (on compact subsets of $((a,b) \times \mathcal{E} \setminus \Delta)$):

$$\lim_{r \to 0} r^\alpha U(t, rs) = \tilde{U}(t, s);$$

[U3$_l$] there exists a function $\tilde{U}$ defined on $(a,b) \times (\mathcal{E} \setminus \Delta)$, such that (on compact subsets of $((a,b) \times \mathcal{E} \setminus \Delta)$):

$$\lim_{|x| \to 0} \left[U(t,x) + M(t) \log |x|\right] = \tilde{U}(t, s);$$

We then split the lagrangian as the sum of the kinetic and potential terms:

$$L(t, x, \dot{x}) = K(\dot{x}) + U(t, x) \qquad\qquad K(\dot{x}) := \frac{1}{2} M\dot{x} \cdot \dot{x}\ .$$

K is the kinetic energy and the total energy of the system

$$h(t) = K - U\ .$$

15 Isolatedness of collisions instants

Theorem: *Let $x : (a, b) \to \mathcal{X}$ be a generalized solution of the N–body problem. Then collision instants are isolated in (a, b). Furthermore, if (a, b) is the finite maximal extension interval of x and no escape in finite time occurs then the number of collision instants is finite.*

Some remarks:

- A generalized solution does not solve the Euler–Lagrange equation in a distributional sense (the force field can not be locally integrable). Moreover, a priori its action needs not to be finite: one needs to prove it.
- *a priori* our solution can have a huge set of collision instants (more than countable, but null measure);
- there may be accumulation of partial collisions at a collision having more bodies;
- there is no *a priori* bound on the total action on the whole interval (a, b), nor on the energy;
- we have very weak assumptions on the potential and only a one side inequality on the radial component
- the theorem extends to the logarithmic potentials.

Locally minimal solutions do not solve the differential equation in the sense of distributions, but many qualitative properties can be deduce from the fact that they can be arbitrarily approximated with solutions of regular problems.

Proposition. *Let $\bar{x}$ be a locally minimizing solutions. Then, there exists a family of smoothed potentials and corresponding solutions such that, up to subsequences, as* $\varepsilon \to 0$,

- $U_\varepsilon(t, x_\varepsilon) \to U(t, \bar{x})$ *almost everywhere and in* L^1;
- $x_\varepsilon \to \bar{x}$ *uniformly;*
- $\dot{x}_\varepsilon \to \dot{\bar{x}}$ *in* L^2;
- $\dot{x}_\varepsilon \to \dot{\bar{x}}$ *almost everywhere;*
- $\dfrac{\partial U_\varepsilon}{\partial t}(t, x_\varepsilon) \to \dfrac{\partial U}{\partial t}(t, \bar{x})$ *almost everywhere and in* L^1.

As a consequence, the following energy estimates hold:

Corollary. *Let $\bar{x}$ be a locally minimizing solution on (a,b). Suppose that*

$$\limsup_{t \to b^-} I(\bar{x}(t)) < +\infty, \qquad \textit{and} \qquad \liminf_{t \to b^-} \dot{I}(\bar{x}(t)) < +\infty.$$

Then, if $-\infty < a < \tau < b < +\infty$ there hold

- $\displaystyle\int_\tau^b U(t, \bar{x}(t))\, dt < +\infty;$
- $\displaystyle\left| \int_\tau^b h(t)\, dt \right| < +\infty;$
- $\displaystyle\int_\tau^b K(\dot{\bar{x}}(t))\, dt < +\infty;$
- $\|h\|_\infty < +\infty$ *on* $[\tau, b)$.
- $\displaystyle\lim_{t \to b^-} \bar{x}(t)$ *exists.*

17 Energy estimates for generalized solutions.

There exists a $\delta > 0$ such that the following holds.

Proposition. *Let $\bar{x}$ be a generalized solution, let $\tau \notin \bar{x}^{-1}(\Delta)$ be a point of convergence of the energy sequence, and let $0 < \delta < b - \tau$. Then the associated sequence of locally minimizing solutions has the following properties*

- $\limsup_{n \to +\infty} \int_\tau^{\tau+\delta/2} U(t, x_n(t))\, dt < +\infty$.;
- $\limsup_{n \to +\infty} \int_\tau^{\tau+\delta/2} |\dot{h}_n|\, dt < +\infty$.;
- $\limsup_{n \to +\infty} \int_\tau^{\tau+\delta/2} K(\dot{x}_n(t))\, dt < +\infty$.;

By repeatedly applying this result we can cover every compact subinterval of (a, b) and then, passing to the limit as n tends to infinity, we easily obtain next result, which states that generalized solutions are actually much more regular, together with their energies. Moreover, we can pass into the limit in Lagrange–Jacobi distributional inequality **??**, and applying Fatou's Lemma in its right–hand side, we obtain the validity of the Lagrange–Jacobi inequality also for generalized solutions.

Corollary. *Let $\bar{x}$ be a generalized solution on (a, b). Then*

- $U(t, \bar{x}(t)) \in L^1_{loc}(a, b)$.;
- $h \in BV_{loc}(a, b)$;
- $\bar{x} \in H^1_{loc}(a, b)$.;
- $\dot{I}(\bar{x}) \in BV_{loc}(a, b)$ *and the following inequality holds in the distributional sense*

$$\frac{1}{2}\ddot{I}(\bar{x}(t)) \geq 2h(t) + (2 - \tilde{\alpha})U(t, \bar{x}(t)) - C_2 \ .$$

Finally, one can extend the estimates up to the end point of the interval.

Corollary. *Let $\bar{x}$ be a generalized solution on (a,b) and let x_n be the approximating sequence of locally minimal solutions. Suppose that*

$$\limsup_{t\to b^-} I(\bar{x}(t)) < +\infty, \qquad \textit{and} \qquad \liminf_{t\to b^-} \dot{I}(\bar{x}(t)) < +\infty.$$

Then, if $-\infty < a < \tau < b < +\infty$ there hold

- $\displaystyle \limsup_{n\to+\infty} \int_\tau^b U(t, x_n(t))\, dt < +\infty$;
- $\displaystyle \limsup_{n\to+\infty} \int_\tau^b |\dot{h}_n| + |h_n|\, dt < +\infty$;
- $\displaystyle \limsup_{n\to+\infty} \int_\tau^b K(\dot{x}_n(t))\, dt < +\infty$;
- $\|h\|_\infty < +\infty$ *on* $[\tau, b)$;
- $\displaystyle \lim_{t\to b^-} \bar{x}(t)$ *exists.*

Conservation laws

Even though generalized solutions do not not satisfy the differential equations in a distributional sense, they satisfy a number of conservation laws. Let x be a generalized solutions on (a, b). Then

- the action $\mathcal{A}(x, [a, b])$ on (a, b) is finite
- the energy h is bounded and belongs to the Sobolev space $W^{1,1}((a, b), \mathbb{R})$.
- Lagrange–Jacobi inequality holds in the sense of measures:

$$\frac{1}{2}\ddot{I}(\bar{x}(t)) \geq 2h(t)(\bar{x}(t)) + (2-\alpha)U(t, \bar{x}(t)) - C_2|\bar{x}|^\gamma U(t, \bar{x}), \qquad \forall t \in (a, b).$$

- A monotonicity formula holds (extending Sundman's inequality): let us consider the angular energy

$$\Gamma_\alpha(r, s) := r^\alpha \left[\frac{1}{2}r^2|\dot{s}|^2 - U(t, rs)\right]$$

the Γ_α is bounded variation and

$$\frac{d}{dt}\Gamma_\alpha(r, s) \geq -\frac{2-\alpha}{2}r^{1+\alpha}\dot{r}|\dot{s}|^2 - C_1 r^\alpha U(t, rs) + C_2 r^{\alpha+\gamma}\frac{\dot{r}}{r}U(t, rs).$$

19 Generalized Sundman–Sperling estimates

Sundman, Wintner, Sperling, Pollard, Saari, Simò, Diacu, Elbialy.

- The following asymptotic estimates hold:

$$r \sim (\kappa t)^{\frac{2}{2+\alpha}} \qquad \left(r \sim |t|\sqrt{-\log(|t|)}\right)$$

$$K \sim U \sim \frac{1}{4-2\alpha}\ddot{I} \sim \frac{2}{(2+\alpha)^2}\kappa^2(\kappa t)^{\frac{-2\alpha}{2+\alpha}} \qquad (\sim -\log|t|)\,.$$

- Let s be the *normalized configuration* of the colliding cluster $s = x/r$. Then

$$\lim_{t\to 0} r^2|\dot{s}|^2 = 0$$

$$\lim_{t\to 0} U(s(t)) = b < +\infty$$

- Moreover there exists the angular blow–up, that is angular scaled family $(s(\lambda t))_\lambda$ is pre-compact for the topology of uniform convergence on compact sets of $\mathbb{R} \setminus \{0\}$.

As a consequence we have the vanishing of the total angular momentum and the absence of partial collisions in a neighbourhood of the total collision.

20 Dissipation (Mc Gehee revisited)

The asymptotic estimates follow from a monotonicity formula which is basically equivalent to Sundman inequality. Let us think to a homogeneous potential U_α. The quickest way to see this dissipation is to perform the change of variables

$$\rho = r^{\frac{2-\alpha}{4}}, \quad \rho' = \frac{2-\alpha}{4} r^{-\frac{2+\alpha}{4}} r'$$

to obtain the action functional depending on (ρ, s)

$$\mathcal{A}(\rho, s) = \int_0^{\tau^*} \frac{1}{2}\left(\frac{4}{2-\alpha}\right)^2 (\rho')^2 + \rho^2\left(\frac{1}{2}|s'|^2 + U(s)\right) - \lambda\rho^\beta d\tau,$$

where $\beta := 2(2+\alpha)/(2-\alpha) > 2$. Here we have reparametrized the time as

$$dt = r^{\frac{2+\alpha}{2}} d\tau,$$

one proves that

$$\tau^* = \int_0^1 r^{-\frac{2+\alpha}{2}} dt = +\infty.$$

This is the coupling between a Duffing equation and the N-body angular system.

21 Blow-ups

For every $\lambda > 0$ let

$$x^{\lambda}(t) = \lambda^{-2/(2+\alpha)} x(\lambda t)$$

If $\{\lambda_n\}_n$ is a sequence of positive real numbers such that $s(\lambda_n)$ converges to a normalized configuration $\bar{s}$, then $\forall t \in (0,1) : \lim_{n\to\infty} s(\lambda_n t) = \lim_{n\to\infty} s(\lambda_n) = \bar{s}$. Hence the rescaled sequence will converge uniformly to the *blow-up* of $x(t)$ relative to the colliding cluster $\mathbf{k} \subset \mathbf{n}$ (in $t = 0$).

- The blow-up $\bar{x}$ is parabolic: where a *parabolic collision trajectory* for the cluster $\mathbf{k}$ is the path

  $$\bar{x}_i(t) = |t|^{2/(2+\alpha)} \xi_i, \quad i \in \mathbf{k},\ t \in \mathbb{R}$$

  where $\xi = (\xi_i)_{i\in\mathbf{k}}$ is a central configuration with k bodies.

- **Proposition;** The sequences x^{λ_n} and $\dfrac{dx^{\lambda_n}}{dt}$ converge to the blow-up $\bar{x}$ and its derivative $\dot{\bar{x}}$ respectively, in the H^1–topology. Moreover $\bar{x}$ is a minimizing trajectory in the sense of Morse.

  $$\int_0^T [\mathcal{L}(\bar{x} + \varphi) - \mathcal{L}(\bar{q}) \geq 0]\, dt.$$

  for any compactly supported variation φ.

Logarithmic type potentials

It is not possible to define a blow–up suitable for logarithmic type potentials. Indeed, the natural scaling should be $\bar{x}^{\lambda_n}(t) := \lambda_n^{-1}\bar{x}(\lambda_n t)$, which does not converge, since, by the asymptotics $|x(t)| \simeq |t|\sqrt{-\log|t|}$, we have:

$$\lim_{\lambda_n \to 0} |\bar{x}^{\lambda_n}(t)| = \lim_{\lambda_n \to 0} \frac{r(\lambda_n t)}{\lambda_n t\sqrt{-2M(0)\log(\lambda_n t)}} t\sqrt{-2M(0)\log(\lambda_n t)} = +\infty$$

for every $t > 0$.

On the other, hand, looking at the differential equation, the (right) blow–up should be:

$$\bar{q}(t) := t\bar{s}, \quad i \in \mathbf{k},$$

where $\bar{s}$ is a central configuration for the system limit of a sequence $s(\lambda_n)$ where $(\lambda_n)_n$ is such that $\lambda_n \to 0$. This limiting function is the pointwise limit of the normalized sequence

$$\bar{x}^{\lambda_n}(t) := \frac{1}{\lambda_n\sqrt{-2M(0)\log\lambda_n}}\bar{x}(\lambda_n t).$$

Unfortunately this path is not locally minimal for the limiting problem, indeed since, the sequence $(\ddot{\bar{x}}^{\lambda_n})_n$ converges to 0 as n tends to $+\infty$, and hence this blow-up minimizes only the kinetic part of the action functional.

23 Absence of collision for locally minimal paths

As a matter of fact, solutions to the Newtonian n–body problem which are minimals for the action are, very likely, free of any collision. This fact was observed by the construction of suitable local variation arguments for the 2 and 3–body cases by Serra and Terracini (1992 and 1994). The 4–body case was treated afterward by Dell'Antonio (non really rigorously) and then by A. Venturelli in his PhD thesis. In general, the proof goes by the sake of the contradiction and involves the construction of a suitable variation that lowers the action in presence of a collision. A recent breakthrough in this direction is due of the neat idea, due to C. Marchal, of averaging over a family of variations parameterized on a sphere. The method of averaged variations for Newtonian potentials has been developed and exposed by Chenciner, and then extended to α–homogeneous potentials and various constrained minimization problems by Ferrario and Terracini. This argument can be used in many of the known cases to prove that minimizing trajectories are collisionless.

Of course, in some specific situations, other arguments can be useful, such as level estimates, on the infimum of the action on colliding paths [Bessi and Coti Zelati, Chenciner and Montgomery, Chen]. However, these argument require global conditions on the potentials and can not be applied in the present setting, where we work under local assumptions about the singularities.

We are in a position to prove the absence of collisions for locally minimal solutions when the potentials have quasi–homogeneous or logarithmic singularities. The first case is simpler, because one can take advantage the blow–up technique already exploited by Ferrario and Terracini. On the other hand, when dealing with logarithmic potentials, the blow–up technique is no longer available and we conclude proving directly some averaging estimates that can be used to show the non minimality of large classes of colliding motions.

Problems:

- Anisotropic and logarithmic potentials.
- Study the contributions to the Morse index given by the possible collisions (Barutello, Secchi, (2006).

Let $\tilde{U}$ be the $\mathcal{C}^1$ function defined on $(a,b)\times(\mathbb{R}^k\setminus\Delta)$ in the following way:

$$\tilde{U}(t,x)=|x|^{-\alpha}\lim_{r\to 0} r^{\alpha}\tilde{U}(t,rx/|x|).$$

With a slight abuse of notation, we denote $\tilde{U}(x)=\tilde{U}(t^*,x)$. $\tilde{U}$ is homogeneous of degree $-\alpha$.

[U6] there is a 2–dimensional linear subspace of $V^{\perp}_{\mu(\xi)}$, say W, where $\tilde{U}$ is rotationally invariant:

$$\tilde{U}(e^{i\theta}w)=\tilde{U}(w),\qquad \forall w\in W,\forall\theta\in[0,2\pi];$$

[U7] for every $x\in\mathbb{R}^k$ and $\delta\in W$ there holds

$$\tilde{U}(x+\delta)\leq\tilde{U}\left(\left(\frac{\tilde{U}(\pi_W(x))}{\tilde{U}(x)}\right)^{1/\alpha}\pi_W(x)+\left(\frac{\tilde{U}(x)}{\tilde{U}(\pi_W(x))}\right)^{1/\alpha}\delta\right)$$

where π_W denotes the orthogonal projection onto W.

Theorem. *In addition to [U0], [U1], [U2h], [U3h], [U4h], [U5], assume that, for all $\xi \in \Delta$ [U6] and [U7] hold. Then generalized solutions do not have collisions at the time t^*.*

Remark. As our potential $\tilde{U}$ is homogeneous of degree $-\alpha$ the function

$$\varphi(x) = \tilde{U}^{-1/\alpha}(x)$$

is a non negative, homogeneous of degree one function, having now Δ as zero set. In most of our applications φ will be indeed a quadratic form. Assume that φ^2 splits in the following way:

$$\varphi^2(x) = K|\pi_W(x)|^2 + \varphi^2(\pi_{W^\perp}(x))$$

for some positive constant K. Then [U6] and [U7] are satisfied. Indeed, denoting $w = \pi_W(x)$ and $z = x - w$ we have, for every $\delta \in W$,

$$\begin{aligned}
\varphi^2(x+\delta) &= K|w+\delta|^2 + \varphi^2(z) \\
&= K\left|\frac{\varphi(x)}{\varphi(w)}w + \frac{\varphi(w)}{\varphi(x)}\delta\right|^2 + K\frac{\varphi^2(z)}{\varphi^2(x)}|\delta|^2 \\
&\geq K\left|\frac{\varphi(x)}{\varphi(w)}w + \frac{\varphi(w)}{\varphi(x)}\delta\right|^2 = \varphi^2\left(\frac{\varphi(x)}{\varphi(w)}w + \frac{\varphi(w)}{\varphi(x)}\delta\right),
\end{aligned}$$

which is obviously equivalent to [U7].

Proposition. *Assume* $\tilde{U}(x) = \mathcal{Q}^{-\alpha/2}(x)$ *for some non negative quadratic form* $\mathcal{Q}(x) = \langle Ax, x\rangle$. *Then assumptions [U6] and [U7] are satisfied whenever* W *is included in an eigenspace of* A *associated with a multiple eigenvalue.*

Given two potentials satisfying [U6] and [U7] for a common subspace W, their sum enjoys the same properties.

On the other hand, the class of potentials satisfying [U6] and [U7] is not stable with respect to the sum of potentials. In order to deal with a class of potentials which is closed with respect to the sum, we introduce the following variant of the last Theorem.

Theorem. *In addition to [U0], [U1], [U2h], [U3h], [U4h], [U5], assume that* $\tilde{U}$ *has the form*

$$\tilde{U}(x) = \sum_{\nu=1}^{N} \frac{K_\nu}{(\mathrm{dist}(x, V_\nu))^\alpha}$$

where K_ν *are positive constants and* V_ν *is a family of linear subspaces, with* $\mathrm{codim}(V_\nu) \geq 2$, *for every* $\nu = 1, \ldots, N$. *Then locally minimizing trajectories do not have collisions at the time* t^*.

These two theorems extend to logarithmic potentials.

25 Neumann boundary conditions and G–equivariant minimizers

Our analysis allows to prove that minimizers to the fixed–ends (Bolza) problems are free of collisions: indeed all the variations of our class have compact support. However, other type of boundary conditions (generalized Neumann) can be treated in the same way. Indeed, consider a trajectory which is a (local) minimizer of the action among all paths satisfying the boundary conditions

$$x(0) \in X^0 \qquad x(T) \in X^1,$$

where X^0 and X^1 are two given linear subspaces of the configuration space. Consider a (locally) minimizing path $\bar{x}$: of course it has not interior collisions.

- to exclude boundary collisions we have to be sure that the class of variations preserve the boundary condition;

this can be achieved by imposing assumptions [U6] and [U7] to be fulfilled also by the restriction of the potential to the boundary subspaces X^i.

The analysis of boundary conditions was a key point in the paper [Ferrario and T.], where symmetric periodic trajectories where constructed by reflections about given subspaces. Our results can be used to prove the absence of collisions also for G–equivariant (local) minimizers, provided the group G satisfies the Rotating Circle Property introduced in [FT]. Hence, existence of G–equivariant collisionless periodic solutions can be proved for the wide class of symmetry groups described in [FT, Ferrario, BFT], for a much larger class of interacting potentials, including quasi–homogeneous and logarithmic ones. On the other hand, our results can be applied to prove that G–equivariant minimals are collisionless for many relevant symmetry groups violating the rotating circle property, such as the groups of rotations recently introduced by Ferrario.

Let G_0 be the isotropy group at the collision time, then the blow–up procedure implies the existence of q, a G_0-equivariant minimizing parabolic collision trajectory.

The *standard variation* associated to δ and T is defined as

$$v^\delta(t) = \begin{cases} \delta & \text{if } 0 \le |t| \le T - |\delta| \\ (T-t)\frac{\delta}{|\delta|} & \text{if } T - |\delta| \le |t| \le T \\ 0 & \text{if } |t| \ge T. \end{cases}$$

Our next goal is to find a G_0-equivariant standard variation v^δ such that the trajectory $q + v^\delta$ does not have a collision at $t = 0$ and

$$\Delta\mathcal{A} := \int_{-\infty}^{+\infty} [\mathcal{L}_{\mathbf{k}}(q + v^\delta) - \mathcal{L}_{\mathbf{k}}(q)] dt < 0.$$

Introduce the function

$$S(\xi, \delta) = \int_0^{+\infty} \left(\frac{1}{\left|\xi t^{2/(2+\alpha)} - \delta\right|^\alpha} - \frac{1}{\left|\xi t^{2/(2+\alpha)}\right|^\alpha} \right) dt$$

where $\xi, \delta \in \mathbb{R}^2$.

Theorem: *Let $q = \{q\}_i = \{t^{2/(2+\alpha)}\xi_i\}$, $i = 1, \ldots, k$ be a parabolic collision trajectory and v^δ a G_0-equivariant standard variation. Then, as $\delta \to 0$*

$$\Delta\mathcal{A} = 2|\delta|^{1-\alpha/2} \sum_{\substack{i<j \\ i,j\in\mathbf{k}}} m_i m_j S(\xi_i - \xi_j, \frac{\delta_i - \delta_j}{|\delta|}) + O(|\delta|).$$

We observe that

$$S(\lambda\xi, \mu\delta) = |\lambda|^{-1-\alpha/2} |\mu|^{1-\alpha/2} S(\xi, \delta)$$

and hence the sign of S depends on the angle between ξ and δ. Let

$$\Phi(\vartheta) = \int_0^{+\infty} \frac{1}{\left(t^{\frac{4}{\alpha+2}} - 2\cos\vartheta t^{\frac{2}{\alpha+2}} + 1\right)^{\alpha/2}} - \frac{1}{t^{\frac{2\alpha}{\alpha+2}}} dt, \quad \alpha \in (0, 2)$$

$\Phi(\theta)$ represents the potential differential needed for displacing the colliding particle from zero to $e^{i\theta}$. Expanding, we find

$$\Phi(\vartheta) = \frac{\alpha(\alpha+2)}{2} \left\{ \frac{1}{\alpha - 2} \beta\left(\frac{\alpha+2}{4}, \frac{\alpha+2}{4}\right) + \frac{1}{\alpha} \sum_{k=1}^{+\infty} \binom{-\alpha/2}{k} (-1)^k 2^{k-1} (\cos\vartheta)^k \beta\left(\frac{\alpha}{4} - \frac{1}{2} + \frac{k}{2}, \frac{\alpha}{4} + \frac{1}{2} + \frac{k}{2}\right) \right\}.$$

27 Some properties of Φ

The value of $\Phi(\theta)$ ranges from $+\infty$ to some negative value, depending on α. However, thanks to some harmonic analysis one can prove that suitable *averages* are always negative: the first inequality is particularity useful for dealing with reflected triple collisions from the Lagrange central configuration:

$$\Phi(\frac{2\pi}{3}+\gamma)+\Phi(\frac{2\pi}{3}-\gamma)<0, \quad \forall\gamma\in[0,\pi/2].$$

A key remark was made by Christian Marchal: being the Newton potential a harmonic map averaging it on a sphere results in a truncation in the interior. In fact, is not so much a matter of harmonicity. A crucial estimate was proved in [FT] about the averages of Φ on circles:

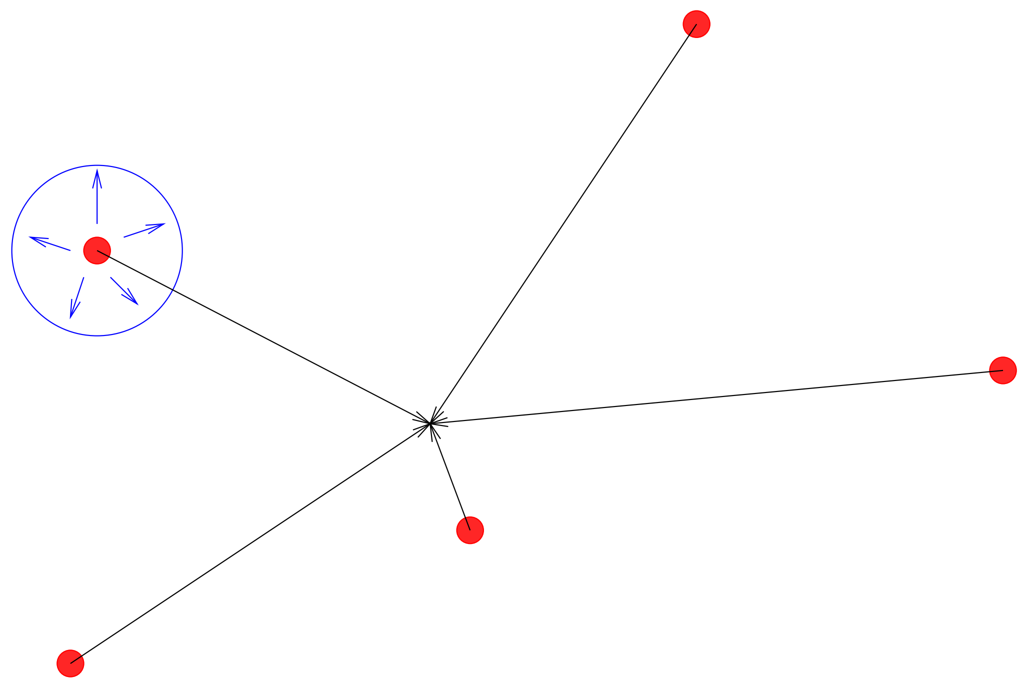
For every $\alpha>0$, $\xi\in\mathbb{R}^3\setminus\{0\}$ and for every circle $\mathbb{S}\subset\mathbb{R}^d$ with center in 0,

$$\tilde{S}(\xi,\mathbb{S})=\frac{1}{|\mathbb{S}|}\int_{\mathbb{S}}S(\xi,\delta)d\delta=|\xi|^{-1-\alpha/2}\,|\delta|^{1-\alpha/2}\frac{1}{2\pi}\int_0^{2\pi}\Phi(\theta)d\theta<0.$$

Consider $\xi=x_i-x_j$ and δ ranging in a circle. Then we obtain the principle, a generalization of the result announced in :

CHENCINER, A., Action minimizing solutions of the Newtonian n–body problem: from homology to symmetry, August 2002, *ICM, Peking*

Marchal's Principle



It is more convenient (from the point of view of the integral of the potential on the time line) to replace one of the point particles with a homogeneous circle of same mass and fixed radius which is moving keeping its center in the position of the original particle

If the action of G on $\mathbb{T}$ and $\mathcal{X}$ fulfills some conditions (computable) then (local) minimizers of the action functional $\mathcal{A}^G$ in $\Lambda^G \subset \Lambda$ do not have collisions.

29 The function $\tilde{S}(\xi, \mathbb{S})$ as a hypergeometric combination

We can write $\tilde{S}(\xi, \mathbb{S})$ in terms of hypergeometric functions as follows:

$$\tilde{S}(\xi, \mathbb{S}) = \frac{1}{2\pi} \int_0^{2\pi} \left(\int_0^1 + \int_1^{+\infty} \right) \left[\frac{1}{|\xi t^{2/(2+\alpha)} + \delta|^\alpha} - \frac{1}{|t^{2/(2+\alpha)} \xi|^\alpha} \right] dt \, d\theta$$

We have

$$\tilde{S}(\xi, \mathbb{S}) = {}_3F_2 \left(\begin{matrix} \alpha/2, & \alpha/2, & (2+\alpha)/4; \\ & 1, & (6+\alpha)/4; \end{matrix} \, 1 \right) - \frac{2+\alpha}{2-\alpha} +$$
$$\frac{2+\alpha}{2-\alpha} \left(1 - {}_3F_2 \left(\begin{matrix} \alpha/2, & \alpha/2, & (\alpha-2)/4; \\ & 1, & (\alpha+2)/4; \end{matrix} \, 1 \right) \right).$$

They are nearly-poised (of the second kind) hypergeometric functions evaluated in 1. They are balanced (i.e. Saalschützian) if and only if $\alpha = 1$.

$$\tilde{S}(\xi, \mathbb{S}) dt = \frac{2+\alpha}{4} \sum_{k=0}^{\infty} \left[\binom{-\alpha/2}{k}^2 \frac{1}{k + \frac{\alpha+2}{4}} \left(\frac{(\alpha/2+k)^2}{(1+k)^2} + 1 \right) \right] - \frac{2+\alpha}{2-\alpha}.$$

The rotating circle property

For a group H acting orthogonally on $\mathbb{R}^d$, a circle $\mathbb{S} \subset \mathbb{R}^d$ (with center in 0) is termed *rotating under H* if $\mathbb{S}$ is invariant under H (that is, for every $g \in H$ $g\mathbb{S} = \mathbb{S}$) and for every $g \in H$ the restriction $g|\mathbb{S} \colon \mathbb{S} \to \mathbb{S}$ is a rotation (the identity is meant as a rotation of angle 0).

Let $i \in \mathbf{n}$ be an index and $H \subset G$ a subgroup. A circle $\mathbb{S} \subset \mathbb{R}^d = V$ (with center in 0) is called *rotating for i under H* if $\mathbb{S}$ is rotating under H and

$$\mathbb{S} \subset V^{H_i} \subset V = \mathbb{R}^d,$$

where $H_i \subset H$ denotes the isotropy subgroup of the index i in H relative to the action of H on the index set $\mathbf{n}$ induced by restriction (that is, the isotropy $H_i = \{g \in H \mid gi = i\}$).

A group G acts with the *rotating circle property* if for every $\mathbb{T}$-isotropy subgroup $G_t \subset G$ and for at least $n - 1$ indexes $i \in \mathbf{n}$ there exists in $\mathbb{R}^d$ a rotating circle $\mathbb{S}$ under G_t for i.

- If the action has the rotating circle property, then for every $g \in G$ the linear map $1 - g$ sends the rotating circle into another circle (thus we can use the averaging trick).
- In most of the known examples the property is fulfilled.
- There are several infinite families with the rotating circle property.

31 Theorems with the RCP

- **Theorem**: Consider a finite group K acting on Λ with the rotating circle property. Then a minimizer of the K-equivariant fixed–ends (Bolza) problem is free of collisions.
- **Corollary**: For every $\alpha > 0$, minimizers of the fixed-ends (Bolza) problem are free of interior collisions.
- **Corollary**: If the action of G on Λ is of cyclic type and $\ker \tau$ has the rotating circle property then any local minimizer of $\mathcal{A}^G$ in Λ^G is collisionless.
- **Corollary**: If the action of G on Λ is of cyclic type and $\ker \tau = 1$ is trivial then any local minimizer of $\mathcal{A}^G$ in Λ^G is collisionless.
- **Theorem**: Consider a finite group G acting on Λ so that every maximal $\mathbb{T}$-isotropy subgroup of G either has the rotating circle property or acts trivially on the index set $\mathbf{n}$. Then any local minimizer of $\mathcal{A}^G$ yields a collision-free periodic solution of the Newton equations for the n-body problem in $\mathbb{R}^d$.