Research statement

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1 Research interests

My research interests encompass various theoretical aspects of nonlinear differential equations, both ordinary and partial derivatives. Generally speaking, I am interested in non trivial solutions, such as self–sustained oscillations or pattern formation, appearing in systems with strong interactions (possibly singular), both in classical and quantum models. More specifically, my favorite research topics include:

- ◇ Variational methods in celestial mechanics:
  - periodic solutions in the N–body problem;
  - parabolic and other selected trajectories;
  - chaotic motions in the N–centers and N–body problems;
  - singular, periodic and regular motions of vortex filaments;
  - collisions and regularization for singular hamiltonian systems;
- ◇ Bifurcations of multi–modal periodic solutions, Birkhoff–Lewis type theorems for chains of interacting particles.
- ◇ Partial Differential Equations:
  - optimal partition problems related to linear and nonlinear eigenvalues;
  - free boundary problems in spatial segregation, regularity of the solutions and the interfaces;
  - applications to competition–diffusion systems in population dynamics and phase segregation of solitary waves of Gross-Pitaevski systems
  - asymptotic behaviour of solutions to many body and multi–vortex linear and nonlinear Schrödinger equations.

In order to know my recent results in some detail, one can download (http://www.matapp.unimib.it/ suster/slides.html) the slides of my talks

- A variational problem for the spatial segregation of reaction–diffusion systems and related problems (Jean Leray international conference, Bedlewo, june 2006)
- On the variational approach to the n-body problem (Dynamics, Topology and Computations, June 2006)
- A function analytic approach to multi-modal periodic trajectories in Fermi-Pasta-Ulam chains (Marseille, february 2008)
- Uniform Hölder bounds and regularity properties of the limiting profile for highly competing nonlinear systems of Schrödinger equations (Postech, South Korea, october 2008)

2 Research plans

2.1 Morse Theory for periodic solutions to singular Hamiltonian Systems

In the recent mathematical literature, the existence of new nontrivial families of periodic trajectories for the N–body problem has been achieved by minimizing the Lagrangian action in spaces of symmetric loops. In the next future, I will try to develop a Morse theoretical approach to periodic solutions to singular hamiltonian systems that takes into account also the contribution of collision solutions. Such a study involves the extension of the gradient flow at the singularity, via some suitable monotonicity lemma, and a topological balance between collision and non collision paths. This approach should enlighten the links between the structure of the collision solutions and the cellular decomposition of the action functional associated with the periodic trajectories. A simplified, still highly nontrivial, problem is the planar N–center problem: in such a case, indeed, collisions can be regularized through the Levi–Civita space–time change of coordinates. A first attempt will be to relate the existence of topologically non trivial periodic trajectories with the Morse complex.

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[BT1] Barutello V., Terracini S., *Action Minimizing Orbits in the N-body problem with simple choreography contstraint,* Nonlinearity 17 (2004), 2015-2039

[T] S. Terracini, *On the variational approach to the periodic N-body problem*, Celestial Mechanics and Dynamical Astronomy , 95 (2006), 1-4, 3-25

[ABT] G. Arioli, V. Barutello and S. Terracini, *A new branch of mountain pass solutions to the choreographical 3-body problem*, Comm. Math. Phys, 268 (2006) 439-463

[BT2] Barutello V, Terracini S., *Double choreographical solutions for n-body type problems*, Celestial Mechanics and Dynamical Astronomy , 95 (2006), 1-4, 67–80

[TV] Terracini S., Venturelli A., *Symmetric trajectories for the* $2N$*-body problem with equal masses*, Arch. Rat. Mech. Anal, 184 (2007) n.3 , 465–493

[BFT] V. Barutello, Ferrario D.L., Terracini S., *Symmetry groups of the planar three-body problems and action-minimizing trajectories*, Arch. Rational Mech. Anal. 190 (2008), 189-226

[BFT2] V. Barutello, Ferrario D.L., Terracini S., *On the singularities of generalized solutions to* n*–body type problem*, Int Math Res Notices.2008 (2008), rnn069-78

2.2 Limiting profiles of mixtures of Bose–Einsitein condensates with strong interaction

Let us consider a family of solutions to nonlinear Schr̈dinger equations of the form

$$\begin{cases} -\Delta u + \lambda u = \omega_1 u^3 - \beta u v^2 \\ -\Delta v + \mu v = \omega_2 v^3 - \beta u^2 v \ . \end{cases} \qquad (1)$$

Such systems arise in different physical applications, such as the study of standing waves in a binary mixture of Bose-Einstein condensates in two (or more) different hyperfine states. While the sign of the parameter ω_i distinguishes between the focusing and defocusing behaviour of a single component, the sign of β determines the type of interplay between the two states. When positive, the two states are in competition and repel each other. I am concerned with diverging interspecific competition rates. The limiting behaviour is known for the ground state solutions: as $\beta \to +\infty$ the wave amplitudes segregate, that is, their supports tend to be disjoint. This phenomenon, called phase separation, has been studied in the literature both in the focusing and the defocusing cases. As far as the excited states are concerned, the recent literature shows that other families of solutions exist for large β's. The asymptotic behaviour of such families of solutions is known and it can be proved the convergence (in Hölder norms, up to subsequences) to a Lipschitz continuous limiting profile satisfying

$$\begin{cases} -\Delta u + \lambda u &= \omega_1 u^3 \quad \text{in } \{u > 0\}, \\ -\Delta v + \mu v &= \omega_2 v^3 \quad \text{in } \{v > 0\}. \end{cases} \tag{2}$$

I believe that the limiting profile should satisfy more stringent conditions at the interface, involving the gradients of the components. Moreover, the (free) interface should be proved to enjoy regularity properties, both in the case of two competing waves and in that of many. Of course, a further very interesting point of investigation concerns the stability of phase segregated solutions for the full systems on nonlinear Schrödinger equations. This requires a completely new approach for, while the free boundary theory for elliptic and parabolic problems is now well developed, the main technical tools (monotonicity formulæ) fail to hold in the case of hyperbolic equations.

[CTV1] Conti M., Terracini S., Verzini G., *Nehari method for PDE's and competing species systems*,Ann. Inst. H. Poincaré Anal. Non Linéaire 19 (2002), no. 6, 871–888,

[CTV2] Conti M., Terracini S., Verzini G., *An optimal partition problem related to some nonlinear eigenvalueproblems*, J. Functional Analysis, 198 (2003) 160-196

[CTV3] Conti M., Terracini S., Verzini G., *A variational problem for the spatial segregation of reaction–diffusion systems*, Indiana Univ. Math. J. 54 (2005), no. 3, 779–815.

[CTV4] Conti M., Terracini S., Verzini G., *On a class of optimal partition problems related to the Fucik spectrum and the monotonicity formulae*, Calc. Var. 22 (2005), n. 1, 45-72

[CTV5] Conti M., Terracini S., Verzini G., *Asymptotic estimates for the spatial segregation of competitive systems*, Adv. Math. 195 (2005), no. 2, 524–560.

[CTV6] Conti M., Terracini S., Verzini G., Uniqueness and Least Energy Property for Solutions to Strongly Competing Systems, *Interfaces and Free Boundaries*, 8 (2006), 437–446

[HHT] B. Helffer, T. Hoffmann-Ostenhof and S. Terracini, *Nodal domains and spectral minimal partitions*, Ann. Inst. H. Poincaré Anal. Non Linéaire, to appear

[GT] S. Terracini and G. Verzini, *Multipulse phases in k–mixtures of Bose–Einstein condensates*, to appear on Arch. Rational Mech. and Anal.

[NT] B. Noris and Terracini S., *Nodal sets of magnetic Schrödinger operators of Aharonov Bohm type and energy minimizing partitions*, preprint (2008)

[NTTV] B. Noris, H. Tavares, Terracini S. and G. Verzini, *Uniform Hölder bounds for nonlinear Schrödinger systems with strong competition*, preprint (2008)

2.3 Schrödinger operators with singular electro-magnetic potentials: exact behaviour of the eigenfunctions

In some recent papers, we studied Schrödinger equations with singular potentials whose poles are located according to some symmetric structure (e.g. on the vertices of regular polygons or distributed along circles), wondering how the symmetry affects their mutual interaction. In [FMT1, FMT2, FMT3], positivity and spectral properties of a class of Schrödinger operators with multipolar inverse-square potentials have been investigated, with and without radial symmetry. In particular we established a necessary and sufficient condition on the masses of singularities for the existence of at least a configuration of poles ensuring the positivity of the associated quadratic form. In the case of Schrödinger operators with multiple dipole potentials, positivity, semiboundedness and localization of binding is established in dependence of the dipole orientation. In particular the problem of the best associated Hardy constant and the spectral and essential self-adjointness properties of this type of operators has been analyzed. All these aspects are deeply related with the exact behaviour of the eigenfunctions at the singular set: in fact, all solutions develop a singularity when the electro-magnetic potentials are at some critical threshold and have the same degree of homogeneity than the laplacian. To this aim a general Cauchy type formula ha been recently estabilshed for eigenfunctions with a general electro-magnetic potential with a one point singularity at the critical threshold([FFT]). I would like to extend these results to the case of operators associated with many-body interactions and see what happens at the thermodynamic limit.

[FT1] Felli, V., Terracini, S., *Nonlinear Schrödinger equations with symmetric multipolar potentials*, Calc. Var. 27 (2006), 25-58

[FT2] Felli V., Terracini S., *Elliptic Equations with multi-singular inverse-square potentials and critical nonlinearity*, Comm. Partial Differential Equations 31 (2006), no. 1-3, 469–495.

[FMT1] V. Felli, E.M. Marchini, and S. Terracini, *On Schrödinger operators with multipolar inverse-square potentials*, J. of Functional Analysis, 250 (2007), 265–316

[FMT2] V. Felli, E.M. Marchini, and S. Terracini, *On Schrödinger operators with multisingular inverse-square anisotropic potentials*, Discrete and Continuous Dynamical Systems A, 21 (2008), 285-313

[FFT] V. Felli, A. Ferrero and S. Terracini, *Asymptotic behavior of solutions to Schrödinger equations near an isolated singularity of the electromagnetic potential*, preprint (2008)

[FMT3] V. Felli, E.M. Marchini, and S. Terracini, *On Schrödinger operators with multisingular inverse-square anisotropic potentials*, Indiana Univ. Math. J., to appear