ELLIPTIC EQUATIONS WITH MULTI-SINGULAR INVERSE-SQUARE POTENTIALS AND CRITICAL NONLINEARITY

VERONICA FELLI AND SUSANNA TERRACINI

ABSTRACT. This paper deals with a class of nonlinear elliptic equations involving a critical power-nonlinearity as well as a potential featuring multiple inverse square singularities. We show that existence of solutions heavily depends on the strength and the location of the singularities. We associate to the problem the corresponding Rayleigh quotient and give both sufficient and necessary conditions on masses and location of singularities for the minimum to be achieved. Both the cases of whole $\mathbb{R}^N$ and bounded domains are taken into account.

1. Introduction and statement of the main results

This paper deals with a class of nonlinear elliptic equations involving a critical power-nonlinearity as well as a potential featuring multiple inverse square singularities:

$$\begin{cases} -\Delta v - \displaystyle\sum_{i=1}^{k} \frac{\lambda_i}{|x-a_i|^2}\, v = v^{2^*-1}, \\ v > 0 \quad \text{in } \mathbb{R}^N \setminus \{a_1, \dots, a_k\}, \end{cases} \tag{1}$$

where $N \geq 3$, $k \in \mathbb{N}$, $(\lambda_1, \lambda_2, \dots, \lambda_k) \in \mathbb{R}^k$, $(a_1, a_2, \dots, a_k) \in \mathbb{R}^{kN}$, and $2^* = \frac{2N}{N-2}$. Among all possible solutions of the problem, we are interested in those having the smallest energy, termed *ground states*. These solutions minimize the Rayleigh quotient associated with problem (1):

$$S(\lambda_1, \lambda_2, \dots, \lambda_k) = \inf_{u \in \mathcal{D}^{1,2}(\mathbb{R}^N) \setminus \{0\}} \frac{\displaystyle\int_{\mathbb{R}^N} |\nabla u|^2 dx - \sum_{i=1}^{k} \lambda_i \int_{\mathbb{R}^N} \frac{u^2(x)}{|x-a_i|^2}\, dx}{\left(\displaystyle\int_{\mathbb{R}^N} |u|^{2^*} dx \right)^{2/2^*}}, \tag{2}$$

where $\mathcal{D}^{1,2}(\mathbb{R}^N)$ denotes the closure space of $C_0^\infty(\mathbb{R}^N)$ with respect to the norm

$$\|u\|_{\mathcal{D}^{1,2}(\mathbb{R}^N)} := \left(\int_{\mathbb{R}^N} |\nabla u|^2\, dx \right)^{1/2}.$$

While the exponent 2^* appears in the inclusion of the Sobolev space $\mathcal{D}^{1,2}(\mathbb{R}^N)$ into $L^{2^*}(\mathbb{R}^N)$, inverse-square potentials are related to the Hardy inequality (see for instance [18, 15]), which

Supported by Italy MIUR, national project "Variational Methods and Nonlinear Differential Equations".
2000 *Mathematics Subject Classification.* 35J60, 35J20, 35B33.

ensures the inclusion of $\mathcal{D}^{1,2}(\mathbb{R}^N)$ into the weighted space $L^2(\mathbb{R}^N, |x|^{-2}\,dx)$ and

$$\int_{\mathbb{R}^N} \frac{|u|^2}{|x|^2}\,dx \leq C_N \int_{\mathbb{R}^N} |\nabla u|^2\,dx, \tag{3}$$

where $C_N = \left(\frac{2}{N-2}\right)^2$ is optimal and not attained.

Problem (1) with only one singularity has been first studied in [27] where it is completely solved. More precisely it is shown that if $\lambda \in [0, (N-2)^2/4)$ then problem

$$\begin{cases} -\Delta u = \dfrac{\lambda}{|x|^2} u + u^{2^*-1}, \quad x \in \mathbb{R}^N, \\ u > 0 \text{ in } \mathbb{R}^N \setminus \{0\}, \text{ and } u \in \mathcal{D}^{1,2}(\mathbb{R}^N), \end{cases} \tag{4}$$

has exactly a one-dimensional $\mathcal{C}^2$ manifold of positive solutions given by

$$Z_\lambda = \left\{ w_\mu^\lambda(x) = \mu^{-\frac{N-2}{2}} w^{(\lambda)}\Big(\frac{x}{\mu}\Big),\ \mu > 0 \right\}, \tag{5}$$

where we denote

$$w^{(\lambda)}(x) = \frac{\big(N(N-2)\nu_\lambda^2\big)^{\frac{N-2}{4}}}{\big(|x|^{1-\nu_\lambda}(1+|x|^{2\nu_\lambda})\big)^{\frac{N-2}{2}}}, \quad \text{and} \quad \nu_\lambda = \left(1 - \frac{4\lambda}{(N-2)^2}\right)^{1/2}. \tag{6}$$

As a matter of facts, all solutions of (4) minimize the associated Rayleigh quotient and the minimum can be computed as:

$$S(\lambda) := \inf_{u \in \mathcal{D}^{1,2}(\mathbb{R}^N)\setminus\{0\}} \frac{Q_\lambda(u)}{\big(\int_{\mathbb{R}^N} |u|^{2^*}dx\big)^{2/2^*}} = \frac{Q_\lambda(w_\mu^\lambda)}{\big(\int_{\mathbb{R}^N} |w_\mu^\lambda|^{2^*}dx\big)^{2/2^*}} = \left(1 - \frac{4\lambda}{(N-2)^2}\right)^{\frac{N-1}{N}} S, \tag{7}$$

where we denoted the quadratic form $Q_\lambda(u) = \int_{\mathbb{R}^N} |\nabla u|^2 dx - \lambda \int_{\mathbb{R}^N} \frac{u^2}{|x|^2} dx$, see [27], and S is the best constant in the Sobolev inequality

$$S\|u\|^2_{L^{2^*}(\mathbb{R}^N)} \leq \|u\|^2_{\mathcal{D}^{1,2}(\mathbb{R}^N)}.$$

It turns out that the solvability of (1) is strongly related to the positivity of the quadratic form associated with the singular potential.

Proposition 1.1. *A necessary condition for the solvability of problem (1) is that the quadratic form*

$$\mathcal{Q}(u) = Q_{\lambda_1,\dots,\lambda_k,a_1,\dots,a_k}(u) := \int_{\mathbb{R}^N} |\nabla u|^2 dx - \sum_{i=1}^k \lambda_i \int_{\mathbb{R}^N} \frac{u^2(x)}{|x-a_i|^2}\,dx$$

is positive semidefinite, i.e

$$\mathcal{Q}(u) \geq 0 \quad \textit{for all } u \in \mathcal{D}^{1,2}(\mathbb{R}^N).$$

To prove existence we shall actually require that $\mathcal{Q}$ is positive definite, i.e. there exists a positive constant $\varepsilon = \varepsilon(\lambda_1, \dots, \lambda_k, a_1, \dots, a_k)$ such that

$$\mathcal{Q}(u) \geq \varepsilon(\lambda_1, \dots, \lambda_k, a_1, \dots, a_k) \int_{\mathbb{R}^N} |\nabla u|^2\,dx. \tag{8}$$

In such a case, from Sobolev's inequality

$$S(\lambda_1, \lambda_2, \dots, \lambda_k) \geq \varepsilon(\lambda_1, \dots, \lambda_k, a_1, \dots, a_k)\, S > 0.$$

In general the positivity of $\mathcal{Q}$ depends on the strength and the location of the singularities.

Proposition 1.2. *A sufficient condition for $\mathcal{Q}$ to be positive definite for any choice of $a_1, a_2, \ldots, a_k$ is that*

$$\sum_{\substack{i=1,\ldots,k\\ \lambda_i>0}} \lambda_i < \frac{(N-2)^2}{4}.$$

Conversely, if

$$\sum_{\substack{i=1,\ldots,k\\ \lambda_i>0}} \lambda_i > \frac{(N-2)^2}{4},$$

then there exist points $a_1, a_2, \ldots, a_k$ such that $\mathcal{Q}$ is not positive definite.

The minimization of the Rayleigh quotients is a non trivial issue, as the embedding $\mathcal{D}^{1,2}(\mathbb{R}^N) \hookrightarrow L^{2^*}(\mathbb{R}^N)$ is not compact. In the case of only one singularity, such a difficulty was overcome in [27] by exploiting the invariances of the problem. This argument is no longer available in the multi–singular case; indeed, while problem (4) is invariant by the rescaling $\mu^{-(N-2)/\mu}u(\cdot/\mu)$ and by the Kelvin transform, problem (1) is not, though it is locally almost-invariant by scaling close to each singularity. This can cause the nonexistence of a minimizer in some circumstances:

Theorem 1.3. *Let $k \geq 2$. Then the infimum in (2) is not achieved in each of the following cases:*

$$\begin{aligned}
&\text{(i)} \quad \lambda_i < 0 \quad \textit{for all } i = 1, \ldots, k-1 \quad \textit{and} \quad 0 \leq \lambda_k < \frac{(N-2)^2}{4},\\
&\text{(ii)} \quad \lambda_i > 0 \quad \textit{for all } i = 1, \ldots, k \quad \textit{and} \quad \sum_{i=1}^{k} \lambda_i < \frac{(N-2)^2}{4}.
\end{aligned}$$

To go further in the analysis, we have deepened the study of the behavior of minimizing sequences, with the aid of P. L. Lions Concentration-Compactness [22, 23]. There are three possible reasons for this lack of compactness: there might be concentration of mass at some non-singular point, at one of the singularities or at infinity. Hence, a minimizing sequence can diverge only when $S(\lambda_1, \ldots, \lambda_k)$ takes one of the values S (concentration at a non-singular point), $S(\lambda_i)$ (concentration at the singular point a_i) or $S\big(\sum_{i=1}^{k} \lambda_i\big)$ (concentration at infinity). Next result provides sufficient conditions for the infimum $S(\lambda_1, \ldots, \lambda_k)$ to stay below all the energy thresholds at which the compactness can be lost.

Theorem 1.4. *Assume that* $(\lambda_1, \lambda_2, \dots, \lambda_k) \in \mathbb{R}^k$ *and* $(a_1, a_2, \dots, a_k) \in \mathbb{R}^{kN}$ *satisfy (8) and the following conditions*

$$\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_k < \frac{(N-2)^2}{4}, \tag{9}$$

$$\begin{cases} \displaystyle\sum_{i \neq k} \frac{\lambda_i}{|a_k - a_i|^2} > 0, & \text{if} \quad 0 < \lambda_k \leq \dfrac{N(N-4)}{4}, \\ \displaystyle\sum_{i \neq k} \frac{\lambda_i}{|a_i - a_k|^{\sqrt{(N-2)^2 - 4\lambda_k}}} > 0, & \text{if} \quad \dfrac{N(N-4)}{4} < \lambda_k < \dfrac{(N-2)^2}{4}, \end{cases} \tag{10}$$

$$\sum_{i \neq k} \lambda_i \leq 0. \tag{11}$$

Then the infimum in (2) is achieved. Therefore equation (1) admits a solution in $\mathcal{D}^{1,2}(\mathbb{R}^N)$.

Hence the existence of a minimizer over the whole $\mathbb{R}^N$ heavily depends on the strength and the location of the singularities. The result above applies, for example, when the point carrying the largest positive mass is surrounded by other positive singularities, while negative singularities do appear, but far away. We note that the assumptions of Theorem 1.4 exclude the case of exactly two poles; on the other hand in such a case Theorem 1.3 implies that the infimum in (2) is not achieved.

Surprisingly enough, and in contrast with the case of only one singularity, the phenomenon of loss of compactness becomes less dramatic when working on bounded domains: indeed the infimum can be achieved also in case of all positive masses, as the following existence result shows.

Theorem 1.5. *Assume that* Ω *is a bounded smooth domain,* $\{a_1, a_2, \dots, a_k\} \subset \Omega$,

$$\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_k \leq \frac{N(N-4)}{4}, \quad \text{and} \quad \sum_{i \neq k} \frac{\lambda_i}{|a_k - a_i|^2} > 0,$$

and the quadratic form

$$Q_\Omega(u) =:= \int_\Omega |\nabla u|^2 dx - \sum_{i=1}^k \lambda_i \int_\Omega \frac{u^2(x)}{|x - a_i|^2}\, dx \quad \text{is positive definite.} \tag{12}$$

Then the infimum in

$$S_\Omega(\lambda_1, \lambda_2, \dots, \lambda_k) = \inf_{u \in H_0^1(\Omega) \setminus \{0\}} \frac{\displaystyle\int_\Omega |\nabla u|^2 dx - \sum_{i=1}^k \lambda_i \int_\Omega \frac{u^2(x)}{|x - a_i|^2}\, dx}{\left(\displaystyle\int_\Omega |u|^{2^*} dx\right)^{2/2^*}}, \tag{13}$$

is achieved. Therefore equation

$$\begin{cases} -\Delta u - \displaystyle\sum_{i=1}^{k} \frac{\lambda_i}{|x-a_i|^2}\, v = v^{2^*-1}, \\ v > 0 \quad \text{in } \Omega \setminus \{a_1, \dots, a_k\}, \quad v = 0 \quad \text{on } \partial\Omega, \end{cases} \tag{14}$$

admits a solution in $H^1_0(\Omega)$.

The presence of the bound $\lambda_k \leq N(N-4)/4$ does not come unexpected, as it is related to some peculiar phenomena occurring in concentration and cutting-off of our test functions at the singularities as observed also in [19, 16, 14]. This restriction can be removed letting Ω containing a sufficiently large ball. Let $B(0, R)$ denote the ball $\{x \in \mathbb{R}^N : |x| < R\}$.

Theorem 1.6. *Assume that* $\lambda_1 \leq \dots \leq \lambda_k$*, (12) and* (10) *hold. Then there exists* $R > 0$ *such that if* $\Omega \supset B(0, R)$*, the infimum in* (13) *is achieved. Therefore equation (14) admits a solution in* $H^1_0(\Omega)$.

Singular potentials appear in several fields of applications and have been the object of a wide recent mathematical research. Besides the already mentioned papers [14, 15, 19, 27], we quote, among others, [1, 9, 10, 12, 25, 26]. Equation (1) can deserve as a model for many problems coming from Quantum Mechanics, Chemistry, Cosmology, Astrophysics and Differential Geometry.

Equation (1) is characterized by the presence of inverse-square multi-singular potentials, whose physical relevance is briefly described below. Potentials of the type $1/|x|^2$ arise in many fields, such as quantum mechanics, nuclear physics, molecular physics, and quantum cosmology. The relevance of singular potentials in nonrelativistic quantum mechanics is highlighted in [17], where a classification of spherically symmetric potentials $V(|x|)$ is given by considering the limit

$$\lim_{r \to 0} r^2 V(r). \tag{15}$$

The potential $V(r)$ is said to be *regular* at 0 if the limit in (15) is 0 and *singular* if such a limit is $\pm\infty$. When the limit in (15) is finite and different from 0 (as in the case of inverse square potentials) the potential is said to be a *transition potential*. Moreover we say that the potential V is attractively (respectively repulsively) singular when the limit in (15) is $-\infty$ (respectively $+\infty$).

A simple argument by Landau and Lifshitz (see [17] and[20]) explains why $1/r^2$ can be regarded as the transition threshold in the classification of singular potentials in a nonrelativistic context. Let us consider a particle near the origin in the presence of a potential $1/r^m$. From the *Uncertainty Principle*, its kinetic energy scales like r^{-2}, so that the energy is approximatively given by $r^{-2} + \lambda r^{-m}$. For $\lambda < 0$ and $m > 2$ (attractively singular potential), the energy is not lower-bounded and the particle "falls" to the center. On the other hand, if $m < 2$ the discrete spectrum has a lower bound.

The potential $1/r^2$ also arises in point-dipole interactions in molecular physics (see [21]), where the interactions between the charge of the electron and the dipole moment of the molecule gives rise to long-distance forces and to the presence of an inverse-square potential in the Schrödinger equation for the wave function of the electron. We also mention that inverse-square singular potentials appear in the linearization of standard combustion models leading to blow-up phenomenon (see [3, 15, 28]) and in quantum cosmological models such as the Wheeler-de-Witt equation (see [4]).

In Quantum Chemistry, multi-singular potentials arise for example when considering molecular systems consisting of k nuclei of unit charge located at a finite number of points $a_1, \dots, a_k$ and of k electrons. This type of systems are described by the Hartree-Fock model, where Coulomb multi-singular potentials arise in correspondence to the interactions between the electrons and the fixed nuclei, see [8, 24].

We also mention that Schödinger operators with multipolar inverse square singular potentials are studied in [11], where estimates on resolvant truncated at high frequencies are proved.

The presence of the nonlinear term in equation (1) is motivated by the fact that in some physical problems interaction phenomena (e.g. the presence of many particles interacting in quantum physics or the possible joining or splitting of different universes in quantum cosmology) lead to nonlinear terms, which are power-type in a first approximation. As far as the meaning of the critical exponent is concerned, see [5].

Let us finally remark that (1) has also a geometric motivation as it is related to the Yamabe problem on the sphere $\mathbb{S}^N$. Indeed, if we identify $\mathbb{R}^N$ with $\mathbb{S}^N$ through the stereographic projection and endow $\mathbb{S}^N$ with a metric whose scalar curvature is singular at the north pole and at a finite number of points, then the problem of finding a conformal metric with prescribed scalar curvature 1 leads to solve equation (1), where the unknown v has the meaning of a conformal factor (see [2]).

The paper is organized as follows. In section 2 we prove that the Palais-Smale condition is satisfied below some critical threshold involving S, $S(\lambda_i)$, and $S\big(\sum_{i=1}^k \lambda_i\big)$. Section 3 contains some interaction estimates and the proof of Theorem 1.4. Sections 4, respectively 5, are devoted to the proof of Proposition 1.1, respectively Theorem 1.3. Section 6 deals with multi-singular problems in bounded domains. Finally in the Appendix we prove some technical estimates stated in Section 3.

2. The Palais-Smale condition

Let us introduce the functional

$$J(u) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 dx - \sum_{i=1}^k \frac{\lambda_i}{2}\int_{\mathbb{R}^N} \frac{u^2(x)}{|x-a_i|^2}\, dx - \frac{S(\lambda_1,\lambda_2,\dots,\lambda_k)}{2^*}\int_{\mathbb{R}^N} |u|^{2^*} dx. \tag{16}$$

If u is a critical point of J in $\mathcal{D}^{1,2}(\mathbb{R}^N)$, $u > 0$, then $v = S(\lambda_1,\lambda_2,\dots,\lambda_k)^{1/(2^*-2)}u$ is a solution to equation (1). The following theorem provides a local Palais-Smale condition for J below some critical threshold.

Theorem 2.1. *Assume (8). Let $\{u_n\}_{n\in\mathbb{N}} \subset \mathcal{D}^{1,2}(\mathbb{R}^N)$ be a Palais-Smale sequence for J, namely*

$$\lim_{n\to\infty} J(u_n) = c < \infty \text{ in } \mathbb{R} \quad \text{and} \quad \lim_{n\to\infty} J'(u_n) = 0 \text{ in the dual space } (\mathcal{D}^{1,2}(\mathbb{R}^N))^\star.$$

If

$$c < c^* = \frac{1}{N} S(\lambda_1,\lambda_2,\dots,\lambda_k)^{1-\frac{N}{2}} \min\Big\{S, S(\lambda_1),\dots,S(\lambda_k), S\Big(\sum\nolimits_{j=1}^k \lambda_j\Big)\Big\}^{N/2}, \tag{17}$$

then $\{u_n\}_{n\in\mathbb{N}}$ has a converging subsequence.

Proof. Let $\{u_n\}$ be a Palais-Smale sequence for J, then from (8) there exists some positive constant c_1 such that

$$c_1\|u_n\|^2_{\mathcal{D}^{1,2}(\mathbb{R}^N)} \le \int_{\mathbb{R}^N}|\nabla u|^2dx - \sum_{i=1}^k \lambda_i \int_{\mathbb{R}^N}\frac{u^2(x)}{|x-a_i|^2}\,dx = NJ(u_n) - \frac{N-2}{2}\langle J'(u_n),u_n\rangle$$
$$= Nc + o(\|u_n\|_{\mathcal{D}^{1,2}(\mathbb{R}^N)}) + o(1)$$

hence $\{u_n\}$ is a bounded sequence in $\mathcal{D}^{1,2}(\mathbb{R}^N)$. Then, up to a subsequence, we have

$$u_n \rightharpoonup u_0 \text{ in } \mathcal{D}^{1,2}(\mathbb{R}^N), \quad u_n \to u_0 \text{ almost everywhere, and} \quad u_n \to u_0 \text{ in } L^\alpha_{loc} \text{ for any } \alpha\in[1,2^*).$$

Therefore, from the *Concentration Compactness Principle* by P. L. Lions, (see [22] and [23]), we deduce the existence of a subsequence, still denoted by $\{u_n\}$, for which there exist an at most countable set $\mathcal{J}$, $x_j\in\mathbb{R}^N\setminus\{a_1,\dots,a_k\}$, real numbers μ_{x_j},ν_{x_j}, $j\in\mathcal{J}$, and $\mu_{a_i},\nu_{a_i},\gamma_i$, $i=1,\dots,k$ such that the following convergences hold in the sense of measures

$$|\nabla u_n|^2 \rightharpoonup d\mu \ge |\nabla u_0|^2 + \sum_{i=1}^k \mu_{a_i}\delta_{a_i} + \sum_{j\in\mathcal{J}}\mu_{x_j}\delta_{x_j}, \tag{18}$$

$$|u_n|^{2^*} \rightharpoonup d\nu = |u_0|^{2^*} + \sum_{i=1}^k \nu_{a_i}\delta_{a_i} + \sum_{j\in\mathcal{J}}\nu_{x_j}\delta_{x_j}, \tag{19}$$

$$\lambda_i\frac{u_n^2}{|x-a_i|^2} \rightharpoonup d\gamma_{a_i} = \lambda_i\frac{u_0^2}{|x-a_i|^2} + \gamma_i\delta_{a_i}, \quad \text{for any} \quad i=1,\dots,k. \tag{20}$$

From Sobolev's inequality it follows that

$$S\nu_{x_j}^{\frac{2}{2^*}} \le \mu_{x_j} \text{ for all } j\in\mathcal{J} \quad\text{and}\quad S\nu_{a_i}^{\frac{2}{2^*}} \le \mu_{a_i} \text{ for all } i=1,\dots,k. \tag{21}$$

To study the concentration at infinity of the sequence we also introduce the following quantities

$$\nu_\infty = \lim_{R\to\infty}\limsup_{n\to\infty}\int_{|x|>R}|u_n|^{2^*}dx, \quad \mu_\infty = \lim_{R\to\infty}\limsup_{n\to\infty}\int_{|x|>R}|\nabla u_n|^2dx$$

and

$$\gamma_\infty = \lim_{R\to\infty}\limsup_{n\to\infty}\int_{|x|>R}\Big(\sum_{i=1}^k\lambda_i\Big)\frac{u_n^2}{|x|^2}dx.$$

Claim 1. We claim that

$$\mathcal{J} \quad \text{is finite and for } j\in\mathcal{J} \text{ either } \nu_{x_j}=0 \text{ or } \nu_{x_j}\ge\Big(\frac{S}{S(\lambda_1,\lambda_2,\dots,\lambda_k)}\Big)^{N/2}. \tag{22}$$

For $\varepsilon>0$, let ϕ_j be a smooth cut-off function centered at x_j, $0\le\phi_j(x)\le1$ such that

$$\phi_j(x)=1 \quad \text{if } |x-x_j|\le\frac{\varepsilon}{2}, \quad \phi_j(x)=0 \quad \text{if } |x-x_j|\ge\varepsilon, \quad\text{and}\quad |\nabla\phi_j|\le\frac{4}{\varepsilon}.$$

Testing $J'(u_n)$ with $u_n\phi_j$ we obtain

$$0 = \lim_{n\to\infty}\langle J'(u_n),u_n\phi_j\rangle$$
$$= \lim_{n\to\infty}\bigg[\int_{\mathbb{R}^N}|\nabla u_n|^2\phi_j + \int_{\mathbb{R}^N}u_n\nabla u_n\cdot\nabla\phi_j - \int_{\mathbb{R}^N}\sum_{i=1}^k\frac{\lambda_i u_n^2\phi_j}{|x-a_i|^2} - S(\lambda_1,\lambda_2,\dots,\lambda_k)\int_{\mathbb{R}^N}\phi_j|u_n|^{2^*}\bigg].$$

From (18–20), and since $a_i \notin \operatorname{supp}(\phi_j)$ for all $i = 1, \dots, k$ provided ε is sufficiently small, we find that

$$\lim_{n\to\infty}\int_{\mathbb{R}^N} |\nabla u_n|^2\phi_j = \int_{\mathbb{R}^N}\phi_j\,d\mu, \quad \lim_{n\to\infty}\int_{\mathbb{R}^N}|u_n|^{2^*}\phi_j = \int_{\mathbb{R}^N}\phi_j\,d\nu,$$

and

$$\lim_{n\to\infty}\int_{B_\varepsilon(x_j)}\sum_{i=1}^k \frac{\lambda_i u_n^2\phi_j}{|x-a_i|^2} = \int_{B_\varepsilon(x_j)}\sum_{i=1}^k \frac{\lambda_i u_0^2\phi_j}{|x-a_i|^2}.$$

Taking limits as $\varepsilon \to 0$ we obtain

$$\lim_{\varepsilon\to 0}\lim_{n\to\infty}\left|\int_{\mathbb{R}^N} u_n\nabla u_n\nabla\phi_j\right| \to 0.$$

Hence

$$0 = \lim_{\varepsilon\to 0}\lim_{n\to\infty}\langle J'(u_n), u_n\phi\rangle \geq \mu_{x_j} - S(\lambda_1,\lambda_2,\dots,\lambda_k)\nu_{x_j}.$$

By (21) we have that $S\nu_{x_j}^{\frac{2}{2^*}} \leq \mu_{x_j}$, then we obtain that either $\nu_{x_j} = 0$ or $\nu_{x_j} \geq \big(\frac{S}{S(\lambda_1,\lambda_2,\dots,\lambda_k)}\big)^{N/2}$, which implies that $\mathcal{J}$ is finite. Claim 1 is proved.

Claim 2. We claim that

(23) $$\text{for each} \quad i = 1,2,\dots,k \quad \text{either} \quad \nu_{a_i} = 0 \quad \text{or} \quad \nu_{a_i} \geq \Big(\frac{S(\lambda_i)}{S(\lambda_1,\lambda_2,\dots,\lambda_k)}\Big)^{N/2}.$$

In order to prove claim 2, for each $i = 1,2,\dots,k$ we consider a smooth cut-off function ψ_i satisfying $0 \leq \psi_i(x) \leq 1$,

$$\psi_i(x) = 1 \quad \text{if } |x-a_i| \leq \frac{\varepsilon}{2}, \quad \psi_i(x) = 0 \quad \text{if } |x-a_i| \geq \varepsilon, \quad \text{and} \quad |\nabla\psi_i| \leq \frac{4}{\varepsilon}.$$

From (7) we obtain that

(24) $$\frac{\int_{\mathbb{R}^N}|\nabla(u_n\psi_i)|^2dx - \lambda_i\int_{\mathbb{R}^N}\frac{\psi_i^2u_n^2}{|x-a_i|^2}dx}{\Big(\int_{\mathbb{R}^N}|\psi_iu_n|^{2^*}\Big)^{2/2^*}} \geq S(\lambda_i)$$

hence

$$\int_{\mathbb{R}^N}\psi_i^2|\nabla u_n|^2dx + \int_{\mathbb{R}^N}u_n^2|\nabla\psi_i|^2\,dx + 2\int_{\mathbb{R}^N}u_n\psi_i\nabla u_n\cdot\nabla\psi_i dx$$
$$\geq \lambda_i\int_{\mathbb{R}^N}\frac{\psi_i^2u_n^2}{|x-a_i|^2}\,dx + S(\lambda_i)\Big(\int_{\mathbb{R}^N}|\psi_iu_n|^{2^*}\Big)^{2/2^*}.$$

It is easy to verify that

$$\lim_{\varepsilon\to 0}\limsup_{n\to\infty}\left[\int_{\mathbb{R}^N}u_n^2|\nabla\psi_i|^2\,dx + 2\int_{\mathbb{R}^N}u_n\psi_i\nabla u_n\cdot\nabla\psi\,dx\right] = 0.$$

Then from (18–20) we obtain

(25) $$\mu_{a_i} \geq \gamma_i + S(\lambda_i)\nu_{a_i}^{2/2^*}.$$

Testing $J'(u_n)$ with $u_n\psi_i$ we infer

$$0 = \lim_{n\to\infty} \langle J'(u_n), u_n\psi_i\rangle$$

$$= \lim_{n\to\infty} \left[\int_{\mathbb{R}^N} |\nabla u_n|^2\psi_i + \int_{\mathbb{R}^N} u_n \nabla u_n \cdot \nabla \psi_i - \int_{\mathbb{R}^N} \sum_{j=1}^k \frac{\lambda_j u_n^2 \psi_i}{|x-a_j|^2} - S(\lambda_1, \lambda_2, \dots, \lambda_k) \int_{\mathbb{R}^N} \psi_i |u_n|^{2^*} \right].$$

Hence from (18), (19) and the following fact

$$\lim_{\varepsilon\to 0} \limsup_{n\to\infty} \int_{\mathbb{R}^N} \frac{\lambda_j u_n^2 \psi_i}{|x-a_j|^2}\, dx = \begin{cases} 0 & \text{if } i \neq j \\ \gamma_i & \text{if } i = j \end{cases}$$

(which easily follows from (20)), we deduce that

$$\mu_{a_i} - \gamma_i \leq S(\lambda_1, \lambda_2, \dots, \lambda_k)\nu_{a_i}. \tag{26}$$

From (25) and (26) we conclude that either $\nu_{a_i} = 0$ or $\nu_{a_i} \geq \big(\frac{S(\lambda_i)}{S(\lambda_1,\lambda_2,\dots,\lambda_k)}\big)^{\frac{N}{2}}$. Claim 2 is thereby proved.

Claim 3. We claim that

$$\text{either} \quad \nu_\infty = 0 \quad \text{or} \quad \nu_\infty \geq \left(\frac{S(\sum_{i=1}^k \lambda_i)}{S(\lambda_1, \lambda_2, \dots, \lambda_k)} \right)^{N/2}. \tag{27}$$

In order to prove claim 3, we study the possibility of concentration at ∞. Let ψ be a regular cut-off function such that

$$0 \leq \psi(x) \leq 1, \quad \psi(x) = \begin{cases} 1, & \text{if } |x| > 2R \\ 0, & \text{if } |x| < R, \end{cases} \qquad \text{and} \quad |\nabla\psi| \leq \frac{2}{R}.$$

From (7) we obtain that

$$\frac{\int_{\mathbb{R}^N} |\nabla(u_n\psi)|^2 dx - \Big(\sum_{i=1}^k \lambda_i\Big) \int_{\mathbb{R}^N} \frac{\psi^2 u_n^2}{|x|^2} dx}{\Big(\int_{\mathbb{R}^N} |\psi u_n|^{2^*}\Big)^{2/2^*}} \geq S\Big(\sum_{i=1}^k \lambda_i\Big). \tag{28}$$

Therefore we have

$$\begin{aligned} &\int_{\mathbb{R}^N} \psi^2 |\nabla u_n|^2 dx + \int_{\mathbb{R}^N} u_n^2 |\nabla\psi|^2 dx + 2\int_{\mathbb{R}^N} u_n \psi \nabla u_n \cdot \nabla\psi dx \\ &\qquad \geq \Big(\sum_{i=1}^k \lambda_i\Big) \int_{\mathbb{R}^N} \frac{\psi^2 u_n^2}{|x|^2} dx + S\Big(\sum_{i=1}^k \lambda_i\Big)\Big(\int_{\mathbb{R}^N} |\psi u_n|^{2^*}\Big)^{2/2^*}. \end{aligned} \tag{29}$$

We claim that

$$\lim_{R\to\infty} \limsup_{n\to\infty} \left\{ \int_{\mathbb{R}^N} u_n^2 |\nabla\psi|^2 dx + 2\int_{\mathbb{R}^N} u_n\psi\nabla u_n \cdot \nabla\psi\, dx \right\} = 0.$$

Indeed using Hölder inequality we obtain

$$\int_{\mathbb{R}^N} |u_n||\psi|\nabla u_n||\nabla\psi| dx \leq \left(\int_{R<|x|<2R} |u_n|^2 |\nabla\psi|^2 dx \right)^{1/2} \left(\int_{R<|x|<2R} |\nabla u_n|^2 dx \right)^{1/2}.$$

Hence

$$\begin{aligned}\limsup_{n\to\infty}\int_{\mathbb{R}^N}|u_n||\psi||\nabla u_n||\nabla\psi|\,dx &\leq \text{const}\left(\int_{R<|x|<2R}|u_0|^2|\nabla\psi|^2\,dx\right)^{1/2}\\ &\leq \text{const}\left(\int_{R<|x|<2R}|u_0|^{2^*}\,dx\right)^{1/2^*}\left(\int_{R<|x|<2R}|\nabla\psi|^N\,dx\right)^{1/N}\\ &\leq \text{const}\left(\int_{R<|x|<2R}|u_0|^{2^*}dx\right)^{1/2^*}.\end{aligned}$$

Therefore we conclude that

$$\lim_{R\to\infty}\limsup_{n\to\infty}\int_{\mathbb{R}^N}|u_n||\psi||\nabla u_n||\nabla\psi|\,dx \leq \text{const}\lim_{R\to\infty}\left(\int_{R<|x|<2R}|u_0|^{2^*}dx\right)^{1/2^*}=0.$$

Using the same argument we can prove that

$$\lim_{R\to\infty}\limsup_{n\to\infty}\int_{\mathbb{R}^N}u_n^2|\nabla\psi|^2=0.$$

Then from (29) we infer

$$\mu_\infty-\gamma_\infty\geq S\Big(\textstyle\sum_{i=1}^k\lambda_i\Big)\nu_\infty^{2/2^*}. \tag{30}$$

Testing $J'(u_n)$ with $u_n\psi$ we obtain

$$\begin{aligned}0&=\lim_{n\to\infty}\langle J'(u_n),u_n\psi\rangle\\ &=\lim_{n\to\infty}\left[\int_{\mathbb{R}^N}|\nabla u_n|^2\psi+\int_{\mathbb{R}^N}u_n\nabla u_n\cdot\nabla\psi-\int_{\mathbb{R}^N}\sum_{j=1}^k\frac{\lambda_j u_n^2\psi}{|x-a_j|^2}-S(\lambda_1,\lambda_2,\dots,\lambda_k)\int_{\mathbb{R}^N}\psi|u_n|^{2^*}\right].\end{aligned}$$

Since

$$\left|\frac{u_n^2\psi}{|x-a_j|^2}-\frac{u_n^2\psi}{|x|^2}\right|=\frac{u_n^2\psi}{|x|^2}\,\frac{\big||x|^2-|x-a_j|^2\big|}{|x-a_j|^2}\leq\tilde c\,\frac{u_n^2\psi}{|x|^3}$$

for some constant $\tilde c$ independent of R, and by Hölder's inequality

$$\int_{\mathbb{R}^N}\frac{u_n^2\psi}{|x|^3}\,dx\leq\left(\int_{|x|>R}u_n^{2^*}\right)^{2/2^*}\left(\int_{|x|>R}|x|^{-\frac{3}{2}N}\right)^{2/N}=O(R^{-1}),$$

we deduce that

$$\lim_{R\to+\infty}\limsup_{n\to\infty}\int_{\mathbb{R}^N}\sum_{j=1}^k\frac{\lambda_j u_n^2\psi}{|x-a_j|^2}=\lim_{R\to+\infty}\limsup_{n\to\infty}\int_{\mathbb{R}^N}\sum_{j=1}^k\frac{\lambda_j u_n^2\psi}{|x|^2}=\gamma_\infty. \tag{31}$$

Hence from (18), (19) and (31) we deduce that

$$\mu_\infty-\gamma_\infty\leq S(\lambda_1,\lambda_2,\dots,\lambda_k)\nu_\infty. \tag{32}$$

From (30) and (32) we conclude that either $\nu_\infty=0$ or $\nu_\infty\geq\big(\frac{S(\sum_{i=1}^k\lambda_i)}{S(\lambda_1,\lambda_2,\dots,\lambda_k)}\big)^{\frac{N}{2}}$, thus proving Claim 3.

As a conclusion we obtain

$$\begin{aligned} c &= J(u_n) - \frac{1}{2}\langle J'(u_n), u_n\rangle + o(1) \\ &= \frac{1}{N} S(\lambda_1, \lambda_2, \dots, \lambda_k) \int_{\mathbb{R}^N} |u_n|^{2^*} dx + o(1) \\ &= \frac{S(\lambda_1, \lambda_2, \dots, \lambda_k)}{N} \Big\{ \int_{\mathbb{R}^N} |u_0|^{2^*} dx + \sum_{i=1}^{k} \nu_{a_i} + \nu_\infty + \sum_{j\in\mathcal{J}} \nu_{x_j} \Big\}. \end{aligned} \tag{33}$$

From (17), (33), (22), (23), and (27), we deduce that $\nu_{x_j} = 0$ for any $j \in \mathcal{J}$, $\nu_{a_i} = 0$ for any $i = 1, \dots, k$, and $\nu_\infty = 0$. Then up to a subsequence $u_n \to u_0$ in $\mathcal{D}^{1,2}(\mathbb{R}^N)$. ■

3. Interaction estimates and proof of Theorem 1.4

In order to prove Theorem 1.4 some interaction estimates are needed. To begin with, we state a continuity lemma related to Hardy's inequality, which can be proved in a standard way using continuity properties of convolution and density arguments.

Proposition 3.1. *For any $\phi \in \mathcal{D}^{1,2}(\mathbb{R}^N)$, there holds*

$$\lim_{|\xi|\to 0} \int_{\mathbb{R}^N} \frac{\phi^2(x)}{|x+\xi|^2}\, dx = \int_{\mathbb{R}^N} \frac{\phi^2(x)}{|x|^2}\, dx. \tag{34}$$

As minimizers of problem (7), we consider

$$z_\mu^\lambda(x) = \frac{w_\mu^\lambda(x)}{\big(\int_{\mathbb{R}^N} |w_\mu^\lambda|^{2^*} dx\big)^{1/2^*}} = \alpha_{\lambda,N}\, \mu^{-\frac{N-2}{2}} \left(\Big|\frac{x}{\mu}\Big|^{1-\nu_\lambda} + \Big|\frac{x}{\mu}\Big|^{1+\nu_\lambda} \right)^{-\frac{N-2}{2}} \tag{35}$$

where $\alpha_{\lambda,N} = \big(N(N-2)\nu_\lambda^2\big)^{\frac{N-2}{4}} \|w^{(\lambda)}\|_{L^{2^*}}^{-1}$ is a positive constant depending only on λ and N. A direct consequence of Proposition 3.1 is the following corollary.

Corollary 3.2. *For any $\xi \in \mathbb{R}^N$ and $\lambda \in (-\infty, (N-2)^2/4)$ there holds*

$$\int_{\mathbb{R}^N} \frac{|z_\mu^\lambda(x)|^2}{|x+\xi|^2}\, dx = \int_{\mathbb{R}^N} \frac{|z_\mu^\lambda(x)|^2}{|x|^2}\, dx + o(1) \quad \text{as } \mu \to +\infty. \tag{36}$$

Proof. By the change of variable $x = \mu y$ we have

$$\int_{\mathbb{R}^N} \frac{|z_\mu^\lambda(x)|^2}{|x+\xi|^2}\, dx = \mu^{-(N-2)} \int_{\mathbb{R}^N} \frac{|z_1^\lambda(x/\mu)|^2}{|x+\xi|^2}\, dx = \int_{\mathbb{R}^N} \frac{|z_1^\lambda(y)|^2}{\big|y+\frac{\xi}{\mu}\big|^2}\, dy.$$

Since $\frac{\xi}{\mu} \to 0$ as $\mu \to +\infty$, the corollary follows from Proposition 3.1. ■

As a consequence of the above corollary, we can bound $S(\lambda_1, \dots, \lambda_k)$ from above by $S(\sum_{i=1}^k \lambda_i)$.

Corollary 3.3. *Assume that $(a_1, a_2, \dots, a_k) \in \mathbb{R}^{kN}$ and $(\lambda_1, \lambda_2, \dots, \lambda_k) \in \mathbb{R}^k$ satisfy $\sum_{i=1}^k \lambda_i \in (0, (N-2)^2/4)$. Then*

$$S(\lambda_1, \dots, \lambda_k) \le S(\textstyle\sum_{i=1}^k \lambda_i).$$

Proof. Let us set $\tilde\lambda=\sum_{i=1}^k\lambda_i$. From Corollary 3.2 it follows that

$$S(\lambda_1,\dots,\lambda_k)\le\int_{\mathbb{R}^N}|\nabla z_\mu^{\tilde\lambda}|^2\,dx-\sum_{i=1}^k\lambda_i\int_{\mathbb{R}^N}\frac{|z_\mu^{\tilde\lambda}|^2}{|x-a_i|^2}\,dx$$
$$=\int_{\mathbb{R}^N}|\nabla z_\mu^{\tilde\lambda}|^2\,dx-\Big(\sum_{i=1}^k\lambda_i\Big)\int_{\mathbb{R}^N}\frac{|z_\mu^{\tilde\lambda}|^2}{|x|^2}\,dx+o(1)=S(\textstyle\sum_{i=1}^k\lambda_i)+o(1)$$

as $\mu\to\infty$. Letting $\mu\to\infty$, we obtain the desired estimate. ■

The following lemma describes the behavior of $\int|x+\xi|^{-2}|z_\mu^\lambda|^2$ as $\mu\to0$. The proof is quite technical and is contained in the appendix.

Lemma 3.4. *For any $\xi\in\mathbb{R}^N$ there holds*

$$\int_{\mathbb{R}^N}\frac{|z_\mu^\lambda|^2}{|x+\xi|^2}\,dx=\begin{cases}\frac{\mu^2}{|\xi|^2}\int_{\mathbb{R}^N}|z_1^\lambda|^2\,dx+o\left(\mu^2\right) & \text{if } \lambda<\frac{N(N-4)}{4},\\ \alpha_{\lambda,N}^2\,\frac{\mu^2|\ln\mu|}{|\xi|^2}+o(\mu^2|\ln\mu|) & \text{if } \lambda=\frac{N(N-4)}{4},\\ \alpha_{\lambda,N}^2\,\beta_{\lambda,N}\,\mu^{\sqrt{(N-2)^2-4\lambda}}|\xi|^{-\sqrt{(N-2)^2-4\lambda}}+o\big(\mu^{\sqrt{(N-2)^2-4\lambda}}\big) & \text{if } \lambda>\frac{N(N-4)}{4},\end{cases}$$

as $\mu\to0$, where

$$\beta_{\lambda,N}=\int_{\mathbb{R}^N}\frac{dx}{|x|^2|x-e_1|^{N-2+\sqrt{(N-2)^2-4\lambda}}},\qquad e_1=(1,0,\dots,0)\in\mathbb{R}^N.$$

From Lemma 3.4, we deduce the following corollary which provides sufficient conditions for $S(\lambda_1,\dots,\lambda_k)$ to stay below $S(\lambda_j)$.

Corollary 3.5. *Let $j\in\{1,2,\dots,k\}$. If $\lambda_j>0$ and one of the following assumptions is satisfied*

$$0<\lambda_j\le\frac{N(N-4)}{4}\quad\text{and}\quad\sum_{i\neq j}\frac{\lambda_i}{|a_j-a_i|^2}>0,\tag{37}$$

$$\frac{N(N-4)}{4}<\lambda_j<\frac{(N-2)^2}{4}\quad\text{and}\quad\sum_{i\neq j}\frac{\lambda_i}{|a_j-a_i|^{\sqrt{(N-2)^2-4\lambda_j}}}>0,\tag{38}$$

then

$$S(\lambda_1,\dots,\lambda_k)<S(\lambda_j).$$

PROOF. Under assumption (37), from Lemma 3.4 we obtain

$$
\begin{aligned}
S(\lambda_1,\dots,\lambda_k) &\le \int_{\mathbb{R}^N} |\nabla z_\mu^{\lambda_j}(x-a_j)|^2\,dx - \lambda_j \int_{\mathbb{R}^N} \frac{|z_\mu^{\lambda_j}(x-a_j)|^2}{|x-a_j|^2}\,dx - \sum_{i\neq j}\lambda_i \int_{\mathbb{R}^N} \frac{|z_\mu^{\lambda_j}(x-a_j)|^2}{|x-a_i|^2}\,dx \\
&= \int_{\mathbb{R}^N} |\nabla z_\mu^{\lambda_j}(x)|^2\,dx - \lambda_j \int_{\mathbb{R}^N} \frac{|z_\mu^{\lambda_j}(x)|^2}{|x|^2}\,dx - \sum_{i\neq j}\lambda_i \int_{\mathbb{R}^N} \frac{|z_\mu^{\lambda_j}(x)|^2}{|x+a_j-a_i|^2}\,dx \\
&= S(\lambda_j) - \sum_{i\neq j}\lambda_i \int_{\mathbb{R}^N} \frac{|z_\mu^{\lambda_j}(x)|^2}{|x+a_j-a_i|^2}\,dx \\
&= S(\lambda_j) - \begin{cases} \mu^2 \int_{\mathbb{R}^N} |z_1^{\lambda_j}|^2 \Big(\sum_{i\neq j} \frac{\lambda_i}{|a_j-a_i|^2} + o(1)\Big) & \text{if } \lambda_j < \frac{N(N-4)}{4} \\ \alpha_{\lambda_j,N}^2 \mu^2 |\ln\mu| \Big(\sum_{i\neq j} \frac{\lambda_i}{|a_j-a_i|^2} + o(1)\Big) & \text{if } \lambda_j = \frac{N(N-4)}{4}, \end{cases}
\end{aligned}
$$

as $\mu \to 0$. Taking μ sufficiently small we obtain that $S(\lambda_1,\dots,\lambda_k) < S(\lambda_j)$. Under assumption (38), from Lemma 3.4 we obtain

$$
S(\lambda_1,\dots,\lambda_k) \le S(\lambda_j) - \alpha_{\lambda_j,N}^2 \beta_{\lambda_j,N} \mu^{\sqrt{(N-2)^2-4\lambda}} \bigg(\sum_{i\neq j} \frac{\lambda_i}{|a_i-a_j|^{\sqrt{(N-2)^2-4\lambda}}} + o(1) \bigg).
$$

Again, taking μ sufficiently small we obtain that $S(\lambda_1,\dots,\lambda_k) < S(\lambda_j)$. ∎

Proof of Theorem 1.4. By (8) the quadratic form $\mathcal{Q}$ is positive definite and provides an equivalent norm on $\mathcal{D}^{1,2}(\mathbb{R}^N)$. Let $\{u_n\}_n \subset \mathcal{D}^{1,2}(\mathbb{R}^N)$ be a minimizing sequence for (2). From the homogeneity of the quotient there is no restriction requiring $\|u_n\|_{L^{2^*}(\mathbb{R}^N)} = 1$, while from Ekeland's variational principle we can assume that the sequence has the Palais-Smale property, i.e for any $v \in \mathcal{D}^{1,2}(\mathbb{R}^N)$

$$
\int_{\mathbb{R}^N} \nabla u_n \cdot \nabla v - \sum_{i=1}^k \lambda_i \int_{\mathbb{R}^N} \frac{u_n(x)v(x)}{|x-a_i|^2}\,dx - S(\lambda_1,\lambda_2,\dots,\lambda_k) \int_{\mathbb{R}^N} |u_n|^{2^*-2} u_n v\,dx = o\big(\|v\|_{\mathcal{D}^{1,2}(\mathbb{R}^N)}\big).
$$

Hence $J'(u_n) \to 0$ in $(\mathcal{D}^{1,2}(\mathbb{R}^N))^\star$ and

$$
J(u_n) \to \Big(\frac{1}{2} - \frac{1}{2^*}\Big) S(\lambda_1,\lambda_2,\dots,\lambda_k) = \frac{1}{N} S(\lambda_1,\lambda_2,\dots,\lambda_k).
$$

From assumption (10) and Corollary 3.5, we infer that

$$
S(\lambda_1,\dots,\lambda_k) < S(\lambda_k). \tag{39}
$$

Since the function $\lambda \mapsto S(\lambda)$ is decreasing, from assumption (9) we deduce that

$$
S(\lambda_k) \le S(\lambda_i) \quad \text{for all} \quad i = 1,\dots,k-1 \quad \text{and} \quad S(\lambda_k) < S(0) = S, \tag{40}
$$

and from assumption (11) we deduce that $\sum_{i=1}^k \lambda_i \le \lambda_k$ hence (again by monotonicity of $\lambda \mapsto S(\lambda)$) we obtain

$$
S(\lambda_k) \le S\Big(\sum_{i=1}^k \lambda_i\Big). \tag{41}
$$

Gathering (39), (40), and (41), we finally have

$$S(\lambda_1,\dots,\lambda_k) < \min\Big\{S, S(\lambda_1),\dots,S(\lambda_k), S\Big(\sum\nolimits_{j=1}^k \lambda_j\Big)\Big\}$$

and hence

$$\frac{1}{N}S(\lambda_1,\lambda_2,\dots,\lambda_k) < \frac{1}{N}S(\lambda_1,\lambda_2,\dots,\lambda_k)^{1-\frac{N}{2}}\min\Big\{S, S(\lambda_1),\dots,S(\lambda_k), S\Big(\sum\nolimits_{j=1}^k \lambda_j\Big)\Big\}^{N/2}.$$

Therefore Theorem 2.1 applies and we can conclude that $\{u_n\}_{n\in\mathbb{N}}$ has a subsequence strongly converging to some $u_0\in\mathcal{D}^{1,2}(\mathbb{R}^N)$, which satisfies $J(u_0)=\frac{1}{N}S(\lambda_1,\lambda_2,\dots,\lambda_k)$. Hence u_0 achieves the infimum in (2). Since $J(u_0)=J(|u_0|)$, we have that also $|u_0|$ is a minimizer in (2) and then $v_0=S(\lambda_1,\lambda_2,\dots,\lambda_k)^{1/(2^*-2)}|u_0|$ is a nonnegative solution to equation (1). The maximum principle in $\mathbb{R}^N\setminus\{a_1,\dots,a_k\}$ implies that $v_0>0$ in $\mathbb{R}^N\setminus\{a_1,\dots,a_k\}$. ■

4. Proof of Propositions 1.1 and 1.2

Proof of Proposition 1.1. Let u be a positive solution to problem (1). Let $\phi\in C^\infty_C(\mathbb{R}^N\setminus\{a_1,\dots,a_k\})$, i.e. let ϕ be a smooth function with a compact support A which is disjoint from the set of the singularities. By classical regularity results, u is smooth in $\mathbb{R}^N\setminus\{a_1,\dots,a_k\}$; in particular u is bounded from below by a positive constant on the set A. Hence we can test equation (1) with the function $\frac{\phi^2}{u}$ thus obtaining

$$2\int_{\mathbb{R}^N}\frac{\phi}{u}\nabla\phi\cdot\nabla u\,dx-\int_{\mathbb{R}^N}\frac{\phi^2}{u^2}|\nabla u|^2\,dx-\sum_{i=1}^k\lambda_i\int_{\mathbb{R}^N}\frac{\phi^2}{|x-a_i|^2}\,dx-\int_{\mathbb{R}^N}\phi^2u^{2^*-2}\,dx=0.$$

Since

$$2\,\frac{\phi}{u}\,\nabla\phi\cdot\nabla u-\frac{\phi^2}{u^2}\,|\nabla u|^2\le|\nabla\phi|^2$$

we deduce

$$\int_{\mathbb{R}^N}|\nabla\phi|^2-\sum_{i=1}^k\lambda_i\int_{\mathbb{R}^N}\frac{\phi^2}{|x-a_i|^2}\,dx\ge\int_{\mathbb{R}^N}\phi^2u^{2^*-2}\,dx\ge0\quad\text{for all }\phi\in C^\infty_C(\mathbb{R}^N\setminus\{a_1,\dots,a_k\}).$$

Since $C^\infty_C(\mathbb{R}^N\setminus\{a_1,\dots,a_k\})$ is dense in $\mathcal{D}^{1,2}(\mathbb{R}^N)$ (see [7, Lemma 2.1]), we obtain

$$\mathcal{Q}(\phi)=\int_{\mathbb{R}^N}|\nabla\phi|^2-\sum_{i=1}^k\lambda_i\int_{\mathbb{R}^N}\frac{\phi^2}{|x-a_i|^2}\,dx\ge0\quad\text{for all }\phi\in\mathcal{D}^{1,2}(\mathbb{R}^N),$$

namely $\mathcal{Q}$ is positive semidefinite. ■

Proof of Proposition 1.2. From Hardy's inequality, it follows that

$$\mathcal{Q}(u)\ge\bigg(1-\frac{4}{(N-2)^2}\sum_{\lambda_i>0}\lambda_i\bigg)\int_{\mathbb{R}^N}|\nabla u|^2\,dx. \tag{42}$$

Hence a sufficient condition for $\mathcal{Q}$ to be positive definite is that

$$\sum_{\substack{i=1,\dots,k\\ \lambda_i>0}}\lambda_i<\frac{(N-2)^2}{4}.$$

Assume now that $\sum_{\lambda_i>0}\lambda_i > (N-2)^2/4$. From optimality of the constant $(N-2)^2/4$ in Hardy's inequality, we have that there exists some function $\phi \in C_0^\infty(\mathbb{R}^N)$ such that

$$\int_{\mathbb{R}^N} |\nabla\phi|^2 - \sum_{\lambda_i>0}\lambda_i \int_{\mathbb{R}^N} \frac{\phi^2}{|x|^2}\,dx < 0.$$

For any $\mu > 0$, consider the function $\phi_\mu(x) = \mu^{-\frac{N-2}{2}}\phi(x/\mu)$. A change of variable yields

$$\int_{\mathbb{R}^N} |\nabla\phi_\mu|^2 - \sum_{\lambda_i>0}\lambda_i \int_{\mathbb{R}^N} \frac{\phi_\mu^2}{|x-a_i|^2}\,dx = \int_{\mathbb{R}^N} |\nabla\phi|^2 - \sum_{\lambda_i>0}\lambda_i \int_{\mathbb{R}^N} \frac{\phi^2}{\left|x-\frac{a_i}{\mu}\right|^2}\,dx \quad \text{for all } \mu > 0.$$

Letting $\mu \to \infty$, Proposition 3.1 yields

$$\int_{\mathbb{R}^N} |\nabla\phi_\mu|^2 - \sum_{\lambda_i>0}\lambda_i \int_{\mathbb{R}^N} \frac{\phi_\mu^2}{|x-a_i|^2}\,dx \longrightarrow \int_{\mathbb{R}^N} |\nabla\phi|^2 - \sum_{\lambda_i>0}\lambda_i \int_{\mathbb{R}^N} \frac{\phi^2}{|x|^2}\,dx < 0$$

therefore there exists some large $\bar\mu$ such that the function $\psi = \phi_{\bar\mu}$ satisfies

$$\int_{\mathbb{R}^N} |\nabla\psi|^2 - \sum_{\lambda_i>0}\lambda_i \int_{\mathbb{R}^N} \frac{\psi^2}{|x-a_i|^2}\,dx < 0.$$

We now notice that since ψ has compact support, i.e. $\operatorname{supp}\psi \subset B(0,R)$,

$$\int_{\mathbb{R}^N} \frac{\psi^2}{|x-a|^2}\,dx \le \frac{1}{(|a|-R)^2}\int_{B(0,R)} \psi^2\,dx \longrightarrow 0 \quad \text{as } |a| \to \infty,$$

hence it is possible to locate the poles carrying negative masses far away from $\operatorname{supp}\psi$ in order to get

$$\int_{\mathbb{R}^N} |\nabla\psi|^2 - \sum_{i=1}^k \lambda_i \int_{\mathbb{R}^N} \frac{\psi^2}{|x-a_i|^2}\,dx < 0,$$

thus proving the second part of Proposition 1.2. ■

5. Proof of Theorem 1.3

We start by showing that if all masses λ_i are positive, then the inequality of Corollary 3.3 is indeed an equality.

Proposition 5.1. *Assume that $\lambda_i > 0$ for all $i = 1, \dots, k$ and $\sum_{i=1}^k \lambda_i < \frac{(N-2)^2}{4}$. Then*

$$S(\lambda_1, \dots, \lambda_k) = S\big(\textstyle\sum_{i=1}^k \lambda_i\big).$$

Proof. For any $u \in \mathcal{D}^{1,2}(\mathbb{R}^N)$, $u \ge 0$ a.e., we consider the Schwarz symmetrization of u defined as

$$u^*(x) := \inf\big\{t > 0 : \ \big|\{y \in \mathbb{R}^N : \ u(y) > t\}\big| \le \omega_N |x|^N\big\} \tag{43}$$

where $|\cdot|$ denotes the Lebesgue measure of $\mathbb{R}^N$ and ω_N is the volume of the standard unit N-ball. From [29, Theorem 21.8], it follows that

$$\int_{\mathbb{R}^N} \frac{u^2}{|x-a_i|^2}\,dx \le \int_{\mathbb{R}^N} (u^*(x))^2 \left[\left(\frac{1}{|x-a_i|}\right)^*\right]^2 dx.$$

A direct calculation shows that $\left(\frac{1}{|x-a_i|}\right)^* = \frac{1}{|x|}$, hence

$$\int_{\mathbb{R}^N} \frac{u^2}{|x-a_i|^2}\,dx \leq \int_{\mathbb{R}^N} \frac{(u^*(x))^2}{|x|^2}\,dx \quad \text{for any } u \geq 0 \text{ a.e, } u \in \mathcal{D}^{1,2}(\mathbb{R}^N). \tag{44}$$

Moreover

$$\int_{\mathbb{R}^N} |u^*|^p = \int_{\mathbb{R}^N} |u|^p, \tag{45}$$

see for example [29, Corollary 21.7], and, by the Pólya-Szegö inequality

$$\int_{\mathbb{R}^N} |\nabla u^*|^2 \leq \int_{\mathbb{R}^N} |\nabla u|^2, \tag{46}$$

which together with (44) imply that for all $u \in \mathcal{D}^{1,2}(\mathbb{R}^N)$, $u \geq 0$ a.e.

$$\begin{aligned}\frac{\int_{\mathbb{R}^N} |\nabla u|^2 dx - \sum_{i=1}^k \lambda_i \int_{\mathbb{R}^N} \frac{u^2(x)}{|x-a_i|^2}\,dx}{\left(\int_{\mathbb{R}^N} |u|^{2^*} dx\right)^{2/2^*}} &\geq \frac{\int_{\mathbb{R}^N} |\nabla u^*|^2 dx - \sum_{i=1}^k \lambda_i \int_{\mathbb{R}^N} \frac{|u^*(x)|^2}{|x|^2}\,dx}{\left(\int_{\mathbb{R}^N} |u^*|^{2^*} dx\right)^{2/2^*}} \\ &\geq S\Big(\sum\nolimits_{i=1}^k \lambda_i\Big).\end{aligned} \tag{47}$$

Since the Rayleigh quotient above remains unchanged when replacing u with $|u|$, we have that

$$S(\lambda_1, \lambda_2, \dots, \lambda_k) = \inf_{u \in \mathcal{D}^{1,2}(\mathbb{R}^N)\setminus\{0\},\ u \geq 0} \frac{\int_{\mathbb{R}^N} |\nabla u|^2 dx - \sum_{i=1}^k \lambda_i \int_{\mathbb{R}^N} \frac{u^2(x)}{|x-a_i|^2}\,dx}{\left(\int_{\mathbb{R}^N} |u|^{2^*} dx\right)^{2/2^*}}$$

hence passing to the infimum in (47), we obtain $S(\lambda_1, \lambda_2, \dots, \lambda_k) \geq S(\sum_{i=1}^k \lambda_i)$, which together with the estimate of Corollary 3.3 gives equality $S(\lambda_1, \lambda_2, \dots, \lambda_k) = S(\sum_{i=1}^k \lambda_i)$. ■

We are now in position to prove Theorem 1.3.

Proof of Theorem 1.3. We first consider case (i). In such a case $S(\lambda_1, \lambda_2, \dots, \lambda_k) \geq S(\lambda_k)$. On the other hand by Lemma 3.4, it follows that

$$\begin{aligned}S(\lambda_1, \lambda_2, \dots, \lambda_k) &\leq \int_{\mathbb{R}^N} |\nabla z_\mu^{\lambda_k}(x)|^2\,dx - \lambda_k \int_{\mathbb{R}^N} \frac{|z_\mu^{\lambda_k}(x)|^2}{|x-a_k|^2}\,dx - \sum_{i=1}^{k-1} \lambda_i \int_{\mathbb{R}^N} \frac{|z_\mu^{\lambda_k}(x)|^2}{|x-a_i|^2}\,dx \\ &= S(\lambda_k) + o(1) \quad \text{as } \mu \to 0.\end{aligned}$$

Letting $\mu \to 0$, we obtain $S(\lambda_1, \lambda_2, \dots, \lambda_k) = S(\lambda_k)$. Assume by contradiction that the infimum in (2) is attained by some function $u_0 \in \mathcal{D}^{1,2}(\mathbb{R}^N) \setminus \{0\}$. Then

$$S(\lambda_1, \lambda_2, \dots, \lambda_k) = \frac{\displaystyle\int_{\mathbb{R}^N} |\nabla u_0|^2 dx - \lambda_k \int_{\mathbb{R}^N} \frac{u_0^2(x)}{|x-a_k|^2}\,dx - \sum_{i=1}^{k-1} \lambda_i \int_{\mathbb{R}^N} \frac{u_0^2(x)}{|x-a_i|^2}\,dx}{\left(\displaystyle\int_{\mathbb{R}^N} |u_0|^{2^*} dx\right)^{2/2^*}}$$

$$= S(\lambda_k) \le \frac{\displaystyle\int_{\mathbb{R}^N} |\nabla u_0|^2 dx - \lambda_k \int_{\mathbb{R}^N} \frac{u_0^2(x)}{|x-a_k|^2}\,dx}{\left(\displaystyle\int_{\mathbb{R}^N} |u_0|^{2^*} dx\right)^{2/2^*}}$$

which implies

$$\sum_{i=1}^{k-1} \lambda_i \int_{\mathbb{R}^N} \frac{u_0^2(x)}{|x-a_i|^2}\,dx \ge 0$$

giving rise to a contradiction with assumption (i).

In case (ii), we also argue by contradiction. Assume that the infimum in (2) is attained by some function $u_0 \in \mathcal{D}^{1,2}(\mathbb{R}^N) \setminus \{0\}$. We may assume that $u_0 \ge 0$ a.e. in $\mathbb{R}^N$, otherwise we take $|u_0|$ which also is a minimizer in (2). Hence we can consider the Schwarz symmetrization u_0^*, see (43). From (47), we have that

$$S(\lambda_1, \lambda_2, \dots, \lambda_k) = \frac{\displaystyle\int_{\mathbb{R}^N} |\nabla u_0|^2 dx - \sum_{i=1}^{k} \lambda_i \int_{\mathbb{R}^N} \frac{u_0^2(x)}{|x-a_i|^2}\,dx}{\left(\displaystyle\int_{\mathbb{R}^N} |u_0|^{2^*} dx\right)^{2/2^*}} \tag{48}$$

$$\ge \frac{\displaystyle\int_{\mathbb{R}^N} |\nabla u_0^*|^2 dx - \sum_{i=1}^{k} \lambda_i \int_{\mathbb{R}^N} \frac{|u_0^*(x)|^2}{|x|^2}\,dx}{\left(\displaystyle\int_{\mathbb{R}^N} |u_0^*|^{2^*} dx\right)^{2/2^*}} \ge S\Big(\sum_{i=1}^{k} \lambda_i\Big).$$

From (48) and Proposition 5.1, we deduce that all inequalities in (48) are indeed equalities. In particular

$$\frac{\displaystyle\int_{\mathbb{R}^N} |\nabla u_0|^2 dx - \sum_{i=1}^{k} \lambda_i \int_{\mathbb{R}^N} \frac{u_0^2(x)}{|x-a_i|^2}\,dx}{\left(\displaystyle\int_{\mathbb{R}^N} |u_0|^{2^*} dx\right)^{2/2^*}} = \frac{\displaystyle\int_{\mathbb{R}^N} |\nabla u_0^*|^2 dx - \sum_{i=1}^{k} \lambda_i \int_{\mathbb{R}^N} \frac{|u_0^*(x)|^2}{|x|^2}\,dx}{\left(\displaystyle\int_{\mathbb{R}^N} |u_0^*|^{2^*} dx\right)^{2/2^*}} \tag{49}$$

$$= S\Big(\sum_{i=1}^{k} \lambda_i\Big).$$

From (49) and (45) it follows that

$$\int_{\mathbb{R}^N} |\nabla u_0|^2 dx - \sum_{i=1}^k \lambda_i \int_{\mathbb{R}^N} \frac{u_0^2(x)}{|x-a_i|^2}\, dx = \int_{\mathbb{R}^N} |\nabla u_0^*|^2 dx - \sum_{i=1}^k \lambda_i \int_{\mathbb{R}^N} \frac{|u_0^*(x)|^2}{|x|^2}\, dx$$

hence in view of (44) and (46) we obtain

$$0 \le \int_{\mathbb{R}^N} |\nabla u_0|^2 dx - \int_{\mathbb{R}^N} |\nabla u_0^*|^2 dx = \sum_{i=1}^k \lambda_i \int_{\mathbb{R}^N} \frac{u_0^2(x)}{|x-a_i|^2}\, dx - \sum_{i=1}^k \lambda_i \int_{\mathbb{R}^N} \frac{|u_0^*(x)|^2}{|x|^2}\, dx \le 0.$$

Therefore

$$\int_{\mathbb{R}^N} |\nabla u_0|^2 dx = \int_{\mathbb{R}^N} |\nabla u_0^*|^2 dx. \tag{50}$$

On the other hand from (49), it follows that u_0^* is a minimizer of (7) and solves equation (4) with $\lambda = \sum_{i=1}^k \lambda_i$. Hence in view of the classification of solutions to (4) given in [27], u_0^* must be equal to w_μ^λ for some $\mu > 0$ where $\lambda = \sum_{i=1}^k \lambda_i$ and w_μ^λ is defined in (5). In particular $u_0^*(|x|)$ is strictly decreasing and hence

$$\big|\{x \in \mathbb{R}^N : \ \nabla u_0^*(x) = 0\}\big| = 0. \tag{51}$$

(50) and (51) allow to use [6, Theorem 1.1] to conclude that there exists some point $x_0 \in \mathbb{R}^N$ such that $u_0 = u_0^*(\cdot - x_0)$, namely u_0 is spherically symmetric with respect to x_0. Since u_0 is a minimizer in (2), then $v_0 = S(\lambda_1, \lambda_2, \dots, \lambda_k)^{1/(2^*-2)} u_0$ is a solution to equation (1). Consequently $\sum_{i=1}^k \frac{\lambda_i}{|x-a_i|^2}$ must be spherically symmetric with respect to x_0, which gives a contradiction. Hence the infimum in (2) cannot be attained. ■

6. The problem on bounded domains

Let Ω be a bounded smooth domain in $\mathbb{R}^N$, $N \ge 3$. In this section we study equation (14) and the associated minimization problem (13). The corresponding functional is given by

$$J_\Omega(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 dx - \sum_{i=1}^k \frac{\lambda_i}{2} \int_\Omega \frac{u^2(x)}{|x-a_i|^2}\, dx - \frac{S_\Omega(\lambda_1, \lambda_2, \dots, \lambda_k)}{2^*} \int_\Omega |u|^{2^*} dx. \tag{52}$$

The following theorem contains a local Palais-Smale condition.

Theorem 6.1. *Assume that* (12) *holds. Let* $\{u_n\}_{n\in\mathbb{N}} \subset H_0^1(\Omega)$ *be a Palais-Smale sequence for* J_Ω*, namely*

$$\lim_{n\to\infty} J_\Omega(u_n) = c < \infty \ \text{in } \mathbb{R} \quad \text{and} \quad \lim_{n\to\infty} J'_\Omega(u_n) = 0 \ \text{in the dual space } (H_0^1)^\star.$$

If

$$c < c_\Omega^* = \frac{1}{N} S_\Omega(\lambda_1, \lambda_2, \dots, \lambda_k)^{1-\frac{N}{2}} \min\Big\{S, S(\lambda_1), \dots, S(\lambda_k)\Big\}^{N/2},$$

then $\{u_n\}_{n\in\mathbb{N}}$ *has a converging subsequence.*

The proof of the above theorem is analogous to the proof of Theorem 2.1. In this case, due to boundedness of the domain, there is no possibility of loss of mass at infinity, so that the term $S\big(\sum_{j=1}^k \lambda_j\big)$ is not involved in the level at which Palais-Smale condition fails.

Lemma 6.2. *Let $j \in \{1, 2, \dots, k\}$. There holds*

$$S_\Omega(\lambda_1, \dots, \lambda_k) \le S(\lambda_j) + O(\mu^{\nu_{\lambda_j}(N-2)}) \tag{53}$$
$$- \begin{cases} \mu^2 \int_{\mathbb{R}^N} |z_1^{\lambda_j}|^2 \Big(\sum_{i \ne j} \frac{\lambda_i}{|a_j - a_i|^2} + o(1) \Big) & \text{if } \lambda_j < \frac{N(N-4)}{4} \\ \alpha^2_{\lambda_j, N} \mu^2 |\ln \mu| \Big(\sum_{i \ne j} \frac{\lambda_i}{|a_j - a_i|^2} + o(1) \Big) & \text{if } \lambda_j = \frac{N(N-4)}{4} \\ \alpha^2_{\lambda_j, N} \beta_{\lambda, N} \mu^{\nu_{\lambda_j}(N-2)} \Bigg(\sum_{i \ne j} \frac{\lambda_i}{|a_i - a_j|^{\sqrt{(N-2)^2 - 4\lambda}}} + o(1) \Bigg) & \text{if } \lambda_j > \frac{N(N-4)}{4}. \end{cases}$$

Moreover if and

$$0 < \lambda_j \le \frac{N(N-4)}{4}, \quad \text{and} \quad \sum_{i \ne j} \frac{\lambda_i}{|a_j - a_i|^2} > 0$$

then

$$S_\Omega(\lambda_1, \dots, \lambda_k) < S(\lambda_j).$$

Proof. Let ω be an open set such that $\overline{\omega} \subset \Omega$ and $a_j \in \omega$ and let ψ be a smooth cut-off function such that

$$0 \le \psi(x) \le 1, \quad \psi \equiv 0 \quad \text{in } \mathbb{R}^N \setminus \Omega, \quad \psi \equiv 1 \quad \text{in } \omega.$$

Then $\psi(x) z_\mu^{\lambda_j}(x - a_j) \in H_0^1(\Omega)$. (53) follows from

$$S_\Omega(\lambda_1, \dots, \lambda_k)$$
$$\le \frac{\int_{\mathbb{R}^N} |\nabla(\psi(x) z_\mu^{\lambda_j}(x - a_j))|^2\, dx - \lambda_j \int_{\mathbb{R}^N} \frac{|\psi(x) z_\mu^{\lambda_j}(x-a_j)|^2}{|x-a_j|^2}\, dx - \sum_{i \ne j} \lambda_i \int_{\mathbb{R}^N} \frac{|\psi(x) z_\mu^{\lambda_j}(x-a_j)|^2}{|x-a_i|^2}\, dx}{\Big(\int_{\mathbb{R}^N} |\psi(x) z_\mu^{\lambda_j}(x - a_j)|^{2^*}\, dx \Big)^{2/2^*}}$$

and the following estimates

$$\int_{\mathbb{R}^N} |\nabla(\psi(x) z_\mu^{\lambda_j}(x - a_j))|^2\, dx = \int_{\mathbb{R}^N} |\nabla z_\mu^{\lambda_j}(x)|^2\, dx + O(\mu^{\nu_{\lambda_j}(N-2)}) \tag{54}$$

$$\int_{\mathbb{R}^N} \frac{|\psi(x) z_\mu^{\lambda_j}(x - a_j)|^2}{|x - a_j|^2}\, dx = \int_{\mathbb{R}^N} \frac{|z_\mu^{\lambda_j}(x)|^2}{|x|^2}\, dx + O(\mu^{\nu_{\lambda_j}(N-2)}) \tag{55}$$

$$\int_{\mathbb{R}^N} \frac{|\psi(x) z_\mu^{\lambda_j}(x - a_j)|^2}{|x - a_i|^2}\, dx = \int_{\mathbb{R}^N} \frac{|z_\mu^{\lambda_j}(x)|^2}{|x + (a_j - a_i)|^2}\, dx + O(\mu^{\nu_{\lambda_j}(N-2)}) \tag{56}$$

$$\bigg(\int_{\mathbb{R}^N} |\psi(x) z_\mu^{\lambda_j}(x - a_j)|^{2^*}\, dx \bigg)^{2/2^*} = 1 + O(\mu^{\nu_{\lambda_j}(N-2)}). \tag{57}$$

Let us prove (54). We have that

$$\int_{\mathbb{R}^N} |\nabla(\psi(x) z_\mu^{\lambda_j}(x - a_j))|^2\, dx = \int_{\mathbb{R}^N} \psi(x)^2 |\nabla z_\mu^{\lambda_j}(x - a_j)|^2\, dx \tag{58}$$
$$+ \int_{\mathbb{R}^N} |z_\mu^{\lambda_j}(x - a_j)|^2 |\nabla \psi(x)|^2\, dx + 2 \int_{\mathbb{R}^N} \psi(x) z_\mu^{\lambda_j}(x - a_j) \nabla \psi(x) \cdot \nabla z_\mu^{\lambda_j}(x - a_j)\, dx.$$

In view of (35) we have

$$\left|\int_{\mathbb{R}^N}\psi(x)^2|\nabla z_\mu^{\lambda_j}(x-a_j)|^2\,dx-\int_{\mathbb{R}^N}|\nabla z_\mu^{\lambda_j}(x-a_j)|^2\,dx\right| \tag{59}$$
$$=\mu^{-N}\int_{\mathbb{R}^N\setminus\omega}(1-\psi^2(x))\Big|\nabla z_1^{\lambda_j}\Big(\frac{x-a_j}{\mu}\Big)\Big|^2\,dx$$
$$=\int_{\mu^{-1}((\mathbb{R}^N\setminus\omega)-a_j)}(1-\psi^2(\mu y+a_j))|\nabla z_1^{\lambda_j}(y)|^2\,dy$$
$$\le \mathrm{const}\int_{\mu^{-1}r}^{+\infty}s^{-1-\nu_{\lambda_j}N+2\nu_{\lambda_j}}\,ds=\mathrm{const}\,\mu^{\nu_{\lambda_j}(N-2)},$$

$$\int_{\mathbb{R}^N}|z_\mu^{\lambda_j}(x-a_j)|^2|\nabla\psi(x)|^2\,dx\le\mathrm{const}\int_{\mu^{-1}((\Omega\setminus\omega)-a_j)}|z_1^{\lambda_j}(y)|^2\,\mu^2|\nabla\psi(\mu y+a_j)|^2\,dy \tag{60}$$
$$\le\mathrm{const}\Big(\int_{\mu^{-1}((\Omega\setminus\omega)-a_j)}|z_1^{\lambda_j}(y)|^{2^*}\,dy\Big)^{2/2^*}\mu^2\Big(\int_{\mu^{-1}((\Omega\setminus\omega)-a_j)}|\nabla\psi(\mu y+a_j)|^N\,dy\Big)^{2/N}$$
$$\le\mathrm{const}\Big(\int_{\mu^{-1}r}^{\mu^{-1}R}s^{-1-\nu_{\lambda_j}N}\,ds\Big)^{2/2^*}=\mathrm{const}\,\mu^{\nu_{\lambda_j}(N-2)},$$

and

$$\int_{\mathbb{R}^N}\psi(x)z_\mu^{\lambda_j}(x-a_j)\nabla\psi(x)\cdot\nabla z_\mu^{\lambda_j}(x-a_j)\,dx \tag{61}$$
$$\le\mathrm{const}\,\mu\int_{\mu^{-1}r}^{\mu^{-1}R}s^{-\nu_{\lambda_j}(N-2)}\,ds=\mathrm{const}\,\mu^{\nu_{\lambda_j}(N-2)}.$$

Estimate (54) follows from (59–61). The proof of (55–57) is analogous. To show that $S_\Omega(\lambda_1,\dots,\lambda_k)<S(\lambda_j)$, it is enough to observe that if $\lambda_j<\frac{N(N-4)}{4}$ then $\nu_{\lambda_j}(N-2)>2$ and hence $O(\mu^{\nu_{\lambda_j}(N-2)})=o(\mu^2)$ as $\mu\to0^+$, while if $\lambda_j=\frac{N(N-4)}{4}$ then $\nu_{\lambda_j}(N-2)=2$ and hence $O(\mu^{\nu_{\lambda_j}(N-2)})=o(\mu^2|\ln\mu|)$. Taking μ sufficiently small, we obtain $S_\Omega(\lambda_1,\dots,\lambda_k)<S(\lambda_j)$. ■

Proof of Theorem 1.5. It follows from Theorem 6.1 and Lemma 6.2, arguing as in the proof of Theorem 1.4. ■

Proof of Theorem 1.6. Using a density argument it is easy to prove that

(62) for any $\varepsilon>0$ there exists $\bar R>0$ such that if $\Omega\supset B(0,R)$ then $S_\Omega(\lambda_1,\dots,\lambda_k)<S(\lambda_1,\dots,\lambda_k)+\varepsilon$.

Theorem 1.6 follows from Theorem 6.1, Corollary 3.5, and (62). ■

7. Appendix

Proof of Lemma 3.4. Set $\gamma_\lambda = 1-\sqrt{1-\frac{4\lambda}{(N-2)^2}}$. For $\lambda < \frac{N(N-4)}{4}$ we have that $\gamma_\lambda < \frac{N-4}{N-2}$ and $z_1^\lambda \in L^2(\mathbb{R}^N)$. Note that

$$\int_{\mathbb{R}^N} \frac{|z_\mu^\lambda|^2}{|x+\xi|^2}\,dx = \mu^2 \int_{\mathbb{R}^N} \frac{|z_1^\lambda|^2}{|\mu x+\xi|^2}\,dx = \mu^2 \int_{|x|<\frac{|\xi|}{2\mu}} \frac{|z_1^\lambda|^2}{|\mu x+\xi|^2}\,dx + \alpha_{\lambda,N}^2\,\mu^{(1-\gamma_\lambda)(N-2)} \int_{|x-\xi|\geq\frac{|\xi|}{2}} \frac{\left(\mu^{2(1-\gamma_\lambda)}+|x-\xi|^{2(1-\gamma_\lambda)}\right)^{-(N-2)}}{|x|^2|x-\xi|^{\gamma_\lambda(N-2)}}\,dx. \tag{63}$$

Since

$$\begin{aligned}
&\left|\int_{|x|<\frac{|\xi|}{2\mu}} |z_1^\lambda(x)|^2\left[\frac{1}{|\mu x+\xi|^2}-\frac{1}{|\xi|^2}\right]dx\right| \\
&= \alpha_{\lambda,N}^2\left|\int_{|x|<\frac{|\xi|}{2\mu}} |x|^{-(N-2)}\left(|x|^{1-\gamma_\lambda}+|x|^{\gamma_\lambda-1}\right)^{-(N-2)}\frac{-\mu^2|x|^2-2\mu\xi\cdot x}{|\xi|^2|\mu x+\xi|^2}\right| \\
&\leq \frac{\alpha_{\lambda,N}^2\,\mu^2}{|\xi|^2}\int_{|x|<\frac{|\xi|}{2\mu}} \frac{\left(|x|^{1-\gamma_\lambda}+|x|^{\gamma_\lambda-1}\right)^{-(N-2)}|x|^2}{|x|^{N-2}|\mu x+\xi|^2} + \frac{2\alpha_{\lambda,N}^2\,\mu}{|\xi|}\int_{|x|<\frac{|\xi|}{2\mu}} \frac{\left(|x|^{1-\gamma_\lambda}+|x|^{\gamma_\lambda-1}\right)^{-(N-2)}|x|}{|x|^{N-2}|\mu x+\xi|^2} \\
&\leq \frac{4\alpha_{\lambda,N}^2\,\mu^2}{|\xi|^4}\int_0^{\frac{|\xi|}{2\mu}} \frac{r^3}{\left(r^{1-\gamma_\lambda}+r^{\gamma_\lambda-1}\right)^{N-2}}\,dr + \frac{8\alpha_{\lambda,N}^2\,\mu}{|\xi|^3}\int_0^{\frac{|\xi|}{2\mu}} \frac{r^2}{\left(r^{1-\gamma_\lambda}+r^{\gamma_\lambda-1}\right)^{N-2}}\,dr
\end{aligned}$$

and

$$\int_0^{\frac{|\xi|}{2\mu}} \frac{r^3}{\left(r^{1-\gamma_\lambda}+r^{\gamma_\lambda-1}\right)^{N-2}}\,dr = \begin{cases} O(1) & \text{if } \gamma_\lambda < \frac{N-6}{N-2} \\ O(|\log\mu|) & \text{if } \gamma_\lambda = \frac{N-6}{N-2} \\ O(\mu^{(1-\gamma_\lambda)(N-2)-4}) & \text{if } \gamma_\lambda > \frac{N-6}{N-2} \end{cases}$$

$$\int_0^{\frac{|\xi|}{2\mu}} \frac{r^2}{\left(r^{1-\gamma_\lambda}+r^{\gamma_\lambda-1}\right)^{N-2}}\,dr = \begin{cases} O(1) & \text{if } \gamma_\lambda < \frac{N-5}{N-2} \\ O(|\log\mu|) & \text{if } \gamma_\lambda = \frac{N-5}{N-2} \\ O(\mu^{N-5-\gamma_\lambda(N-2)}) & \text{if } \gamma_\lambda > \frac{N-5}{N-2} \end{cases}$$

we deduce that

$$\int_{|x|<\frac{|\xi|}{2\mu}} |z_1^\lambda(x)|^2\left[\frac{1}{|\mu x+\xi|^2}-\frac{1}{|\xi|^2}\right]dx = o(1)$$

as $\mu\to 0$ and hence, since $z_1^\lambda \in L^2(\mathbb{R}^N)$,

$$\int_{|x|<\frac{|\xi|}{2\mu}} |z_1^\lambda(x)|^2\frac{dx}{|\mu x+\xi|^2} = \frac{1}{|\xi|^2}\int_{|x|<\frac{|\xi|}{2\mu}} |z_1^\lambda(x)|^2\,dx + o(1) = \frac{1}{|\xi|^2}\int_{\mathbb{R}^N} |z_1^\lambda(x)|^2\,dx + o(1). \tag{64}$$

On the other hand we have

$$\int_{|x-\xi|\geq\frac{|\xi|}{2}} \frac{\big(\mu^{2(1-\gamma_\lambda)}+|x-\xi|^{2(1-\gamma_\lambda)}\big)^{-(N-2)}}{|x|^2|x-\xi|^{\gamma_\lambda(N-2)}}\,dx \tag{65}$$

$$\leq \int_{\substack{|x-\xi|\geq|\xi|/2\\ |x|<2|\xi|}} \frac{2^{(N-2)(2-\gamma_\lambda)}}{|\xi|^{(N-2)(2-\gamma_\lambda)}}\frac{dx}{|x|^2} + \int_{\substack{|x-\xi|\geq|\xi|/2\\ |x|>2|\xi|}} \frac{4^{(N-2)(2-\gamma_\lambda)}}{|x|^{(N-2)(2-\gamma_\lambda)}}\frac{dx}{|x|^2}$$

$$\leq \frac{2^{(N-2)(2-\gamma_\lambda)}}{|\xi|^{(N-2)(2-\gamma_\lambda)}}\int_0^{2|\xi|} r^{N-3}\,dr + 4^{(N-2)(2-\gamma_\lambda)}\int_{2|\xi|}^{\infty}\frac{dr}{r^{N-1-\gamma_\lambda(N-2)}} = O(1).$$

From (63), (64), and (65) we deduce that

$$\int_{\mathbb{R}^N}\frac{|z_\mu^\lambda|^2}{|x+\xi|^2}\,dx = \frac{\mu^2}{|\xi|^2}\int_{\mathbb{R}^N}|z_1^\lambda(x)|^2\,dx + o(\mu^2).$$

For $\lambda = \frac{N(N-4)}{4}$ we have that $\gamma_\lambda = \frac{N-4}{N-2}$ and

$$\int_{\mathbb{R}^N}\frac{|z_\mu^\lambda|^2}{|x+\xi|^2}\,dx = \mu^2\int_{\mathbb{R}^N}\frac{|z_1^\lambda|^2}{|\mu x+\xi|^2}\,dx = \mu^2\int_{|x|<\frac{|\xi|}{2\mu}}\frac{|z_1^\lambda|^2}{|\mu x+\xi|^2}\,dx \tag{66}$$

$$+\,\alpha_{\lambda,N}^2\,\mu^2\int_{|x-\xi|\geq\frac{|\xi|}{2}}\frac{\big(\mu^{4/(N-2)}+|x-\xi|^{4/(N-2)}\big)^{-(N-2)}}{|x|^2|x-\xi|^{N-4}}\,dx.$$

Since

$$\left|\int_{|x|<\frac{|\xi|}{2\mu}}|z_1^\lambda(x)|^2\left[\frac{1}{|\mu x+\xi|^2}-\frac{1}{|\xi|^2}\right]dx\right|$$

$$\leq \frac{4\alpha_{\lambda,N}^2\mu^2}{|\xi|^4}\int_0^{\frac{|\xi|}{2\mu}}\frac{r^3}{\big(r^{\frac{2}{N-2}}+r^{-\frac{2}{N-2}}\big)^{N-2}}\,dr + \frac{8\alpha_{\lambda,N}^2\mu}{|\xi|^3}\int_0^{\frac{|\xi|}{2\mu}}\frac{r^2}{\big(r^{\frac{2}{N-2}}+r^{-\frac{2}{N-2}}\big)^{N-2}}\,dr = O(1),$$

we deduce that

$$\int_{|x|<\frac{|\xi|}{2\mu}}|z_1^\lambda(x)|^2\frac{dx}{|\mu x+\xi|^2} = \frac{1}{|\xi|^2}\int_{|x|<\frac{|\xi|}{2\mu}}|z_1^\lambda(x)|^2\,dx + O(1). \tag{67}$$

On the other hand

$$\int_{|x|<\frac{|\xi|}{2\mu}}|z_1^\lambda(x)|^2\,dx = \alpha_{\lambda,N}^2\int_0^{\frac{|\xi|}{2\mu}}\frac{r^3\,dr}{(1+r^{\frac{4}{N-2}})^{N-2}}$$

$$=\alpha_{\lambda,N}^2\left[\int_1^{\frac{|\xi|}{2\mu}}\frac{dr}{r}+\int_0^1\frac{r^3\,dr}{(1+r^{\frac{4}{N-2}})^{N-2}}+\int_1^{\frac{|\xi|}{2\mu}}\left[\frac{r^3}{(1+r^{\frac{4}{N-2}})^{N-2}}-\frac{1}{r}\right]dr\right].$$

Since

$$\left|\frac{r^3}{(1+r^{\frac{4}{N-2}})^{N-2}}-\frac{1}{r}\right|\sim\frac{(N-2)}{r^{\frac{N+2}{N-2}}}\quad\text{as}\quad r\to+\infty,$$

we can conclude that

$$\int_{|x|<\frac{|\xi|}{2\mu}}|z_1^\lambda(x)|^2\,dx = \alpha_{\lambda,N}^2\,|\ln\mu| + O(1). \tag{68}$$

From (67) and (68) we then deduce that

$$\int_{|x|<\frac{|\xi|}{2\mu}} |z_1^\lambda(x)|^2 \frac{dx}{|\mu x+\xi|^2} = \frac{\alpha_{\lambda,N}^2}{|\xi|^2}|\ln\mu| + O(1). \tag{69}$$

As above (see (65))

$$\int_{|x-\xi|\geq\frac{|\xi|}{2}} \frac{\big(\mu^{2(1-\gamma_\lambda)}+|x-\xi|^{2(1-\gamma_\lambda)}\big)^{-(N-2)}}{|x|^2|x-\xi|^{\gamma_\lambda(N-2)}}\,dx = O(1). \tag{70}$$

Gathering (66), (69), and (70) we deduce that

$$\int_{\mathbb{R}^N} \frac{|z_\mu^\lambda|^2}{|x+\xi|^2}\,dx = \alpha_{\lambda,N}^2\,\frac{\mu^2|\ln\mu|}{|\xi|^2} + o(\mu^2|\ln\mu|).$$

For $\lambda > \frac{N(N-4)}{4}$ we have that $\gamma_\lambda > \frac{N-4}{N-2}$ and

(71)

$$\begin{aligned}
\int_{\mathbb{R}^N} \frac{|z_\mu^\lambda|^2}{|x+\xi|^2}\,dx &= \alpha_{\lambda,N}^2\,\mu^{(1-\gamma_\lambda)(N-2)} \int_{\mathbb{R}^N} \frac{\big(\mu^{2(1-\gamma_\lambda)}+|x-\xi|^{2(1-\gamma_\lambda)}\big)^{-(N-2)}}{|x|^2|x-\xi|^{\gamma_\lambda(N-2)}}\,dx\\
&= \alpha_{\lambda,N}^2\,\mu^{(1-\gamma_\lambda)(N-2)} \int_{\mathbb{R}^N} \frac{dx}{|x|^2|x-\xi|^{(N-2)(2-\gamma_\lambda)}}\\
&\quad+ \alpha_{\lambda,N}^2\,\mu^{(1-\gamma_\lambda)(N-2)} \int_{\mathbb{R}^N} \left[\frac{\big(\mu^{2(1-\gamma_\lambda)}+|x-\xi|^{2(1-\gamma_\lambda)}\big)^{-(N-2)}}{|x|^2|x-\xi|^{\gamma_\lambda(N-2)}} - \frac{1}{|x|^2|x-\xi|^{(N-2)(2-\gamma_\lambda)}}\right]dx.
\end{aligned}$$

From the elementary inequality

$$|(a+b)^s - a^s| \leq C(a^{s-1}b + b^s)$$

which holds for some $C = C(s) > 0$ where $s \geq 1$, and for any $a, b \geq 0$, it follows that

$$\begin{aligned}
&\left|\frac{\big(\mu^{2(1-\gamma_\lambda)}+|x-\xi|^{2(1-\gamma_\lambda)}\big)^{-(N-2)}}{|x|^2|x-\xi|^{\gamma_\lambda(N-2)}} - \frac{1}{|x|^2|x-\xi|^{(N-2)(2-\gamma_\lambda)}}\right|\\
&\quad\leq \text{const}\left[\frac{\mu^{2(1-\gamma_\lambda)}\big(\mu^{2(1-\gamma_\lambda)}+|x-\xi|^{2(1-\gamma_\lambda)}\big)^{-(N-2)}}{|x|^2|x-\xi|^{2+\gamma_\lambda(N-4)}}\right.\\
&\qquad\left.+\frac{\mu^{2(1-\gamma_\lambda)(N-2)}\big(\mu^{2(1-\gamma_\lambda)}+|x-\xi|^{2(1-\gamma_\lambda)}\big)^{-(N-2)}}{|x|^2|x-\xi|^{(2-\gamma_\lambda)(N-2)}}\right].
\end{aligned} \tag{72}$$

Since

$$\begin{aligned}
&\int_{\mathbb{R}^N} \frac{\mu^{2(1-\gamma_\lambda)}\big(\mu^{2(1-\gamma_\lambda)}+|x-\xi|^{2(1-\gamma_\lambda)}\big)^{-(N-2)}}{|x|^2|x-\xi|^{2+\gamma_\lambda(N-4)}}\\
&\quad\leq \text{const}\Bigg[\mu^{2(1-\gamma_\lambda)}\int_{\substack{|x-\xi|>\frac{|\xi|}{2}\\ |x|<2|\xi|}}\frac{dx}{|x|^2}+\mu^{2(1-\gamma_\lambda)}\int_{\substack{|x-\xi|>\frac{|\xi|}{2}\\ |x|>2|\xi|}}\frac{dx}{|x|^{N(2-\gamma_\lambda)}}\\
&\qquad\qquad+\mu^{\gamma_\lambda(N-2)-N+4}\int_0^{\frac{|\xi|}{2\mu}}\frac{r^{N-1}\,dr}{r^{2+\gamma_\lambda(N-4)}\big(1+r^{2(1-\gamma_\lambda)}\big)^{N-2}}\Bigg]\\
&\quad= O(\mu^{2(1-\gamma_\lambda)})+O(\mu^{\gamma_\lambda(N-2)-N+4})=o(1)\quad\text{as}\quad \mu\to 0^+
\end{aligned}$$

and

$$\begin{aligned}
&\int_{\mathbb{R}^N} \frac{\mu^{2(1-\gamma_\lambda)(N-2)}\big(\mu^{2(1-\gamma_\lambda)}+|x-\xi|^{2(1-\gamma_\lambda)}\big)^{-(N-2)}}{|x|^2|x-\xi|^{(2-\gamma_\lambda)(N-2)}}\\
&\quad\leq \text{const}\Bigg[\mu^{2(1-\gamma_\lambda)(N-2)}\int_{\substack{|x-\xi|>\frac{|\xi|}{2}\\ |x|<2|\xi|}}\frac{dx}{|x|^2}+\mu^{2(1-\gamma_\lambda)(N-2)}\int_{\substack{|x-\xi|>\frac{|\xi|}{2}\\ |x|>2|\xi|}}\frac{dx}{|x|^{4N-6-3\gamma_\lambda(N-2)}}\\
&\qquad+\mu^{\gamma_\lambda(N-2)-N+4}\int_0^{\frac{|\xi|}{2\mu}}\frac{r^{N-1}\,dr}{r^{(2-\gamma_\lambda)(N-2)}\big(1+r^{2(1-\gamma_\lambda)}\big)^{N-2}}\Bigg]\\
&\quad\leq \text{const}\Bigg[\mu^{2(1-\gamma_\lambda)(N-2)}+\mu^{\gamma_\lambda(N-2)-N+4}+\mu^{\gamma_\lambda(N-2)-N+4}\int_1^{\frac{|\xi|}{2\mu}}\frac{dr}{r^{-3\gamma_\lambda(N-2)+3N-7}}\Bigg]\\
&\quad\leq \text{const}\big[\mu^{2(1-\gamma_\lambda)(N-2)}+\mu^{\gamma_\lambda(N-2)-N+4}+\mu^{2(1-\gamma_\lambda)(N-2)}\big]=o(1)\quad\text{as}\quad \mu\to 0^+
\end{aligned}$$

from (71) and (72) we deduce that

$$\int_{\mathbb{R}^N}\frac{|z_\mu^\lambda|^2}{|x+\xi|^2}\,dx=\alpha_{\lambda,N}^2\,\mu^{(1-\gamma_\lambda)(N-2)}\int_{\mathbb{R}^N}\frac{dx}{|x|^2|x-\xi|^{(N-2)(2-\gamma_\lambda)}}+o\big(\mu^{(1-\gamma_\lambda)(N-2)}\big). \tag{73}$$

Note that the function

$$\varphi(\xi):=\int_{\mathbb{R}^N}\frac{dx}{|x|^2|x-\xi|^{(N-2)(2-\gamma_\lambda)}}$$

is invariant by rotation and homogeneous, more precisely

$$\varphi(\eta\xi)=\eta^{-\sqrt{(N-2)^2-4\lambda}}\varphi(\xi),$$

hence

$$\varphi(\xi)=|\xi|^{-\sqrt{(N-2)^2-4\lambda}}\varphi(\xi/|\xi|)=|\xi|^{-\sqrt{(N-2)^2-4\lambda}}\varphi(e_1). \tag{74}$$

(73) and (74) yield the required estimate for $\lambda>\frac{N(N-4)}{4}$. ∎

References

[1] B. Abdellaoui, V. Felli, I. Peral, *Existence and multiplicity for perturbations of an equation involving Hardy inequality and critical Sobolev exponent in the whole* $\mathbb{R}^N$, Adv. Differential Equations, 9 (2004), 481–508.

[2] T. Aubin, *Some nonlinear problems in Riemannian geometry,* Springer Monographs in Mathematics, Springer-Verlag, Berlin, 1998.

[3] P. Baras, J. A. Goldstein, *The heat equation with a singular potential,* Trans. Amer. Math. Soc., 284 (1984), no. 1, 121–139.

[4] H. Berestycki, M. J. Esteban, *Existence and bifurcation of solutions for an elliptic degenerate problem,* J. Differential Equations, 134 (1997), no. 1, 1–25.

[5] H. Brezis, L. Nirenberg, *Positive solutions of nonlinear elliptic equations involving critical exponents,* Comm. Pure. Appl. Math., 36 (1983), no. 4, 437–477.

[6] J. E. Brothers, W. P. Ziemer, *Minimal rearrangements of Sobolev functions,* 15th winter school in abstract analysis (Srní, 1987), Acta Univ. Carolin. Math. Phys, 28 (1987), no. 2, 13–24.

[7] F. Catrina, Z.-Q. Wang, *On the Caffarelli-Kohn-Nirenberg inequalities: sharp constants, existence (and nonexistence), and symmetry of extremal functions*, Comm. Pure Appl. Math., 54 (2001), no. 2, 229–258.

[8] I. Catto, C. Le Bris, P. L. Lions, *On the thermodynamic limit for Hartree-Fock type models,* Ann. Inst. H. Poincaré Anal. Non Linaire, 18 (2001), no. 6, 687–760.

[9] M. Conti, S. Crotti, D. Pardo, *On the existence of positive solutions for a class of singular elliptic equations,* Adv. Differential Equations, 3 (1998), no. 1, 111–132.

[10] M. Conti, S. Terracini, G. Verzini, *Nodal solutions to a class of nonstandard superlinear equations on* $\mathbb{R}^N$*,* Adv. Differential Equations, 7 (2002), no. 3, 297–318.

[11] T. Duyckaerts, *Opérateur de Schrödinger avec potentiel singulier multipolaire,* Bulletin de la SMF, to appear.

[12] H. Egnell, *Elliptic boundary value problems with singular coefficients and critical nonlinearities,* Indiana Univ. Math., J. 38 (1989), no. 2, 235–251.

[13] V. Felli, M. Schneider, *Perturbation results of critical elliptic equations of Caffarelli-Kohn-Nirenberg type,* J. Diff. Equations, 191 (2003), no. 1, 121–142.

[14] A. Ferrero, F. Gazzola, *Existence of solutions for singular critical growth semilinear elliptic equations,* J. Differential Equations, 177 (2001), no. 2, 494–522.

[15] J. García Azorero, I. Peral, *Hardy Inequalities and some critical elliptic and parabolic problems,* J. Diff. Equations, 144 (1998), no. 2, 441–476.

[16] F. Gazzola, H. C. Grunau, *On the role of space dimension* $n = 2 + 2\sqrt{2}$ *in the semilinear Brezis-Nirenberg eigenvalue problem,* Analysis (Munich) 20 (2000), no. 4, 395–399.

[17] W. M. Frank, D. J. Land, R. M. Spector, *Singular potentials,* Rev. Modern Phys., 43 (1971), no. 1, 36–98.

[18] G. H. Hardy, J. E. Littlewood, G. Pólya, *Inequalities,* Reprint of the 1952 edition, Cambridge Mathematical Library. Cambridge University Press, Cambridge, 1988.

[19] E. Jannelli, *The role played by space dimension in elliptic critical problems,* J. Differential Equations, 156 (1999), no. 2, 407–426.

[20] L. D. Landau, E. M. Lifshitz, *Quantum mechanics,* Pergamon Press Ltd., London-Paris, 1965.

[21] J. M. Lévy-Leblond, *Electron capture by polar molecules,* Phys. Rev., 153 (1967), no. 1, 1–4.

[22] P. L. Lions, *The concentration-compactness principle in the calculus of variations. The limit case, part 1,* Rev. Matemática Iberoamericana, 1 (1985), no. 1, 145–201.

[23] P. L. Lions, *The concentration-compactness principle in the calculus of variations. The limit case, part 2,* Rev. Matemática Iberoamericana, 1 (1985), no. 2, 45–121.

[24] P. L. Lions, *Hartree-Fock and related equations,* Nonlinear partial differential equations and their applications, Collège de France Seminar, Vol. IX (Paris, 1985–1986), 304–333, Pitman Res. Notes Math. Ser., 181, Longman Sci. Tech., Harlow, 1988.

[25] D. Ruiz, M. Willem, *Elliptic problems with critical exponents and Hardy potentials,* J. Differential Equations, 190 (2003), no. 2, 524–538.

[26] D. Smets, *Nonlinear Schrödinger equations with Hardy potential and critical nonlinearities,* Trans. AMS, to appear.

[27] S. Terracini, *On positive entire solutions to a class of equations with singular coefficient and critical exponent,* Adv. Diff. Equa., 1 (1996), no. 2, 241–264.

[28] J. L. Vazquez, E. Zuazua, *The Hardy inequality and the asymptotic behavior of the heat equation with an inverse-square potential,* J. Funct. Anal., 173 (2000), no. 1, 103–153.
[29] M. Willem, *Analyse fonctionnelle élémentaire*, Cassini Éditeurs. Paris, 2003.

Università di Milano Bicocca, Dipartimento di Matematica e Applicazioni, Via Cozzi 53, 20125 Milano.
E-mail address: felli@matapp.unimib.it, suster@matapp.unimib.it.