

# ON SCHRÖDINGER OPERATORS WITH MULTISINGULAR INVERSE-SQUARE ANISOTROPIC POTENTIALS

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ABSTRACT. Localization of binding, positivity, and essential self-adjointness properties of a class of Schrödinger operators with many anisotropic inverse square singularities are discussed.

## 1. INTRODUCTION AND STATEMENT OF THE MAIN RESULTS

The present paper means to analyze spectral properties of Schrödinger operators with many anisotropic inverse square singularities. The interest in such a class of operators arises in nonrelativistic molecular physics, where the interaction between an electric charge and the dipole moment of a molecule gives rise to an inverse square potential with an anisotropic coupling strength. More precisely, the Schrödinger equation for the wave function of an electron interacting with a polar molecule (supposed to be point-like) can be written as

$$\left(-\frac{\hbar^2}{2m}\Delta + e\frac{x\cdot\mathbf{D}}{|x|^3} - E\right)\Psi = 0,$$

where  $e$  and  $m$  denote respectively the charge and the mass of the electron and  $\mathbf{D}$  is the dipole moment of the molecule, see [21].

In crystalline matter, the presence of many dipoles leads to consider multisingular Schrödinger operators of the form

$$(1) \quad -\Delta - \sum_{i=1}^k \frac{\lambda_i (x - a_i) \cdot \mathbf{d}_i}{|x - a_i|^3},$$

where  $k \in \mathbb{N}$ ,  $(a_1, \dots, a_k) \in \mathbb{R}^{kN}$ ,  $N \geq 3$ ,  $a_i \neq a_j$  for  $i \neq j$ ,  $(\lambda_1, \dots, \lambda_k) \in \mathbb{R}^k$ ,  $(\mathbf{d}_1, \dots, \mathbf{d}_k) \in \mathbb{R}^{kN}$ ,  $\lambda_i > 0$  and  $|\mathbf{d}_i| = 1$  for any  $i = 1, \dots, k$ .

From the mathematical point of view, potentials of the form  $\frac{\lambda(x\cdot\mathbf{d})}{|x|^3}$  are purely angular multiples of radial inverse-square functions and, consequently, share many features with inverse square radial potentials; in particular, having the same order of homogeneity, they satisfy a Hardy-type inequality, and can be regarded as critical due to their lack of inclusion in the Kato class.

A rich literature deals with Schrödinger equations and operators with isotropic Hardy-type singular potentials, both in the case of one pole, see e.g. [1, 8, 10, 16, 18, 27, 30, 32], and in the case of many singularities, see [4, 5, 7, 9, 11, 14, 15]. Concerning the case of anisotropic potentials, we mention that the asymptotic behavior near the singularity of solutions to equations associated

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to dipole-type Schrödinger operators is described in [12]. The asymptotic analysis performed in [12] turns out to be an important tool in the present paper which aims to discuss fundamental properties of Schrödinger operators of the form (1), such as positivity, essential self-adjointness, and spectral properties, following the techniques developed in [11] for Schrödinger operators with multipolar inverse-square potentials.

A first purpose of the present paper is to study the effect of the configurations of singularities and the orientations of dipoles on the positivity properties of the quadratic form associated to the operator in (1) and defined as

$$Q_{\lambda_1, \dots, \lambda_k, \mathbf{d}_1, \dots, \mathbf{d}_k, a_1, \dots, a_k} : \mathcal{D}^{1,2}(\mathbb{R}^N) \rightarrow \mathbb{R},$$

$$Q_{\lambda_1, \dots, \lambda_k, \mathbf{d}_1, \dots, \mathbf{d}_k, a_1, \dots, a_k}(u) := \int_{\mathbb{R}^N} |\nabla u(x)|^2 dx - \sum_{i=1}^k \frac{\lambda_i (x - a_i) \cdot \mathbf{d}_i}{|x - a_i|^3} u^2(x) dx,$$

where  $\mathcal{D}^{1,2}(\mathbb{R}^N)$  is the functional space given by the completion of  $C_c^\infty(\mathbb{R}^N)$  with respect to the Dirichlet norm

$$\|u\|_{\mathcal{D}^{1,2}(\mathbb{R}^N)} := \left( \int_{\mathbb{R}^N} |\nabla u(x)|^2 dx \right)^{1/2}.$$

We recall that a quadratic form  $Q : \mathcal{D}^{1,2}(\mathbb{R}^N) \rightarrow \mathbb{R}$  is said to be *positive definite* if

$$\inf_{\mathcal{D}^{1,2}(\mathbb{R}^N) \setminus \{0\}} \frac{Q(u)}{\|u\|_{\mathcal{D}^{1,2}(\mathbb{R}^N)}^2} > 0.$$

In the case of a simple dipole operator  $-\Delta - \frac{\lambda(x \cdot \mathbf{d})}{|x|^3}$ , the positivity depends on the value of  $\lambda$  with respect to the threshold

$$(2) \quad \Lambda_N := \sup_{u \in \mathcal{D}^{1,2}(\mathbb{R}^N) \setminus \{0\}} \frac{\int_{\mathbb{R}^N} \frac{x \cdot \mathbf{d}}{|x|^3} u^2(x) dx}{\int_{\mathbb{R}^N} |\nabla u(x)|^2 dx}.$$

We notice that, by rotation invariance,  $\Lambda_N$  does not depend on the unit vector  $\mathbf{d}$  and, by classical Hardy's inequality,  $\Lambda_N < 4/(N-2)^2$ . In particular  $\Lambda_N$  is the best constant in the following Hardy type inequality

$$(3) \quad \int_{\mathbb{R}^N} \frac{x \cdot \mathbf{d}}{|x|^3} u^2(x) dx \leq \Lambda_N \int_{\mathbb{R}^N} |\nabla u(x)|^2 dx \quad \text{for all } u \in \mathcal{D}^{1,2}(\mathbb{R}^N) \text{ and for any unit vector } \mathbf{d}.$$

Some numerical approximations of  $\Lambda_N$  can be found in [12, Table 1].

It is easy to verify that the quadratic form associated to  $-\Delta - \frac{\lambda(x \cdot \mathbf{d})}{|x|^3}$  is positive definite in  $\mathcal{D}^{1,2}(\mathbb{R}^N)$  if and only if  $|\lambda| < \Lambda_N^{-1}$ . Furthermore, the positivity condition for a one dipole operator can be expressed as a condition on the first eigenvalue of the angular component of the operator on the unit sphere  $\mathbb{S}^{N-1}$ . Indeed, letting

$$\mu_1^\lambda = \min_{\psi \in H^1(\mathbb{S}^{N-1}) \setminus \{0\}} \frac{\int_{\mathbb{S}^{N-1}} |\nabla_{\mathbb{S}^{N-1}} \psi(\theta)|^2 dV(\theta) - \lambda \int_{\mathbb{S}^{N-1}} (\theta \cdot \mathbf{d}) \psi^2(\theta) dV(\theta)}{\int_{\mathbb{S}^{N-1}} \psi^2(\theta) dV(\theta)}$$

be the first eigenvalue of the operator  $-\Delta_{\mathbb{S}^{N-1}} - \lambda(\theta \cdot \mathbf{d})$  on  $\mathbb{S}^{N-1}$ , it was proved in [12, Lemma 2.5] that the quadratic form associated to  $-\Delta - \frac{\lambda(x \cdot \mathbf{d})}{|x|^3}$  is positive definite if and only if  $\mu_1^\lambda > -\left(\frac{N-2}{2}\right)^2$ .

The analysis of spectral properties of Schrödinger operators with multiple isotropic inverse square singularities performed in [11] highlighted how the positivity of the associated quadratic form depends on the location and the strength of singularities. In the case of multiple anisotropic singularities, the problem of positivity becomes a more delicate issue, being the interaction between two dipoles strongly affected by their mutual orientation. Indeed, while in the isotropic case the interaction between two singularities becomes weaker and weaker as the distance between them increases, in the anisotropic case it is possible to orientate the dipoles in such a way that the interaction is quite strong even if they are very far away from each other. Unlike the isotropic case in which the interaction between two poles is either attractive or repulsive depending on the sign of coefficients, in the anisotropic one the constructive or destructive character of the interaction is determined by the mutual position and orientation.

The following proposition yields a sufficient condition on the magnitudes for the quadratic form to be positive definite for any localization and orientation of the dipoles.

**Proposition 1.1.** *A sufficient condition for  $Q_{\lambda_1, \dots, \lambda_k, \mathbf{d}_1, \dots, \mathbf{d}_k, a_1, \dots, a_k}$  to be positive definite for any choice of  $a_1, a_2, \dots, a_k$  and  $\mathbf{d}_1, \dots, \mathbf{d}_k$  is that*

$$\sum_{i=1}^k \lambda_i < \Lambda_N^{-1}.$$

*Conversely, if  $\sum_{i=1}^k \lambda_i > \Lambda_N^{-1}$  then there exist points  $a_1, a_2, \dots, a_k$  and unit vectors  $\mathbf{d}_1, \dots, \mathbf{d}_k$  such that  $Q_{\lambda_1, \dots, \lambda_k, \mathbf{d}_1, \dots, \mathbf{d}_k, a_1, \dots, a_k}$  is not positive definite.*

In this paper we deal with a more general class of Schrödinger operators with locally anisotropic inverse-square singularities including those with dipole-type potentials introduced in (1). More precisely, we are interested in operators with potentials exhibiting many singularities which are locally  $L^\infty$ -angular multiples of radial inverse-square potentials.

For any  $h \in L^\infty(\mathbb{S}^{N-1})$ , let  $\mu_1(h)$  be the first eigenvalue of the operator  $-\Delta_{\mathbb{S}^{N-1}} - h(\theta)$  on  $\mathbb{S}^{N-1}$ , i.e.

$$\mu_1(h) = \min_{\psi \in H^1(\mathbb{S}^{N-1}) \setminus \{0\}} \frac{\int_{\mathbb{S}^{N-1}} |\nabla_{\mathbb{S}^{N-1}} \psi(\theta)|^2 dV(\theta) - \int_{\mathbb{S}^{N-1}} h(\theta) \psi^2(\theta) dV(\theta)}{\int_{\mathbb{S}^{N-1}} \psi^2(\theta) dV(\theta)}.$$

We recall that  $\mu_1(h)$  is simple and attained by a smooth positive eigenfunction  $\psi_1^h$  such that  $\min_{\mathbb{S}^{N-1}} \psi_1^h > 0$ . Moreover, if, for some  $\lambda \in \mathbb{R}$ ,  $h(\theta) = \lambda$  for a.e.  $\theta \in \mathbb{S}^{N-1}$ , then  $\mu_1(h) = -\lambda$ . On the other hand, if  $h$  is not constant, then

$$-\operatorname{ess\,sup}_{\mathbb{S}^{N-1}} h < \mu_1(h) < - \int_{\mathbb{S}^{N-1}} h(\theta) dV(\theta),$$

see [12]. The quadratic form associated to  $-\Delta - \frac{h(x/|x|)}{|x|^2}$  is positive definite if and only if

$$(4) \quad \Lambda_N(h) := \sup_{u \in \mathcal{D}^{1,2}(\mathbb{R}^N) \setminus \{0\}} \frac{\int_{\mathbb{R}^N} \frac{h(x/|x|)}{|x|^2} u^2(x) dx}{\int_{\mathbb{R}^N} |\nabla u(x)|^2 dx} < 1,$$

or, equivalently, if and only if  $\mu_1(h) > -(N-2)^2/4$ , see [12, Lemma 2.5]. Furthermore, it is easy to verify that

$$\Lambda_N(h) \geq 0 \quad \text{for all } h \in L^\infty(\mathbb{S}^{N-1})$$

and

$$\Lambda_N(h) = 0 \quad \text{if and only if} \quad h \leq 0 \text{ a.e. on } \mathbb{S}^{N-1}.$$

A necessary condition on the angular coefficients for positivity of the quadratic form associated with multiple dipole-type potentials for at least a configuration of singularities is that each single dipole-type local subsystem is positive definite, as the following proposition clarifies.

**Proposition 1.2.** *Let  $h_1, \dots, h_k, h_\infty \in L^\infty(\mathbb{S}^{N-1})$ ,  $W \in L^{\frac{N}{2}}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ , and  $R, r_i \in \mathbb{R}^+$ ,  $i = 1, \dots, k$ . If there exists a configuration of poles  $\{a_1, \dots, a_k\}$  such that  $a_i \neq a_j$  for  $i \neq j$  and the quadratic form*

$$\begin{aligned} u \in \mathcal{D}^{1,2}(\mathbb{R}^N) \mapsto & \int_{\mathbb{R}^N} |\nabla u(x)|^2 dx - \sum_{i=1}^k \int_{B(a_i, r_i)} \frac{h_i\left(\frac{x-a_i}{|x-a_i|}\right)}{|x-a_i|^2} u^2(x) dx \\ & - \int_{\mathbb{R}^N \setminus B(0, R)} \frac{h_\infty\left(\frac{x}{|x|}\right)}{|x|^2} u^2(x) dx - \int_{\mathbb{R}^N} W(x) u^2(x) dx \end{aligned}$$

is positive definite, then

$$(5) \quad \mu_1(h_i) > -\frac{(N-2)^2}{4} \quad \text{for any } i = 1, \dots, k, \infty.$$

By virtue of Proposition 1.2, the following class of anisotropic multiple inverse square potentials provides a suitable framework for the analysis of coercivity conditions for Schrödinger dipole-type operators:

$$\mathcal{V} := \left\{ \begin{array}{l} V(x) = \sum_{i=1}^k \chi_{B(a_i, r_i)}(x) \frac{h_i\left(\frac{x-a_i}{|x-a_i|}\right)}{|x-a_i|^2} + \chi_{\mathbb{R}^N \setminus B(0, R)}(x) \frac{h_\infty\left(\frac{x}{|x|}\right)}{|x|^2} + W(x) : k \in \mathbb{N}, \\ r_i, R \in \mathbb{R}^+, a_i \in \mathbb{R}^N, a_i \neq a_j \text{ for } i \neq j, W \in L^{N/2}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N), \\ h_i \in L^\infty(\mathbb{S}^{N-1}), \mu_1(h_i) > -\frac{(N-2)^2}{4} \text{ for any } i = 1, \dots, k, \infty \end{array} \right\}.$$

If  $V = \sum_{i=1}^k \frac{\lambda_i(x-a_i) \cdot \mathbf{d}_i}{|x-a_i|^3}$ , then  $V \in \mathcal{V}$  with  $h_i(\theta) = \lambda_i \theta \cdot \mathbf{d}_i$  and  $h_\infty(\theta) = \theta \cdot (\sum_{i=1}^k \lambda_i \mathbf{d}_i)$ .

By Hardy's and Sobolev's inequalities, it follows that, for any  $V \in \mathcal{V}$ , the first eigenvalue  $\mu(V)$  of the operator  $-\Delta - V$  in  $\mathcal{D}^{1,2}(\mathbb{R}^N)$  is finite, namely

$$\mu(V) := \inf_{u \in \mathcal{D}^{1,2}(\mathbb{R}^N) \setminus \{0\}} \frac{\int_{\mathbb{R}^N} (|\nabla u(x)|^2 - V(x)u^2(x)) dx}{\int_{\mathbb{R}^N} |\nabla u(x)|^2 dx} > -\infty.$$

If the potentials are supported in sufficiently small neighborhoods of singularities, then condition (5) turns out to be also sufficient for positivity.

**Lemma 1.3. [Shattering of singularities]** *Let  $a_1, a_2, \dots, a_k \in \mathbb{R}^N$ ,  $a_i \neq a_j$  for  $i \neq j$ , and  $h_1, \dots, h_k, h_\infty \in L^\infty(\mathbb{S}^{N-1})$  with  $\mu_1(h_i) > -(N-2)^2/4$ , for  $i = 1, \dots, k, \infty$ . Then there exist  $\mathcal{U}_1, \dots, \mathcal{U}_k, \mathcal{U}_\infty \subset \mathbb{R}^N$  such that  $\mathcal{U}_i$  is a neighborhood of  $a_i$  for every  $i = 1, \dots, k$ ,  $\mathcal{U}_\infty$  is a*

neighborhood of  $\infty$ , and the quadratic form associated to the operator

$$-\Delta - \sum_{i=1}^k \chi_{\mathcal{U}_i}(x) \frac{h_i\left(\frac{x-a_i}{|x-a_i|}\right)}{|x-a_i|^2} - \chi_{\mathcal{U}_\infty}(x) \frac{h_\infty\left(\frac{x}{|x|}\right)}{|x|^2}$$

is positive definite.

An analogous result will be proved also for potentials with infinitely many dipole-type singularities localized in sufficiently small neighborhoods of equidistant poles, see Lemma 3.8.

Lemma 1.3 implies that Schrödinger operators with potentials in  $\mathcal{V}$  are compact perturbations of positive operators, as stated in the following corollary.

**Corollary 1.4.** *For any  $V \in \mathcal{V}$ , there exist  $\tilde{V} \in \mathcal{V}$  and  $\tilde{W} \in L^{N/2}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$  such that  $\mu(\tilde{V}) > 0$  and  $V(x) = \tilde{V}(x) + \tilde{W}(x)$ .*

The analysis of stability of positivity of Schrödinger operators leads to the problem of *localization of binding* raised by Sigal and Ouchinnokov [23]: if  $-\Delta - V_1$  and  $-\Delta - V_2$  are positive operators, is  $-\Delta - V_1 - V_2(\cdot - y)$  positive for  $|y|$  large? An affirmative answer to the above question can be found in [29] for compactly supported potentials and in [24] for potentials in the Kato class. When dealing with potentials with an inverse square singularity at infinity, the problem becomes more delicate, due to the interaction of singularities which overlap at infinity and a *localization of binding* type result requires the additional assumption of some control of the resulting singularity at infinity. Indeed, if, for  $j = 1, 2$ ,

$$V_j = \sum_{i=1}^{k_j} \chi_{B(a_i^j, r_i^j)}(x) \frac{h_i^j\left(\frac{x-a_i^j}{|x-a_i^j|}\right)}{|x-a_i^j|^2} + \chi_{\mathbb{R}^N \setminus B(0, R_j)}(x) \frac{h_\infty^j\left(\frac{x}{|x|}\right)}{|x|^2} + W_j(x) \in \mathcal{V},$$

then a necessary condition for positivity of  $-\Delta - V_1 - V_2(\cdot - y)$  for some  $y$  is that

$$(6) \quad \mu_1(h_\infty^1 + h_\infty^2) > -\left(\frac{N-2}{2}\right)^2,$$

see Proposition 5.2. In [11], the authors proved that assumption (6) is also sufficient for *localization of binding* when singularities are locally isotropic, see Theorem 5.1. A lack of isotropy could produce the failure of localization of binding even under assumption (6); in section 5, we will construct two anisotropic potentials in the class  $\mathcal{V}$  satisfying (6) for which no localization of binding result holds true, see Example 5.3. On the other hand, it is still possible to prove the following *localization of binding* type result under a stronger control on the singularities at infinity than (6).

**Theorem 1.5.** *Let*

$$V_1(x) = \sum_{i=1}^{k_1} \chi_{B(a_i^1, r_i^1)}(x) \frac{h_i^1\left(\frac{x-a_i^1}{|x-a_i^1|}\right)}{|x-a_i^1|^2} + \chi_{\mathbb{R}^N \setminus B(0, R_1)}(x) \frac{h_\infty^1\left(\frac{x}{|x|}\right)}{|x|^2} + W_1(x) \in \mathcal{V},$$

$$V_2(x) = \sum_{i=1}^{k_2} \chi_{B(a_i^2, r_i^2)}(x) \frac{h_i^2\left(\frac{x-a_i^2}{|x-a_i^2|}\right)}{|x-a_i^2|^2} + \chi_{\mathbb{R}^N \setminus B(0, R_2)}(x) \frac{h_\infty^2\left(\frac{x}{|x|}\right)}{|x|^2} + W_2(x) \in \mathcal{V}.$$

*Assume that  $\mu(V_1), \mu(V_2) > 0$ , and  $\text{ess sup}_{\mathbb{S}^{N-1}}(h_1^\infty)^+ + \text{ess sup}_{\mathbb{S}^{N-1}}(h_2^\infty)^+ < (N-2)^2/4$ . Then, there exists  $R > 0$  such that  $\mu(V_1 + V_2(\cdot - y)) > 0$  for every  $y \in \mathbb{R}^N$  with  $|y| \geq R$ .*

By Corollary 1.4, Schrödinger operators with potentials in  $\mathcal{V}$  are semi-bounded in  $L^2(\mathbb{R}^N)$ , i.e.

$$\nu_1(V) := \inf_{u \in H^1(\mathbb{R}^N) \setminus \{0\}} \frac{\int_{\mathbb{R}^N} (|\nabla u(x)|^2 - V(x)u^2(x)) dx}{\int_{\mathbb{R}^N} |u(x)|^2 dx} \geq -\|\widetilde{W}\|_{L^\infty(\mathbb{R}^N)} > -\infty,$$

thus the class  $\mathcal{V}$  provides a quite natural setting to study the spectral properties of multisingular dipole Schrödinger operators in  $L^2(\mathbb{R}^N)$ .

A further key property of Schrödinger operators which turns out to be very sensitive to the presence of singular terms is the *essential self-adjointness*, namely the existence of a unique self-adjoint extension. Semi-bounded Schrödinger operators are essentially self-adjoint whenever the potential is not too singular (see [26]). On the other hand, inverse square potentials exhibit a quite strong singularity which makes the problem of essential self-adjointness nontrivial. We mention that essential self-adjointness in the case of Hardy type potentials was discussed in [19] for the one-pole case and in [11] for many poles. The following theorem provides a necessary and sufficient condition on the magnitudes of dipole moments for the essential self-adjointness of multisingular dipole Schrödinger operators. An extension to the case of infinitely many dipole-type singularities distributed on reticular structures is contained in Theorem 6.3.

**Theorem 1.6.** *Let*

$$V(x) = \sum_{i=1}^k \chi_{B(a_i, r_i)}(x) \frac{h_i \left( \frac{x-a_i}{|x-a_i|} \right)}{|x-a_i|^2} + \chi_{\mathbb{R}^N \setminus B(0, R)}(x) \frac{h_\infty \left( \frac{x}{|x|} \right)}{|x|^2} + W(x) \in \mathcal{V}.$$

*Then the Schrödinger operator  $-\Delta - V$  is essentially self-adjoint in  $C_c^\infty(\mathbb{R}^N \setminus \{a_1, \dots, a_k\})$  if and only if  $\mu_1(h_i) \geq -\left(\frac{N-2}{2}\right)^2 + 1$ , for all  $i = 1, \dots, k$ .*

The proof of the above theorem is based on the asymptotic analysis performed in [12], where the exact behavior near the poles of solutions to Schrödinger equations with dipole-type singular potentials is evaluated. From Theorem 1.6, it follows that, if  $V \in \mathcal{V}$  with  $\mu_1(h_i) \geq -\left(\frac{N-2}{2}\right)^2 + 1$ , for all  $i = 1, \dots, k$ , then the *Friedrichs extension*  $(-\Delta - V)^F$  defined as

$$(7) \quad D((-\Delta - V)^F) = \{u \in H^1(\mathbb{R}^N) : \Delta u - Vu \in L^2(\mathbb{R}^N)\}, \quad u \mapsto -\Delta u - Vu,$$

is the unique self-adjoint extension. On the other hand, if  $\mu_1(h_i) < -\left(\frac{N-2}{2}\right)^2 + 1$  for some  $i$ , then  $-\Delta - V$  admits many self-adjoint extensions, among which the Friedrichs extension is the only one whose domain is included in  $H^1(\mathbb{R}^N)$ , namely it is the unique self-adjoint extension to which we can associate a natural quadratic form.

The paper is organized as follows. Section 2 contains the proofs of Propositions 1.1 and 1.2. In section 3 we prove a positivity criterion in the spirit of the Allegretto-Piepenbrink theory, which is used to prove Lemma 1.3 and its reticular version (see Lemma 3.8); a key tool in the proof of Lemma 1.3 is the analysis of positivity of potentials obtained as juxtaposition of potentials with different singularity rate, which is a nontrivial issue due to the lack of isotropy, see Lemmas 3.4, 3.5, and 3.6. In section 4 we discuss the stability of positivity with respect to perturbations of the potentials with singularities localized at dipolar-shaped neighborhoods either of a dipole or of infinity. In section 5 we prove Theorem 1.5 and show that condition (6) is no sufficient for localization of binding by constructing a suitable example. Section 6 is devoted to the proof of Theorem 1.6 and of its reticular counterpart.

**Notation.** We list below some notation used throughout the paper.

- $B(a, r)$  denotes the ball  $\{x \in \mathbb{R}^N : |x - a| < r\}$  in  $\mathbb{R}^N$  with center at  $a$  and radius  $r$ .
- $\mathbb{R}^+ := (0, +\infty)$  is the half line of positive real numbers.
- For any  $A \subset \mathbb{R}^N$ ,  $\chi_A$  denotes the characteristic function of  $A$ .
- $S$  is the best constant in the Sobolev inequality  $S\|u\|_{L^{2^*}(\mathbb{R}^N)}^2 \leq \|u\|_{\mathcal{D}^{1,2}(\mathbb{R}^N)}^2$ .
- For all  $t \in \mathbb{R}$ ,  $t^+ := \max\{t, 0\}$  (respectively  $t^- := \max\{-t, 0\}$ ) denotes the positive (respectively negative) part of  $t$ .
- For all functions  $f : \mathbb{R}^N \rightarrow \mathbb{R}$ ,  $\text{supp } f$  denotes the support of  $f$ , i.e. the closure of the set of points where  $f$  is non zero.
- $\omega_N$  denotes the volume of the unit ball in  $\mathbb{R}^N$ .
- For any open set  $\Omega \subset \mathbb{R}^N$ ,  $\mathcal{D}'(\Omega)$  denotes the space of distributions in  $\Omega$ .
- For any  $f \in L^\infty(A)$ , with either  $A = \mathbb{S}^{N-1}$  or  $A = \mathbb{R}^N$ , we denote the essential supremum of  $f$  in  $A$  as  $\text{ess sup}_A f := \inf\{\alpha \in \mathbb{R} : f(x) \leq \alpha \text{ for a.e. } x \in A\}$ , while the essential infimum of  $f$  in  $A$  is denoted as  $\text{ess inf}_A f := -\text{ess sup}_A(-f)$ .

## 2. PROOF OF PROPOSITIONS 1.1 AND 1.2

This section is devoted to the proof of Propositions 1.1 and 1.2.

**Proof of Proposition 1.1.** From (2), it follows that

$$(8) \quad Q_{\lambda_1, \dots, \lambda_k, \mathbf{d}_1, \dots, \mathbf{d}_k, a_1, \dots, a_k} \geq \left(1 - \Lambda_N \sum_{i=1}^k \lambda_i\right) \int_{\mathbb{R}^N} |\nabla u|^2 dx.$$

Hence a sufficient condition for  $Q_{\lambda_1, \dots, \lambda_k, \mathbf{d}_1, \dots, \mathbf{d}_k, a_1, \dots, a_k}$  to be positive definite is that

$$\sum_{i=1}^k \lambda_i < \Lambda_N^{-1}.$$

Assume now that  $\sum_{i=1}^k \lambda_i > \Lambda_N^{-1}$  and fix  $\mathbf{d} \in \mathbb{R}^N$ ,  $|\mathbf{d}| = 1$ . From (2) and density of  $C_c^\infty(\mathbb{R}^N)$  in  $\mathcal{D}^{1,2}(\mathbb{R}^N)$ , there exists some function  $\phi \in C_c^\infty(\mathbb{R}^N)$  such that

$$\int_{\mathbb{R}^N} |\nabla \phi|^2 dx - \left(\sum_{i=1}^k \lambda_i\right) \int_{\mathbb{R}^N} \frac{x \cdot \mathbf{d}}{|x|^3} \phi^2 dx < 0.$$

For any  $\mu > 0$ , consider the function  $\phi_\mu(x) = \mu^{-\frac{N-2}{2}} \phi(x/\mu)$ . A change of variable yields

$$\int_{\mathbb{R}^N} |\nabla \phi_\mu|^2 dx - \sum_{i=1}^k \lambda_i \int_{\mathbb{R}^N} \frac{(x - a_i) \cdot \mathbf{d}}{|x - a_i|^3} \phi_\mu^2 dx = \int_{\mathbb{R}^N} |\nabla \phi|^2 dx - \sum_{i=1}^k \lambda_i \int_{\mathbb{R}^N} \frac{x \cdot \mathbf{d}}{|x|^3} \phi^2 \left(x + \frac{a_i}{\mu}\right) dx$$

for all  $\mu > 0$ . Letting  $\mu \rightarrow \infty$ , the Dominated Convergence Theorem yields

$$\int_{\mathbb{R}^N} |\nabla \phi_\mu|^2 dx - \sum_{i=1}^k \lambda_i \int_{\mathbb{R}^N} \frac{(x - a_i) \cdot \mathbf{d}}{|x - a_i|^3} \phi_\mu^2 dx \longrightarrow \int_{\mathbb{R}^N} |\nabla \phi|^2 dx - \left(\sum_{i=1}^k \lambda_i\right) \int_{\mathbb{R}^N} \frac{x \cdot \mathbf{d}}{|x|^3} \phi^2 dx < 0$$

therefore  $Q_{\lambda_1, \dots, \lambda_k, \mathbf{d}, \dots, \mathbf{d}, a_1, \dots, a_k}(\phi_\mu) < 0$  for  $\mu$  sufficiently large, thus proving the second part of Proposition 1.1.  $\square$

**Proof of Proposition 1.2.** Assume that, for some configuration  $\{a_1, \dots, a_k\}$ , for some  $\varepsilon > 0$ , and for any  $u \in \mathcal{D}^{1,2}(\mathbb{R}^N)$ ,

$$(9) \quad \int_{\mathbb{R}^N} |\nabla u(x)|^2 dx - \sum_{i=1}^k \int_{B(a_i, r_i)} \frac{h_i\left(\frac{x-a_i}{|x-a_i|}\right)}{|x-a_i|^2} u^2(x) dx \\ - \int_{\mathbb{R}^N \setminus B(0, R)} \frac{h_\infty\left(\frac{x}{|x|}\right)}{|x|^2} u^2(x) dx - \int_{\mathbb{R}^N} W(x) u^2(x) dx \geq \varepsilon \int_{\mathbb{R}^N} |\nabla u(x)|^2 dx.$$

Arguing by contradiction, suppose that, for some  $i = 1, \dots, k$ ,  $\mu_1(h_i) \leq -(N-2)^2/4$ , or equivalently that  $\Lambda_N(h_i) \geq 1$ , with  $\Lambda_N(h_i)$  as in (4). Let  $0 < \delta < \varepsilon \Lambda_N(h_i)^{-1}$ . By (4) and density of  $C_c^\infty(\mathbb{R}^N)$  in  $\mathcal{D}^{1,2}(\mathbb{R}^N)$ , there exists  $\phi \in C_c^\infty(\mathbb{R}^N)$  such that

$$(10) \quad \int_{\mathbb{R}^N} |\nabla \phi(x)|^2 dx - \int_{\mathbb{R}^N} \frac{h_i\left(\frac{x}{|x|}\right)}{|x|^2} \phi^2(x) dx < \delta \int_{\mathbb{R}^N} \frac{h_i\left(\frac{x}{|x|}\right)}{|x|^2} \phi^2(x) dx.$$

The rescaled function  $\phi_\mu(x) = \mu^{-(N-2)/2} \phi(x/\mu)$  satisfies

$$(11) \quad \int_{\mathbb{R}^N} |\nabla \phi_\mu(x-a_i)|^2 dx - \sum_{j=1}^k \int_{B(a_j, r_j)} \frac{h_j\left(\frac{x-a_j}{|x-a_j|}\right)}{|x-a_j|^2} \phi_\mu^2(x-a_j) dx \\ - \int_{\mathbb{R}^N \setminus B(0, R)} \frac{h_\infty\left(\frac{x}{|x|}\right)}{|x|^2} \phi_\mu^2(x-a_i) dx - \int_{\mathbb{R}^N} W(x) \phi_\mu^2(x-a_i) dx \\ = \int_{\mathbb{R}^N} |\nabla \phi(x)|^2 dx - \int_{B(0, \frac{r_i}{\mu})} \frac{h_i\left(\frac{x}{|x|}\right)}{|x|^2} \phi^2(x) dx - \int_{\mathbb{R}^N \setminus B\left(\frac{-a_i}{\mu}, \frac{R}{\mu}\right)} \frac{h_\infty\left(\frac{x+a_i/\mu}{|x+a_i/\mu|}\right)}{|x+a_i/\mu|^2} \phi^2(x) dx \\ - \sum_{j \neq i} \int_{B\left(\frac{a_j-a_i}{\mu}, \frac{r_j}{\mu}\right)} \frac{h_j\left(\frac{x-(a_j-a_i)/\mu}{|x-(a_j-a_i)/\mu|}\right)}{|x-(a_j-a_i)/\mu|^2} \phi^2(x) dx - \mu^2 \int_{\mathbb{R}^N} W(\mu x + a_i) \phi^2(x) dx \\ = \int_{\mathbb{R}^N} |\nabla \phi(x)|^2 dx - \int_{\mathbb{R}^N} \frac{h_i\left(\frac{x}{|x|}\right)}{|x|^2} \phi^2(x) dx + o(1), \quad \text{as } \mu \rightarrow 0.$$

Letting  $\mu \rightarrow 0$ , by (10) and (2), we obtain

$$\varepsilon \int_{\mathbb{R}^N} |\nabla \phi(x)|^2 dx \leq \int_{\mathbb{R}^N} |\nabla \phi(x)|^2 dx - \int_{\mathbb{R}^N} \frac{h_i\left(\frac{x}{|x|}\right)}{|x|^2} \phi^2(x) dx \\ < \delta \int_{\mathbb{R}^N} \frac{h_i\left(\frac{x}{|x|}\right)}{|x|^2} \phi^2(x) dx \leq \delta \Lambda_N(h_i) \int_{\mathbb{R}^N} |\nabla \phi(x)|^2 dx$$

thus giving rise to a contradiction.

Suppose now that,  $\mu_1(h_\infty) \leq -\frac{(N-2)^2}{4}$ , or equivalently that  $\Lambda_N(h_\infty) \geq 1$ , with  $\Lambda_N(h_\infty)$  defined in (4). Let  $\delta \in (0, \varepsilon \Lambda_N(h_\infty)^{-1})$ . By definition of  $\Lambda_N(h_\infty)$  and density of  $C_c^\infty(\mathbb{R}^N \setminus \{0\})$  in  $\mathcal{D}^{1,2}(\mathbb{R}^N)$ , there exists  $\phi \in C_c^\infty(\mathbb{R}^N \setminus \{0\})$  such that

$$\int_{\mathbb{R}^N} |\nabla \phi(x)|^2 dx - \int_{\mathbb{R}^N} \frac{h_\infty\left(\frac{x}{|x|}\right)}{|x|^2} \phi^2(x) dx < \delta \int_{\mathbb{R}^N} \frac{h_\infty\left(\frac{x}{|x|}\right)}{|x|^2} \phi^2(x) dx.$$

Since  $\phi \in C_c^\infty(\mathbb{R}^N \setminus \{0\})$ , the rescaled function  $\phi_\mu(x) = \mu^{-(N-2)/2}\phi(x/\mu)$  satisfies

$$\begin{aligned} \int_{\mathbb{R}^N} |\nabla \phi_\mu(x)|^2 dx &- \sum_{i=1}^k \int_{B(a_i, r_i)} \frac{h_i\left(\frac{x-a_i}{|x-a_i|}\right)}{|x-a_i|^2} \phi_\mu^2(x) dx \\ &- \int_{\mathbb{R}^N \setminus B(0, R)} \frac{h_\infty\left(\frac{x}{|x|}\right)}{|x|^2} \phi_\mu^2(x) dx - \int_{\mathbb{R}^N} W(x) \phi_\mu^2(x) dx \\ &= \int_{\mathbb{R}^N} |\nabla \phi(x)|^2 dx - \sum_{i=1}^k \int_{B\left(\frac{a_i}{\mu}, \frac{r_i}{\mu}\right)} \frac{h_i\left(\frac{x-a_i/\mu}{|x-a_i/\mu|}\right)}{|x-a_i/\mu|^2} \phi^2(x) dx \\ &- \int_{\mathbb{R}^N \setminus B\left(0, \frac{R}{\mu}\right)} \frac{h_\infty\left(\frac{x}{|x|}\right)}{|x|^2} \phi^2(x) dx - \mu^2 \int_{\mathbb{R}^N} W(\mu x) \phi^2(x) dx \\ &= \int_{\mathbb{R}^N} |\nabla \phi(x)|^2 dx - \int_{\mathbb{R}^N} \frac{h_\infty\left(\frac{x}{|x|}\right)}{|x|^2} \phi^2(x) dx + o(1), \quad \text{as } \mu \rightarrow \infty. \end{aligned}$$

Letting  $\mu \rightarrow \infty$  and arguing as above, we obtain easily a contradiction.  $\square$

### 3. THE SHATTERING LEMMA

The well-known Allegretto-Piepenbrink theory [2, 25] suggests us a criterion for establishing positivity of Schrödinger operators with potentials in  $\mathcal{V}$ , by relating the existence of positive solutions to a Schrödinger equation with the positivity of the spectrum of the corresponding operator. For analogous criteria for potentials in the Kato class we refer to [6, Theorem 2.12].

**Lemma 3.1.** *Let  $V \in \mathcal{V}$ . Then the two following conditions are equivalent:*

$$(i) \quad \mu(V) := \inf_{u \in \mathcal{D}^{1,2}(\mathbb{R}^N) \setminus \{0\}} \frac{\int_{\mathbb{R}^N} (|\nabla u(x)|^2 - V(x)u^2(x)) dx}{\int_{\mathbb{R}^N} |\nabla u(x)|^2 dx} > 0;$$

(ii) *there exist  $\varepsilon > 0$  and  $\varphi \in \mathcal{D}^{1,2}(\mathbb{R}^N)$ ,  $\varphi > 0$  in  $\mathbb{R}^N \setminus \{a_1, \dots, a_k\}$ , and  $\varphi$  continuous in  $\mathbb{R}^N \setminus \{a_1, \dots, a_k\}$ , such that  $-\Delta\varphi - V\varphi \geq \varepsilon V(x)\varphi$  in  $(\mathcal{D}^{1,2}(\mathbb{R}^N))^*$ , i.e.*

$${}_{(\mathcal{D}^{1,2}(\mathbb{R}^N))^*} \langle -\Delta\varphi - V\varphi - \varepsilon V\varphi, w \rangle_{\mathcal{D}^{1,2}(\mathbb{R}^N)} = \int_{\mathbb{R}^N} [\nabla\varphi \cdot \nabla w - (1 + \varepsilon)V\varphi w] dx \geq 0$$

*for any  $w \in \mathcal{D}^{1,2}(\mathbb{R}^N)$  such that  $w \geq 0$  a.e. in  $\mathbb{R}^N$ .*

Moreover, if (ii) holds,  $\mu(V) \geq \frac{\varepsilon}{1+\varepsilon}$ .

PROOF. Let  $V(x) = \sum_{i=1}^k \chi_{B(a_i, r_i)}(x) \frac{h_i((x-a_i)/|x-a_i|)}{|x-a_i|^2} + \chi_{\mathbb{R}^N \setminus B(0, R)}(x) \frac{h_\infty(x/|x|)}{|x|^2} + W(x) \in \mathcal{V}$  and set  $\bar{h} := \sum_{i=1}^k \|h_i\|_{L^\infty(\mathbb{S}^{N-1})} + \|h_\infty\|_{L^\infty(\mathbb{S}^{N-1})}$ . From Hardy's, Hölder's, and Sobolev's inequalities there holds

$$(12) \quad \int_{\mathbb{R}^N} V(x)u^2(x) dx \leq \left[ \frac{4\bar{h}}{(N-2)^2} + S^{-1}\|W\|_{L^{N/2}(\mathbb{R}^N)} \right] \int_{\mathbb{R}^N} |\nabla u(x)|^2 dx,$$

for every  $u \in \mathcal{D}^{1,2}(\mathbb{R}^N)$ .

Assume that (i) holds. If  $0 < \varepsilon < \frac{\mu(V)}{2} \left[ \frac{4\bar{h}}{(N-2)^2} + S^{-1} \|W\|_{L^{N/2}(\mathbb{R}^N)} \right]^{-1}$ , from (12) it follows that

$$\int_{\mathbb{R}^N} (|\nabla u(x)|^2 - (1 + \varepsilon)V(x)u^2(x)) dx \geq \frac{\mu(V)}{2} \int_{\mathbb{R}^N} |\nabla u(x)|^2 dx.$$

As a consequence, for any fixed  $p \in L^{N/2}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ ,  $p(x) > 0$  a.e. in  $\mathbb{R}^N$ , the infimum

$$\nu_p(V + \varepsilon V) = \inf_{u \in \mathcal{D}^{1,2}(\mathbb{R}^N) \setminus \{0\}} \frac{\int_{\mathbb{R}^N} (|\nabla u(x)|^2 - (1 + \varepsilon)V(x)u^2(x)) dx}{\int_{\mathbb{R}^N} p(x)u^2(x)}$$

is strictly positive and attained by some function  $\varphi \in \mathcal{D}^{1,2}(\mathbb{R}^N) \setminus \{0\}$  satisfying

$$-\Delta \varphi(x) - V(x)\varphi(x) = \varepsilon V(x)\varphi(x) + \nu_p(V + \varepsilon V)p(x)\varphi(x).$$

By evenness we can assume  $\varphi \geq 0$ . Since  $V \in \mathcal{V}$ , the Strong Maximum Principle allows us to conclude that  $\varphi > 0$  in  $\mathbb{R}^N \setminus \{a_1, \dots, a_k\}$ , while standard regularity theory ensures regularity of  $\varphi$  outside the poles. Hence (ii) holds.

Assume now that (ii) holds. For any  $u \in C_c^\infty(\mathbb{R}^N \setminus \{a_1, \dots, a_k\})$ , testing the weak inequality satisfied by  $\varphi$  with  $u^2/\varphi$  we get

$$(1 + \varepsilon) \int_{\mathbb{R}^N} V(x)u^2(x) dx \leq 2 \int_{\mathbb{R}^N} \frac{u(x)}{\varphi(x)} \nabla u(x) \cdot \nabla \varphi(x) dx - \int_{\mathbb{R}^N} \frac{u^2(x)}{\varphi^2(x)} |\nabla \varphi(x)|^2 dx \leq \int_{\mathbb{R}^N} |\nabla u(x)|^2 dx.$$

By density of  $C_c^\infty(\mathbb{R}^N \setminus \{a_1, \dots, a_k\})$  in  $\mathcal{D}^{1,2}(\mathbb{R}^N)$  we deduce that, for every  $u \in \mathcal{D}^{1,2}(\mathbb{R}^N) \setminus \{0\}$ ,

$$\int_{\mathbb{R}^N} V(x)u^2(x) \leq \frac{1}{1 + \varepsilon} \int_{\mathbb{R}^N} |\nabla u(x)|^2 dx,$$

implying

$$(13) \quad \int_{\mathbb{R}^N} (|\nabla u(x)|^2 - V(x)u^2(x)) dx \geq \frac{\varepsilon}{1 + \varepsilon} \int_{\mathbb{R}^N} |\nabla u(x)|^2 dx,$$

and hence  $\mu(V) \geq \frac{\varepsilon}{\varepsilon + 1} > 0$ .  $\square$

The above positivity criterion allows to extend to multiple dipole Schrödinger operators the Shattering Lemma in [11, Lemma 1.3] yielding positivity in the case of singularities localized strictly near the poles.

Let us start by observing that, evaluating the quotient minimized in the definition of  $\mu(V)$  at functions concentrating at the singularities,  $\mu(V)$  can be estimated from above as follows.

**Lemma 3.2.** *For any*

$$(14) \quad V(x) = \sum_{i=1}^k \chi_{B(a_i, r_i)}(x) \frac{h_i\left(\frac{x-a_i}{|x-a_i|}\right)}{|x-a_i|^2} + \chi_{\mathbb{R}^N \setminus B(0, R)}(x) \frac{h_\infty\left(\frac{x}{|x|}\right)}{|x|^2} + W(x) \in \mathcal{V},$$

there holds

$$\mu(V) \leq 1 - \max \{0, \Lambda_N(h_1), \dots, \Lambda_N(h_k), \Lambda_N(h_\infty)\},$$

where  $\Lambda_N(h_i)$  is defined in (4).

PROOF. Let us first consider the case  $\Lambda_N(h_i) = 0$  for every  $i = 1, \dots, k, \infty$ . Let us fix  $u \in C_c^\infty(\mathbb{R}^N)$  and  $P \in \mathbb{R}^N \setminus \{a_1, \dots, a_k\}$ . Letting  $u_\mu(x) = \mu^{-\frac{N-2}{2}} u\left(\frac{x-P}{\mu}\right)$ , for  $\mu$  small there holds

$$\mu(V) \leq 1 - \frac{\int_{\mathbb{R}^N} W(x) u_\mu^2(x) dx}{\int_{\mathbb{R}^N} |\nabla u_\mu(x)|^2 dx} = 1 + o(1) \quad \text{as } \mu \rightarrow 0^+.$$

Letting  $\mu \rightarrow 0^+$  we obtain that  $\mu(V) \leq 1$ .

Assume now that  $\max_{i=1, \dots, k, \infty} \Lambda_N(h_i) > 0$ . Suppose  $\Lambda_N(h_1) \leq \Lambda_N(h_2) \leq \dots \leq \Lambda_N(h_k)$  and let  $\varepsilon > 0$ . From (4) and by density of  $C_c^\infty(\mathbb{R}^N)$  in  $\mathcal{D}^{1,2}(\mathbb{R}^N)$ , there exists  $\phi \in C_c^\infty(\mathbb{R}^N)$  such that

$$\int_{\mathbb{R}^N} |\nabla \phi(x)|^2 dx < \left[ \Lambda_N(h_k)^{-1} + \varepsilon \right] \int_{\mathbb{R}^N} \frac{h_k\left(\frac{x-a_k}{|x-a_k|}\right)}{|x-a_k|^2} \phi^2(x) dx.$$

Letting  $\phi_\mu(x) = \mu^{-\frac{N-2}{2}} \phi\left(\frac{x-a_k}{\mu}\right)$ , for any  $\mu > 0$  there holds

$$\begin{aligned} \mu(V) &\leq 1 - \frac{\int_{B(a_k, r_k)} h_k\left(\frac{x-a_k}{|x-a_k|}\right) |x-a_k|^{-2} \phi_\mu^2(x) dx}{\int_{\mathbb{R}^N} |\nabla \phi_\mu(x)|^2 dx} - \sum_{i=1}^{k-1} \frac{\int_{B(a_i, r_i)} h_i\left(\frac{x-a_i}{|x-a_i|}\right) |x-a_i|^{-2} \phi_\mu^2(x) dx}{\int_{\mathbb{R}^N} |\nabla \phi_\mu(x)|^2 dx} \\ &\quad - \frac{\int_{\mathbb{R}^N \setminus B(0, R)} h_\infty\left(\frac{x}{|x|}\right) |x|^{-2} \phi_\mu^2(x) dx}{\int_{\mathbb{R}^N} |\nabla \phi_\mu(x)|^2 dx} - \frac{\int_{\mathbb{R}^N} W(x) \phi_\mu^2(x) dx}{\int_{\mathbb{R}^N} |\nabla \phi_\mu(x)|^2 dx} \\ &= 1 - \frac{\int_{\mathbb{R}^N} h_k\left(\frac{x}{|x|}\right) |x|^{-2} \phi^2(x) dx}{\int_{\mathbb{R}^N} |\nabla \phi(x)|^2 dx} + o(1) \quad \text{as } \mu \rightarrow 0^+. \end{aligned}$$

Letting  $\mu \rightarrow 0^+$ , by the choice of  $\phi$  we obtain

$$\mu(V) \leq 1 - \left[ \Lambda_N(h_k)^{-1} + \varepsilon \right]^{-1}$$

for any  $\varepsilon > 0$ . Letting  $\varepsilon \rightarrow 0$  we derive that  $\mu(V) \leq 1 - \Lambda_N(h_k)$ . Repeating the same argument, we obtain a function  $\psi_\infty \in C_c^\infty(\mathbb{R}^N \setminus \{0\})$  such that

$$\int_{\mathbb{R}^N} |\nabla \psi_\infty(x)|^2 dx < \left[ \Lambda_N(h_\infty)^{-1} + \varepsilon \right] \int_{\mathbb{R}^N} \frac{h_\infty\left(\frac{x}{|x|}\right)}{|x|^2} \psi_\infty^2(x) dx.$$

Setting  $\psi_\mu(x) = \mu^{-\frac{N-2}{2}} \psi_\infty\left(\frac{x}{\mu}\right)$  and letting  $\mu \rightarrow +\infty$  we obtain also that  $\mu(V) \leq 1 - \Lambda_N(h_\infty)$ . The required estimate is thereby proved.  $\square$

The proof of Lemma 1.3 is immediate if  $\Lambda_N(h_i) = 0$  for all  $i = 1, \dots, k, \infty$ , i.e. if  $h_i \leq 0$  a.e. in  $\mathbb{S}^{N-1}$  for all  $i = 1, \dots, k, \infty$ , as the following lemma states.

**Lemma 3.3.** *Let  $a_1, a_2, \dots, a_k \in \mathbb{R}^N$ ,  $a_i \neq a_j$  for  $i \neq j$ , and  $h_1, \dots, h_k, h_\infty \in L^\infty(\mathbb{S}^{N-1})$  with  $\mu_1(h_i) > -(N-2)^2/4$ , for  $i = 1, \dots, k, \infty$ . Then, if  $\Lambda_N(h_i) = 0$  for all  $i = 1, \dots, k, \infty$ , for every  $\mathcal{U}_1, \dots, \mathcal{U}_k, \mathcal{U}_\infty \subset \mathbb{R}^N$  such that  $\mathcal{U}_i$  is a neighborhood of  $a_i$  for every  $i = 1, \dots, k$  and  $\mathcal{U}_\infty$  is a neighborhood of  $\infty$ , there holds*

$$\mu \left( \sum_{i=1}^k \chi_{\mathcal{U}_i}(x) \frac{h_i\left(\frac{x-a_i}{|x-a_i|}\right)}{|x-a_i|^2} + \chi_{\mathcal{U}_\infty}(x) \frac{h_\infty\left(\frac{x}{|x|}\right)}{|x|^2} \right) = 1.$$

PROOF. The inequality  $\mu\left(\sum_{i=1}^k \chi_{\Omega_i}(x) \frac{h_i((x-a_i)/|x-a_i|)}{|x-a_i|^2} + \chi_{\Omega_\infty}(x) \frac{h_\infty(x/|x|)}{|x|^2}\right) \leq 1$  follows from Lemma 3.2, whereas the reverse inequality comes immediately from the assumption  $h_i \leq 0$  a.e. in  $\mathbb{S}^{N-1}$  for all  $i = 1, \dots, k, \infty$ .  $\square$

When dealing with isotropic potentials, it is quite easy to study the positivity of potentials obtained as juxtaposition of potentials with different singularity rate. If, for example,  $\Omega_1 \subset \Omega_2$ , then the operator  $-\Delta - \frac{\lambda_1}{|x|^2} \chi_{\Omega_1} - \frac{\lambda_2}{|x|^2} \chi_{\Omega_2 \setminus \Omega_1}$  is positive definite whenever  $\max\{\lambda_1, \lambda_2\} < (N-2)^2/4$ . On the other hand, the positivity of a potential obtained as juxtaposition of two potentials with changing sign angular components is more delicate to be established. The analysis we are going to develop shows how juxtaposition of potentials giving rise to positive quadratic forms produces positive operators if their contact region has some particular shape (which resembles a sphere deformed according to dipole coefficients).

For any  $h \in L^\infty(\mathbb{S}^{N-1})$ , let  $\psi_1^h$  denote the positive  $L^2$ -normalized eigenfunction associated to the first eigenvalue  $\mu_1(h)$  of the operator  $-\Delta_{\mathbb{S}^{N-1}} - h(\theta)$  on  $\mathbb{S}^{N-1}$ . Let us notice that, for  $h \equiv 0$ ,  $\psi_1^0 \equiv 1/\sqrt{\omega_N}$ , where  $\omega_N$  denotes the volume of the unit sphere  $\mathbb{S}^{N-1}$ , i.e.  $\omega_N = \int_{\mathbb{S}^{N-1}} dV(\theta)$ .

For  $h_1, h_2 \in L^\infty(\mathbb{S}^{N-1})$ ,  $\sigma > 0$ , and  $R > 0$ , let us denote

$$(15) \quad \mathcal{E}_{h_1, h_2}^{\sigma, R} := \left\{ x \in \mathbb{R}^N : |x| < R \left( \frac{\psi_1^{h_1}(x/|x|)}{\psi_1^{h_2}(x/|x|)} \right)^\sigma \right\}.$$

Let us notice that

$$B\left(0, R \left[ \min_{\theta \in \mathbb{S}^{N-1}} \frac{\psi_1^{h_1}(\theta)}{\psi_1^{h_2}(\theta)} \right]^\sigma\right) \subset \mathcal{E}_{h_1, h_2}^{\sigma, R} \subset B\left(0, R \left[ \max_{\theta \in \mathbb{S}^{N-1}} \frac{\psi_1^{h_1}(\theta)}{\psi_1^{h_2}(\theta)} \right]^\sigma\right).$$

We also set, for  $a \in \mathbb{R}^N$ ,

$$\mathcal{E}_{h_1, h_2}^{\sigma, R}(a) := \{x \in \mathbb{R}^N : x - a \in \mathcal{E}_{h_1, h_2}^{\sigma, R}\}.$$

**Lemma 3.4.** *Let  $h_i \in L^\infty(\mathbb{S}^{N-1})$ ,  $\mu_1(h_i) > -\left(\frac{N-2}{2}\right)^2$ ,  $i = 1, 2$ ,  $\sigma_1, \sigma_2 > 0$  such that*

$$(16) \quad \frac{N-2}{2} < \frac{1}{\sigma_1} < \frac{1}{\sigma_1} + \frac{1}{\sigma_2} < \frac{N-2}{2} + \min_{i=1,2} \sqrt{\frac{(N-2)^2}{4} + \mu_1(h_i)},$$

and  $R_1, R_2 \in (0, +\infty)$  such that  $\mathcal{E}_{h_1, 0}^{\sigma_1, R_1} \subset \mathcal{E}_{h_2, h_1}^{\sigma_2, R_2}$ . Then

(i) the function

$$(17) \quad u(x) = \begin{cases} \omega_N^{-1/2}, & \text{in } \mathcal{E}_{h_1, 0}^{\sigma_1, R_1}, \\ R_1^{1/\sigma_1} |x|^{-\frac{1}{\sigma_1}} \psi_1^{h_1}(x/|x|), & \text{in } \mathcal{E}_{h_2, h_1}^{\sigma_2, R_2} \setminus \mathcal{E}_{h_1, 0}^{\sigma_1, R_1}, \\ R_1^{1/\sigma_1} R_2^{1/\sigma_2} |x|^{-\left(\frac{1}{\sigma_1} + \frac{1}{\sigma_2}\right)} \psi_1^{h_2}(x/|x|), & \text{in } \mathbb{R}^N \setminus \mathcal{E}_{h_2, h_1}^{\sigma_2, R_2}, \end{cases}$$

belongs to  $\mathcal{D}^{1,2}(\mathbb{R}^N)$ ;

(ii) the distribution

$$\mathcal{H} = -\Delta u - \frac{h_1(x/|x|)}{|x|^2} \chi_{\mathcal{E}_{h_2, h_1}^{\sigma_2, R_2} \setminus \mathcal{E}_{h_1, 0}^{\sigma_1, R_1}} u - \frac{h_2(x/|x|)}{|x|^2} \chi_{\mathbb{R}^N \setminus \mathcal{E}_{h_2, h_1}^{\sigma_2, R_2}} u$$

belongs to the dual space  $(\mathcal{D}^{1,2}(\mathbb{R}^N))^*$ ;

(iii) there exists a positive constant  $C$  (depending only on  $N$ ,  $\sigma_1$ ,  $\sigma_2$ ,  $h_1$ , and  $h_2$ ) such that  $\mathcal{H} \geq C \chi_{\mathbb{R}^N \setminus \mathcal{E}_{h_1,0}^{\sigma_1, R_1}} \frac{u}{|x|^2}$  in  $(\mathcal{D}^{1,2}(\mathbb{R}^N))^*$ , i.e.

$$(\mathcal{D}^{1,2}(\mathbb{R}^N))^* \langle \mathcal{H}, w \rangle_{\mathcal{D}^{1,2}(\mathbb{R}^N)} \geq C \int_{\mathbb{R}^N \setminus \mathcal{E}_{h_1,0}^{\sigma_1, R_1}} \frac{u w}{|x|^2}$$

for any  $w \in \mathcal{D}^{1,2}(\mathbb{R}^N)$  such that  $w \geq 0$  a.e. in  $\mathbb{R}^N$ .

PROOF. Let us consider the function

$$u(x) = \begin{cases} \omega_N^{-1/2}, & \text{in } \mathcal{E}_{h_1,0}^{\sigma_1, R_1}, \\ \varphi_1(x), & \text{in } \mathcal{E}_{h_2, h_1}^{\sigma_2, R_2} \setminus \mathcal{E}_{h_1,0}^{\sigma_1, R_1}, \\ \varphi_2(x), & \text{in } \mathbb{R}^N \setminus \mathcal{E}_{h_2, h_1}^{\sigma_2, R_2}, \end{cases}$$

where

$$\varphi_1(x) = R_1^{1/\sigma_1} |x|^{-\frac{1}{\sigma_1}} \psi_1^{h_1}(x/|x|), \quad \text{and} \quad \varphi_2(x) = R_1^{1/\sigma_1} R_2^{1/\sigma_2} |x|^{-\left(\frac{1}{\sigma_1} + \frac{1}{\sigma_2}\right)} \psi_1^{h_2}(x/|x|).$$

By definition of the sets  $\mathcal{E}_{h_2, h_1}^{\sigma_2, R_2}$  and  $\mathcal{E}_{h_1,0}^{\sigma_1, R_1}$ , it follows that  $u \in \mathcal{D}^{1,2}(\mathbb{R}^N) \cap C^0(\mathbb{R}^N)$ . Moreover

$$-\Delta \varphi_1 - \frac{h_1(x/|x|)}{|x|^2} \varphi_1 = \left[ \left(\frac{N-2}{2}\right)^2 + \mu_1(h_1) - \left(\frac{1}{\sigma_1} - \frac{N-2}{2}\right)^2 \right] \frac{\varphi_1}{|x|^2}, \quad \text{in } \mathbb{R}^N \setminus \{0\},$$

$$-\Delta \varphi_2 - \frac{h_2(x/|x|)}{|x|^2} \varphi_2 = \left[ \left(\frac{N-2}{2}\right)^2 + \mu_1(h_2) - \left(\frac{1}{\sigma_1} + \frac{1}{\sigma_2} - \frac{N-2}{2}\right)^2 \right] \frac{\varphi_2}{|x|^2}, \quad \text{in } \mathbb{R}^N \setminus \{0\}.$$

Let us denote as  $\nu_1(x)$  the outward normal derivate to  $\mathcal{E}_{h_1,0}^{\sigma_1, R_1}$  and as  $\nu_2(x)$  the outward normal derivate to  $\mathcal{E}_{h_2, h_1}^{\sigma_2, R_2}$ . A direct calculation shows that

$$\partial \mathcal{E}_{h_1,0}^{\sigma_1, R_1} = \{x \in \mathbb{R}^N : \varphi_1(x) = \omega_N^{-1/2}\} \quad \text{and} \quad \nabla \varphi_1 \cdot \frac{x}{|x|} = -\frac{\varphi_1(x)}{\sigma_1 |x|} < 0 \quad \text{on } \partial \mathcal{E}_{h_1,0}^{\sigma_1, R_1},$$

hence  $\nu_1(x) = -\frac{\nabla \varphi_1(x)}{|\nabla \varphi_1(x)|}$  for all  $x \in \partial \mathcal{E}_{h_1,0}^{\sigma_1, R_1}$ . In a similar way

$$\partial \mathcal{E}_{h_2, h_1}^{\sigma_2, R_2} = \{x \in \mathbb{R}^N : \varphi_1(x) = \varphi_2(x)\} \quad \text{and} \quad \nabla(\varphi_1 - \varphi_2) \cdot \frac{x}{|x|} = \frac{\varphi_2(x)}{\sigma_2 |x|} > 0 \quad \text{on } \partial \mathcal{E}_{h_2, h_1}^{\sigma_2, R_2},$$

hence  $\nu_2(x) = \frac{\nabla(\varphi_1 - \varphi_2)(x)}{|\nabla(\varphi_1 - \varphi_2)(x)|}$  for all  $x \in \partial \mathcal{E}_{h_2, h_1}^{\sigma_2, R_2}$ . Therefore, for any  $w \in \mathcal{D}^{1,2}(\mathbb{R}^N)$ ,  $w \geq 0$  a.e. in  $\mathbb{R}^N$ , there holds

$$\begin{aligned} & (\mathcal{D}^{1,2}(\mathbb{R}^N))^* \left\langle -\Delta u - \frac{h_1(x/|x|)}{|x|^2} \chi_{\mathcal{E}_{h_2, h_1}^{\sigma_2, R_2} \setminus \mathcal{E}_{h_1,0}^{\sigma_1, R_1}} u - \frac{h_2(x/|x|)}{|x|^2} \chi_{\mathbb{R}^N \setminus \mathcal{E}_{h_2, h_1}^{\sigma_2, R_2}} u, w \right\rangle_{\mathcal{D}^{1,2}(\mathbb{R}^N)} \\ &= - \int_{\partial \mathcal{E}_{h_1,0}^{\sigma_1, R_1}} (\nabla \varphi_1 \cdot \nu_1) w \, dS + \int_{\partial \mathcal{E}_{h_2, h_1}^{\sigma_2, R_2}} (\nabla(\varphi_1 - \varphi_2) \cdot \nu_2) w \, dS \\ &+ \left[ \left(\frac{N-2}{2}\right)^2 + \mu_1(h_1) - \left(\frac{1}{\sigma_1} - \frac{N-2}{2}\right)^2 \right] \int_{\mathcal{E}_{h_2, h_1}^{\sigma_2, R_2} \setminus \mathcal{E}_{h_1,0}^{\sigma_1, R_1}} \frac{\varphi_1 w}{|x|^2} \\ &+ \left[ \left(\frac{N-2}{2}\right)^2 + \mu_1(h_2) - \left(\frac{1}{\sigma_1} + \frac{1}{\sigma_2} - \frac{N-2}{2}\right)^2 \right] \int_{\mathbb{R}^N \setminus \mathcal{E}_{h_2, h_1}^{\sigma_2, R_2}} \frac{\varphi_2 w}{|x|^2} \geq C \int_{\mathbb{R}^N \setminus \mathcal{E}_{h_1,0}^{\sigma_1, R_1}} \frac{u w}{|x|^2}, \end{aligned}$$

where  $C = \min \left\{ \left( \frac{N-2}{2} \right)^2 + \mu_1(h_1) - \left( \frac{1}{\sigma_1} - \frac{N-2}{2} \right)^2, \left( \frac{N-2}{2} \right)^2 + \mu_1(h_2) - \left( \frac{1}{\sigma_1} + \frac{1}{\sigma_2} - \frac{N-2}{2} \right)^2 \right\} > 0$ . The proof is complete.  $\square$

The Kelvin's transform yields the counterpart of Lemma 3.4 in the case of singularities located at finite dipoles, as we prove below.

**Lemma 3.5.** *Let  $h_i \in L^\infty(\mathbb{S}^{N-1})$ ,  $\mu_1(h_i) > -\left(\frac{N-2}{2}\right)^2$ ,  $i = 1, 2$ ,  $\sigma_1, \sigma_2 > 0$  satisfying (16), and  $r_1, r_2 \in (0, +\infty)$  such that  $\mathcal{E}_{h_1, h_2}^{\sigma_2, r_2} \subset \mathcal{E}_{0, h_1}^{\sigma_1, r_1}$ . Then*

(i) *the function*

$$(18) \quad v(x) = \begin{cases} |x|^{-(N-2)} \omega_N^{-1/2}, & \text{in } \mathbb{R}^N \setminus \mathcal{E}_{0, h_1}^{\sigma_1, r_1}, \\ r_1^{-1/\sigma_1} |x|^{-(N-2) + \frac{1}{\sigma_1}} \psi_1^{h_1}(x/|x|), & \text{in } \mathcal{E}_{0, h_1}^{\sigma_1, r_1} \setminus \mathcal{E}_{h_1, h_2}^{\sigma_2, r_2}, \\ r_1^{-1/\sigma_1} r_2^{-1/\sigma_2} |x|^{-(N-2) + \frac{1}{\sigma_1} + \frac{1}{\sigma_2}} \psi_1^{h_2}(x/|x|), & \text{in } \mathcal{E}_{h_1, h_2}^{\sigma_2, r_2}, \end{cases}$$

*belongs to  $\mathcal{D}^{1,2}(\mathbb{R}^N)$ ;*

(ii) *the distribution*

$$\mathcal{H} = -\Delta v - \frac{h_1(x/|x|)}{|x|^2} \chi_{\mathcal{E}_{0, h_1}^{\sigma_1, r_1} \setminus \mathcal{E}_{h_1, h_2}^{\sigma_2, r_2}} v - \frac{h_2(x/|x|)}{|x|^2} \chi_{\mathcal{E}_{h_1, h_2}^{\sigma_2, r_2}} v$$

*belongs to the dual space  $(\mathcal{D}^{1,2}(\mathbb{R}^N))^*$ ;*

(iii) *there exists a positive constant  $C$  (depending only on  $N$ ,  $\sigma_1$ ,  $\sigma_2$ ,  $h_1$ , and  $h_2$ ) such that  $\mathcal{H} \geq C \chi_{\mathcal{E}_{0, h_1}^{\sigma_1, r_1}} \frac{v}{|x|^2}$  in  $(\mathcal{D}^{1,2}(\mathbb{R}^N))^*$ , i.e.*

$$(\mathcal{D}^{1,2}(\mathbb{R}^N))^* \langle \mathcal{H}, w \rangle_{\mathcal{D}^{1,2}(\mathbb{R}^N)} \geq C \int_{\mathcal{E}_{0, h_1}^{\sigma_1, r_1}} \frac{v w}{|x|^2}$$

*for any  $w \in \mathcal{D}^{1,2}(\mathbb{R}^N)$  such that  $w \geq 0$  a.e. in  $\mathbb{R}^N$ .*

PROOF. Let us consider the function  $u$  defined in (17) with  $R_1 = 1/r_1$  and  $R_2 = 1/r_2$ . Then the function  $v$  defined in (18) is the Kelvin's transformed of  $u$ , i.e.

$$u(x) = |x|^{-(N-2)} v(x/|x|^2).$$

Since  $\Delta v(x) = |x|^{-N-2} \Delta u(x/|x|^2)$ , the conclusion follows from Lemma 3.4.  $\square$

**Lemma 3.6.** *Let  $h_1, \dots, h_k, h_\infty, H_1, \dots, H_k, H_\infty \in L^\infty(\mathbb{S}^{N-1})$  satisfying*

$$0 < \max_{i=1, \dots, k, \infty} \{ \Lambda_N(h_i), \Lambda_N(H_i) \} < 1,$$

*$0 < \varepsilon < \left( \max_{i=1, \dots, k, \infty} \{ \Lambda_N(h_i), \Lambda_N(H_i) \} \right)^{-1} - 1$ ,  $\tilde{h}_i = (1+\varepsilon)h_i$ ,  $\tilde{H}_i = (1+\varepsilon)H_i$  for all  $i = 1, \dots, k, \infty$ ,  $\sigma_1, \sigma_2 > 0$  such that*

$$\frac{N-2}{2} < \frac{1}{\sigma_1} < \frac{1}{\sigma_1} + \frac{1}{\sigma_2} < \frac{N-2}{2} + \min_{i=1, \dots, k, \infty} \left\{ \frac{N-2}{2}, \sqrt{\frac{(N-2)^2}{4} + \mu_1(\tilde{h}_i)}, \sqrt{\frac{(N-2)^2}{4} + \mu_1(\tilde{H}_i)} \right\},$$

*$\{a_1, a_2, \dots, a_k\} \subset \mathbb{R}^N$ ,  $a_i \neq a_j$  for  $i \neq j$ , and  $R_0 > 0$  such that*

$$\{a_1, a_2, \dots, a_k\} \subset B\left(0, R_0 \left[ \min_{\theta \in \mathbb{S}^{N-1}} \sqrt{\omega_N} \psi_1^{\tilde{h}_\infty}(\theta) \right]^{\sigma_1}\right) \subset \mathcal{E}_{\tilde{h}_\infty, 0}^{\sigma_1, R_0}.$$

Then there exists  $\bar{\delta} > 0$  such that for all  $0 < \delta < \bar{\delta}$ , for any  $R$  such that  $\mathcal{E}_{\tilde{h}_\infty, 0}^{\sigma_1, R_0} \subset \mathcal{E}_{\tilde{H}_\infty, \tilde{h}_\infty}^{\sigma_2, R}$ , and for any  $r > 0$  such that  $\mathcal{E}_{\tilde{h}_i, \tilde{H}_i}^{\sigma_2, r} \subset \mathcal{E}_{0, \tilde{h}_i}^{\sigma_1, \delta}$  for any  $i = 1, \dots, k$ , there holds

$$\mu \left( \sum_{i=1}^k \chi_{\mathcal{E}_{0, \tilde{h}_i}^{\sigma_1, \delta}(a_i) \setminus \mathcal{E}_{\tilde{h}_i, \tilde{H}_i}^{\sigma_2, r}(a_i)}(x) \frac{h_i \left( \frac{x-a_i}{|x-a_i|} \right)}{|x-a_i|^2} + \sum_{i=1}^k \chi_{\mathcal{E}_{\tilde{h}_i, \tilde{H}_i}^{\sigma_2, r}(a_i)}(x) \frac{H_i \left( \frac{x-a_i}{|x-a_i|} \right)}{|x-a_i|^2} \right. \\ \left. + \chi_{\mathcal{E}_{\tilde{H}_\infty, \tilde{h}_\infty}^{\sigma_2, R} \setminus \mathcal{E}_{\tilde{h}_\infty, 0}^{\sigma_1, R_0}} \frac{h_\infty \left( \frac{x}{|x|} \right)}{|x|^2} + \chi_{\mathbb{R}^N \setminus \mathcal{E}_{\tilde{H}_\infty, \tilde{h}_\infty}^{\sigma_2, R}}(x) \frac{H_\infty \left( \frac{x}{|x|} \right)}{|x|^2} \right) \geq \frac{\varepsilon}{\varepsilon + 1}.$$

PROOF. By scaling properties of the operator and in view of Lemma 3.1, to prove the statement it is enough to find  $\varphi \in \mathcal{D}^{1,2}(\mathbb{R}^N)$  positive and continuous outside the singularities such that

$$(19) \quad -\Delta \varphi(x) - \sum_{i=1}^k V_i(x) \varphi(x) - V_\infty(x) \varphi(x) \geq 0 \quad \text{in } (\mathcal{D}^{1,2}(\mathbb{R}^N))^*,$$

where

$$V_\infty(x) = \chi_{\mathcal{E}_{\tilde{H}_\infty, \tilde{h}_\infty}^{\sigma_2, R/\delta} \setminus \mathcal{E}_{\tilde{h}_\infty, 0}^{\sigma_1, R_0/\delta}} \frac{\tilde{h}_\infty(x/|x|)}{|x|^2} + \chi_{\mathbb{R}^N \setminus \mathcal{E}_{\tilde{H}_\infty, \tilde{h}_\infty}^{\sigma_2, R/\delta}} \frac{\tilde{H}_\infty(x/|x|)}{|x|^2}$$

and

$$V_i(x) = \chi_{\mathcal{E}_{0, \tilde{h}_i}^{\sigma_1, 1}(a_i/\delta) \setminus \mathcal{E}_{\tilde{h}_i, \tilde{H}_i}^{\sigma_2, r/\delta}(a_i/\delta)} \frac{\tilde{h}_i \left( \frac{x-(a_i/\delta)}{|x-(a_i/\delta)|} \right)}{|x - \frac{a_i}{\delta}|^2} + \chi_{\mathcal{E}_{\tilde{h}_i, \tilde{H}_i}^{\sigma_2, r/\delta}(a_i/\delta)}(x) \frac{\tilde{H}_i \left( \frac{x-(a_i/\delta)}{|x-(a_i/\delta)|} \right)}{|x - \frac{a_i}{\delta}|^2},$$

and  $\delta > 0$  depends neither on  $R$  nor on  $r$  (but could depend on  $\varepsilon$ ).

Let us consider the function  $\varphi_\infty(x) = u(\delta x)$ , where

$$u(x) = \begin{cases} \omega_N^{-1/2}, & \text{in } \mathcal{E}_{\tilde{h}_\infty, 0}^{\sigma_1, R_0}, \\ R_0^{1/\sigma_1} |x|^{-\frac{1}{\sigma_1}} \psi_1^{\tilde{h}_\infty}(x/|x|), & \text{in } \mathcal{E}_{\tilde{H}_\infty, \tilde{h}_\infty}^{\sigma_2, R} \setminus \mathcal{E}_{\tilde{h}_\infty, 0}^{\sigma_1, R_0}, \\ R_0^{1/\sigma_1} R^{1/\sigma_2} |x|^{-\left(\frac{1}{\sigma_1} + \frac{1}{\sigma_2}\right)} \psi_1^{\tilde{H}_\infty}(x/|x|), & \text{in } \mathbb{R}^N \setminus \mathcal{E}_{\tilde{H}_\infty, \tilde{h}_\infty}^{\sigma_2, R}. \end{cases}$$

From Lemma 3.4, we have that, for some positive constant  $C$  (depending on  $N, \sigma_1, \sigma_2, h_\infty, H_\infty, \varepsilon$ , but independent of  $R$  and  $\delta$ ),

$$-\Delta \varphi_\infty(x) - V_\infty(x) \varphi_\infty(x) \geq C \chi_{\mathbb{R}^N \setminus \mathcal{E}_{\tilde{h}_\infty, 0}^{\sigma_1, R_0/\delta}} \frac{\varphi_\infty(x)}{|x|^2} \quad \text{in } (\mathcal{D}^{1,2}(\mathbb{R}^N))^*.$$

Let us also consider the functions  $\varphi_i(x) = u_i(x - \frac{a_i}{\delta})$ ,  $i = 1, \dots, k$ , where

$$u_i(x) = \begin{cases} |x|^{-(N-2)} \omega_N^{-1/2}, & \text{in } \mathbb{R}^N \setminus \mathcal{E}_{0, \tilde{h}_i}^{\sigma_1, 1}, \\ |x|^{-(N-2) + \frac{1}{\sigma_1}} \psi_1^{\tilde{h}_i}(x/|x|), & \text{in } \mathcal{E}_{0, \tilde{h}_i}^{\sigma_1, 1} \setminus \mathcal{E}_{\tilde{h}_i, \tilde{H}_i}^{\sigma_2, r/\delta}, \\ \left(\frac{r}{\delta}\right)^{-1/\sigma_2} |x|^{-(N-2) + \frac{1}{\sigma_1} + \frac{1}{\sigma_2}} \psi_1^{\tilde{H}_i}(x/|x|), & \text{in } \mathcal{E}_{\tilde{h}_i, \tilde{H}_i}^{\sigma_2, r/\delta}. \end{cases}$$

From Lemma 3.5, we have that, for some positive constant  $\bar{C}$  (depending on  $N, \sigma_1, \sigma_2, h_i, H_i$ , and  $\varepsilon$ , but independent of  $\delta$  and  $r$ ) and for all  $i = 1, \dots, k$ ,

$$-\Delta \varphi_i - V_i \varphi_i \geq \bar{C} \chi_{\mathcal{E}_{0, \tilde{h}_i}^{\sigma_1, 1}(a_i/\delta)} \frac{\varphi_i}{|x - \frac{a_i}{\delta}|^2} \quad \text{in } (\mathcal{D}^{1,2}(\mathbb{R}^N))^*.$$

We notice that, if  $\mathcal{E}_{\tilde{h}_i, \tilde{H}_i}^{\sigma_2, r} \subset \mathcal{E}_{0, \tilde{h}_i}^{\sigma_1, \delta}$ , then  $\frac{r}{\delta} \leq \min_{\mathbb{S}^{N-1}} (\psi_1^0)^{\sigma_1} (\psi_1^{\tilde{H}_i})^{\sigma_2} (\psi_1^{\tilde{h}_i})^{-\sigma_2 - \sigma_1}$ . Hence, by definition of  $\varphi_i$ , there exists some constant  $C_0 > 0$  (depending on  $N$ ,  $\sigma_1$ ,  $\sigma_2$ ,  $h_i$ ,  $H_i$ , and  $\varepsilon$ , but independent of  $\delta$  and  $r$ ) such that, for all  $i = 1, \dots, k$ ,

$$(20) \quad \varphi_i(x) \geq \frac{1}{C_0} \left| x - \frac{a_i}{\delta} \right|^{-(N-2) + \frac{1}{\sigma_1} + \frac{1}{\sigma_2}}, \quad \text{for all } x \in \mathcal{E}_{0, \tilde{h}_i}^{\sigma_1, 1}(a_i/\delta).$$

Moreover, by the definition of  $\varphi_\infty$ ,

$$(21) \quad \varphi_\infty(x) \geq \frac{1}{C_1} |\delta x|^{-\left(\frac{1}{\sigma_1} + \frac{1}{\sigma_2}\right)}, \quad \text{for all } x \in \mathbb{R}^N \setminus \mathcal{E}_{\tilde{h}_\infty, 0}^{\sigma_1, R_0/\delta},$$

where  $C_1 > 0$  depends on  $N$ ,  $\sigma_1$ ,  $\sigma_2$ ,  $h_\infty$ ,  $H_\infty$ ,  $\varepsilon$ , and  $R_0$ , but is independent of  $R$  and  $\delta$ . Let  $\varphi = \sum_{i=1}^k \varphi_i + \eta \varphi_\infty$  for some  $\eta$  such that

$$0 < \eta < \min_{i=1, \dots, k} \left\{ \frac{\bar{C} \sqrt{\omega_N}}{2C_0 \max \{ \|\tilde{h}_i\|_{L^\infty(\mathbb{S}^{N-1})}, \|\tilde{H}_i\|_{L^\infty(\mathbb{S}^{N-1})} \}} \left[ \max_{\theta \in \mathbb{S}^{N-1}} \frac{\omega_N^{-1/2}}{\psi_1^{\tilde{h}_i}(\theta)} \right]^{1 + \frac{\sigma_1}{\sigma_2} - \sigma_1(N-2)} \right\}.$$

Then we have

$$-\Delta \varphi(x) - \sum_{i=1}^k V_i(x) \varphi(x) - V_\infty(x) \varphi(x) \geq f(x)$$

in  $(\mathcal{D}^{1,2}(\mathbb{R}^N))^*$ , where

$$\begin{aligned} f(x) = & \bar{C} \sum_{i=1}^k \chi_{\mathcal{E}_{0, \tilde{h}_i}^{\sigma_1, 1}(a_i/\delta)} \frac{\varphi_i}{\left| x - \frac{a_i}{\delta} \right|^2} + C \eta \chi_{\mathbb{R}^N \setminus \mathcal{E}_{\tilde{h}_\infty, 0}^{\sigma_1, R_0/\delta}} \frac{\varphi_\infty(x)}{|x|^2} - \sum_{\substack{i, j=1 \\ i \neq j}}^k V_i(x) \varphi_j(x) \\ & - \eta \sum_{i=1}^k V_i(x) \varphi_\infty(x) - V_\infty(x) \sum_{i=1}^k \varphi_i(x). \end{aligned}$$

In particular a.e. in the set  $\mathcal{E}_{\tilde{h}_\infty, 0}^{\sigma_1, R_0/\delta} \setminus \bigcup_{i=1}^k \mathcal{E}_{0, \tilde{h}_i}^{\sigma_1, 1}(a_i/\delta)$ , we have that  $f(x) \equiv 0$ . Let us consider  $\mathcal{E}_{0, \tilde{h}_i}^{\sigma_1, 1}(a_i/\delta)$ . Since, for  $\delta$  small,

$$\mathcal{E}_{0, \tilde{h}_i}^{\sigma_1, 1}(a_i/\delta) \subset \mathcal{E}_{\tilde{h}_\infty, 0}^{\sigma_1, R_0/\delta} \quad \text{and} \quad \mathcal{E}_{0, \tilde{h}_i}^{\sigma_1, 1}(a_i/\delta) \subset \mathbb{R}^N \setminus \mathcal{E}_{0, \tilde{h}_j}^{\sigma_1, 1}(a_j/\delta) \text{ for } j \neq i,$$

from (20) it follows that, in  $\mathcal{E}_{0, \tilde{h}_i}^{\sigma_1, 1}(a_i/\delta)$ ,

$$\begin{aligned} f(x) \geq & \bar{C} \chi_{\mathcal{E}_{0, \tilde{h}_i}^{\sigma_1, 1}(a_i/\delta)} \frac{\varphi_i}{\left| x - \frac{a_i}{\delta} \right|^2} - V_i(x) \left( \sum_{\substack{i, j=1 \\ i \neq j}}^k \varphi_j(x) + \eta \varphi_\infty(x) \right) \\ \geq & \left| x - \frac{a_i}{\delta} \right|^{-2} \left[ \frac{\bar{C}}{C_0} \left| x - \frac{a_i}{\delta} \right|^{-(N-2) + \frac{1}{\sigma_1} + \frac{1}{\sigma_2}} \right. \\ & \left. - \omega_N^{-1/2} \max \left\{ \|\tilde{h}_i\|_{L^\infty(\mathbb{S}^{N-1})}, \|\tilde{H}_i\|_{L^\infty(\mathbb{S}^{N-1})} \right\} \left( \sum_{\substack{j=1 \\ j \neq i}}^k \left| x - \frac{a_j}{\delta} \right|^{-(N-2)} + \eta \right) \right]. \end{aligned}$$

It is easy to see that, for  $\delta$  small, in  $\mathcal{E}_{0, \tilde{h}_i}^{\sigma_1, 1}(a_i/\delta)$

$$\begin{aligned} \left| x - \frac{a_i}{\delta} \right|^{-(N-2) + \frac{1}{\sigma_1} + \frac{1}{\sigma_2}} &\geq \left[ \max_{\theta \in \mathbb{S}^{N-1}} \frac{\omega_N^{-1/2}}{\psi_1^{\tilde{h}_i}(\theta)} \right]^{1 + \frac{\sigma_1}{\sigma_2} - \sigma_1(N-2)} \quad \text{and} \\ \left| x - \frac{a_j}{\delta} \right|^{-(N-2)} &\leq \left( \frac{2}{|a_i - a_j|} \right)^{N-2} \delta^{N-2} < \frac{\eta}{k-1}, \end{aligned}$$

and hence the choice of  $\eta$  ensures that  $f \geq 0$  a.e. in  $\mathcal{E}_{0, \tilde{h}_i}^{\sigma_1, 1}(a_i/\delta)$ , provided  $\delta$  is sufficiently small. Let us finally consider  $\mathbb{R}^N \setminus \mathcal{E}_{\tilde{h}_\infty, 0}^{\sigma_1, R_0/\delta}$ . From (21), we deduce that in  $\mathbb{R}^N \setminus \mathcal{E}_{\tilde{h}_\infty, 0}^{\sigma_1, R_0/\delta}$

$$f(x) \geq \frac{1}{|x|^2} \left[ \frac{C\eta}{C_1} |\delta x|^{-(\frac{1}{\sigma_1} + \frac{1}{\sigma_2})} - \omega_N^{-1/2} \max\{\|\tilde{h}_\infty\|_{L^\infty(\mathbb{S}^{N-1})}, \|\tilde{H}_\infty\|_{L^\infty(\mathbb{S}^{N-1})}\} \sum_{i=1}^k \left| x - \frac{a_i}{\delta} \right|^{-(N-2)} \right].$$

It is easy to see that, in  $\mathbb{R}^N \setminus \mathcal{E}_{\tilde{h}_\infty, 0}^{\sigma_1, R_0/\delta}$ ,  $|x - \frac{a_i}{\delta}| \geq (1 - \frac{\alpha}{\beta R_0})|x|$  where  $\alpha = \max_j\{|a_j|\}$  and  $\beta = [\min_{\theta \in \mathbb{S}^{N-1}} \sqrt{\omega_N} \psi_1^{\tilde{h}_\infty}(\theta)]^{\sigma_1}$ , hence

$$\begin{aligned} f(x) &\geq |x|^{-2 - \frac{1}{\sigma_1} - \frac{1}{\sigma_2}} \left[ \frac{C}{C_1} \eta \delta^{-(\frac{1}{\sigma_1} + \frac{1}{\sigma_2})} \right. \\ &\quad \left. - \omega_N^{-1/2} \max\{\|\tilde{h}_\infty\|_{L^\infty(\mathbb{S}^{N-1})}, \|\tilde{H}_\infty\|_{L^\infty(\mathbb{S}^{N-1})}\} k \left( 1 - \frac{\alpha}{\beta R_0} \right)^{-(N-2)} (R_0 \beta)^{-(N-2) + \frac{1}{\sigma_1} + \frac{1}{\sigma_2}} \right] > 0 \end{aligned}$$

a.e. in  $\mathbb{R}^N \setminus \mathcal{E}_{\tilde{h}_\infty, 0}^{\sigma_1, R_0/\delta}$ , provided  $\delta$  is sufficiently small (notice that the choice of  $\delta$  is independent of  $R$  and  $r$ ). The proof is thereby complete.  $\square$

**Lemma 3.7.** *Let  $a_1, a_2, \dots, a_k \in \mathbb{R}^N$ ,  $a_i \neq a_j$  for  $i \neq j$ , and  $h_i \in L^\infty(\mathbb{S}^{N-1})$ ,  $i = 1, \dots, k, \infty$ , with  $\mu_1(h_i) > -(N-2)^2/4$ , for  $i = 1, \dots, k, \infty$ , and  $\max_{i=1, \dots, k, \infty} \Lambda_N(h_i) > 0$ . Then for every  $0 < \alpha < 1 - \max_{i=1, \dots, k, \infty} \Lambda_N(h_i)$  there exist  $\mathcal{U}_1, \dots, \mathcal{U}_k, \mathcal{U}_\infty \subset \mathbb{R}^N$  such that  $\mathcal{U}_i$  is a neighborhood of  $a_i$  for every  $i = 1, \dots, k$ ,  $\mathcal{U}_\infty$  is a neighborhood of  $\infty$ , and*

$$\mu \left( \sum_{i=1}^k \chi_{\mathcal{U}_i}(x) \frac{h_i \left( \frac{x-a_i}{|x-a_i|} \right)}{|x-a_i|^2} + \chi_{\mathcal{U}_\infty}(x) \frac{h_\infty \left( \frac{x}{|x|} \right)}{|x|^2} \right) \geq 1 - \max_{i=1, \dots, k, \infty} \Lambda_N(h_i) - \alpha > 0.$$

PROOF. For any  $0 < \alpha < 1 - \max_{i=1, \dots, k, \infty} \Lambda_N(h_i)$ , the statement follows from Lemma 3.6 with  $\varepsilon = (\alpha + \max_{i=1, \dots, k, \infty} \Lambda_N(h_i))^{-1} - 1$  and  $H_i = h_i$ ,  $i = 1, \dots, k, \infty$ .  $\square$

**Proof of Lemma 1.3.** It follows by Lemmas 3.3 and 3.7.  $\square$

Lemma 1.3 can be extended to infinitely many dipole-type singularities distributed on reticular structures.

**Lemma 3.8. [Shattering of reticular singularities]** *Let  $\{h_n\}_{n \in \mathbb{N}} \subset L^\infty(\mathbb{S}^{N-1})$  such that*

$$(22) \quad \sup_{n \in \mathbb{N}} \|h_n\|_{L^\infty(\mathbb{S}^{N-1})} < +\infty \quad \text{and} \quad 0 < \sup_{n \in \mathbb{N}} \Lambda_N(h_n) < 1,$$

and  $\{a_n\}_n \subset \mathbb{R}^N$  satisfy

$$(23) \quad \sum_{n=1}^{\infty} |a_n|^{-(N-2)} < +\infty, \quad \sum_{\substack{m=1 \\ m \neq n}}^{\infty} |a_m - a_n|^{-(N-2)} \text{ is bounded uniformly in } n,$$

and  $|a_n - a_m| \geq 1$  for all  $n \neq m$ . Then there exist  $\varepsilon > 0$ ,  $\sigma > 0$ , and  $\bar{\delta} > 0$ , such that, for all  $0 < \delta < \bar{\delta}$ ,

$$\inf_{\substack{u \in \mathcal{D}^{1,2}(\mathbb{R}^N) \\ u \neq 0}} \frac{\int_{\mathbb{R}^N} (|\nabla u(x)|^2 - V(x)u^2(x)) dx}{\int_{\mathbb{R}^N} |\nabla u(x)|^2 dx} > 0$$

where

$$V(x) = \sum_{n=1}^{\infty} \frac{h_n \left( \frac{x-a_n}{|x-a_n|} \right)}{|x-a_n|^2} \chi_{\mathcal{E}_{0,(1+\varepsilon)h_n}^{\sigma,\delta}(a_n)}.$$

PROOF. Taking into account the characterization of  $\Lambda_N(h)$  given in [12, Lemma 2.4], a direct calculation yields that, for any  $h \in L^\infty(\mathbb{S}^{N-1})$ ,

$$(24) \quad \Lambda_N(h) \geq -\frac{4}{(N-2)^2} \mu_1(h).$$

Letting  $0 < \varepsilon < (\sup_{n \in \mathbb{N}} \Lambda_N(h_n))^{-1} - 1$  and  $\tilde{h}_n := (1 + \varepsilon)h_n$ , from (24), it follows that

$$\inf_{n \in \mathbb{N}} \mu_1(\tilde{h}_n) > -\left(\frac{N-2}{2}\right)^2,$$

hence there exists  $\sigma > 0$  such that

$$(25) \quad \frac{N-2}{2} < \frac{1}{\sigma} < \frac{N-2}{2} + \min \left\{ \frac{N-2}{2}, \sqrt{\frac{(N-2)^2}{4} + \inf_{n \in \mathbb{N}} \mu_1(\tilde{h}_n)} \right\}.$$

Moreover, classical elliptic regularity theory and bootstrap methods (see also [12, Lemma 2.3]) yield that  $\psi_1^{\tilde{h}_n}$  is bounded in  $C^{0,\alpha}(\mathbb{S}^{N-1})$  uniformly with respect to  $n$ , thus implying that the sequence  $\{\psi_1^{\tilde{h}_n}\}_{n \in \mathbb{N}}$  is equi-continuous and, by Ascoli-Arzelà's Theorem, compact in  $C^0(\mathbb{S}^{N-1})$ . We deduce that, for positive constant  $C > 0$  independent on  $n$ ,

$$(26) \quad \frac{1}{C} \leq \psi_1^{\tilde{h}_n}(\theta) \leq C \quad \text{for all } \theta \in \mathbb{S}^{N-1}.$$

Let

$$\psi_n(x) = \begin{cases} |x|^{-(N-2)+\frac{1}{\sigma}} \psi_1^{\tilde{h}_n}(x/|x|), & \text{in } \mathcal{E}_{0,\tilde{h}_n}^{\sigma,1}, \\ |x|^{-(N-2)} \omega_N^{-1/2}, & \text{in } \mathbb{R}^N \setminus \mathcal{E}_{0,\tilde{h}_n}^{\sigma,1}. \end{cases}$$

Arguing as in Lemma 3.5, one can easily verify that  $\psi_n \in \mathcal{D}^{1,2}(\mathbb{R}^N)$  and

$$-\Delta \psi_n - \frac{\tilde{h}_n(x/|x|)}{|x|^2} \chi_{\mathcal{E}_{0,\tilde{h}_n}^{\sigma,1}} \psi_n \geq \bar{C} \frac{\psi_n}{|x|^2} \chi_{\mathcal{E}_{0,\tilde{h}_n}^{\sigma,1}} \quad \text{in } (\mathcal{D}^{1,2}(\mathbb{R}^N))^*,$$

for some  $\bar{C} > 0$  depending on  $N$ ,  $\inf_{n \in \mathbb{N}} \mu_1(\tilde{h}_n)$ , and  $\sigma$ .

Let  $\varphi(x) = \sum_{n=1}^{\infty} \psi_n(x - \frac{a_n}{\delta})$ . Due to estimate (26), for any open set  $\Omega \subset \mathbb{R}^N$  such  $\bar{\Omega}$  is compact, we have that there exists  $\bar{n}$  such that  $|\psi_n(x - \frac{a_n}{\delta})| = \omega_N^{-1/2} |x - \frac{a_n}{\delta}|^{-(N-2)} \leq \text{const} |\frac{a_n}{\delta}|^{-(N-2)}$  and  $|\nabla \psi_n(x - \frac{a_n}{\delta})| = \omega_N^{-1/2} (N-2) |x - \frac{a_n}{\delta}|^{-(N-1)} \leq \text{const} |\frac{a_n}{\delta}|^{-(N-1)}$  for all  $n \geq \bar{n}$  and  $x \in \Omega$ . Then

$$\varphi|_{\Omega}(x) = \sum_{n=1}^{\bar{n}-1} \psi_n\left(x - \frac{a_n}{\delta}\right) + \sum_{n=\bar{n}}^{\infty} \psi_n\left(x - \frac{a_n}{\delta}\right) \in H^1(\Omega) + L^\infty(\Omega)$$

and  $\nabla(\varphi|_{\Omega}) \in L^2(\Omega)$ . In particular  $\varphi \in L_{\text{loc}}^{2^*}(\mathbb{R}^N)$ ,  $\varphi \in H_{\text{loc}}^1(\mathbb{R}^N)$ , and  $\sum_{n=1}^{\infty} \psi_n(x - \frac{a_n}{\delta})$  converges to  $\varphi$  in  $H_0^1(\Omega)$  for every  $\Omega \Subset \mathbb{R}^N$ . Moreover

$$-\Delta\varphi(x) - \sum_{n=1}^{\infty} \frac{\tilde{h}_n\left(\frac{x - \frac{a_n}{\delta}}{|x - \frac{a_n}{\delta}|}\right)}{|x - \frac{a_n}{\delta}|^2} \chi_{\mathcal{E}_{0, \tilde{h}_n}^{\sigma, 1}(a_n/\delta)} \varphi(x) \geq f(x) \quad \text{in } (H_0^1(\Omega))^* \quad \text{for all } \Omega \Subset \mathbb{R}^N,$$

where

$$f(x) = \bar{C} \sum_{n=1}^{\infty} \frac{\psi_n\left(x - \frac{a_n}{\delta}\right)}{|x - \frac{a_n}{\delta}|^2} \chi_{\mathcal{E}_{0, \tilde{h}_n}^{\sigma, 1}(a_n/\delta)}(x) - \sum_{\substack{n, m=1 \\ m \neq n}}^{\infty} \frac{\tilde{h}_n\left(\frac{x - (a_n/\delta)}{|x - a_n/\delta|}\right)}{|x - \frac{a_n}{\delta}|^2} \chi_{\mathcal{E}_{0, \tilde{h}_n}^{\sigma, 1}(a_n/\delta)} \psi_m\left(x - \frac{a_m}{\delta}\right).$$

A.e. in the set  $\mathbb{R}^N \setminus \bigcup_{n=1}^{\infty} \mathcal{E}_{0, \tilde{h}_n}^{\sigma, 1}(a_n/\delta)$ , we have that  $f(x) = 0$ . From the definition of  $\psi_n$  and estimate (26), it follows that in each  $\mathcal{E}_{0, \tilde{h}_n}^{\sigma, 1}(a_n/\delta)$

$$f(x) \geq C_1 \left| x - \frac{a_n}{\delta} \right|^{-2} \left( C_2 \left| x - \frac{a_n}{\delta} \right|^{-(N-2)+1/\sigma} - \sum_{m \neq n} \left| x - \frac{a_m}{\delta} \right|^{-(N-2)} \right),$$

for some constants  $C_1, C_2$  independent of  $n$ . Since, for small  $\delta$ ,  $\left| x - \frac{a_m}{\delta} \right| \geq \frac{|a_m - a_n|}{\delta} - \text{const} \geq \frac{|a_m - a_n|}{2\delta}$  provided  $\delta$  small enough, we deduce that

$$\sum_{m \neq n} \left| x - \frac{a_m}{\delta} \right|^{-(N-2)} \leq (2\delta)^{N-2} \sum_{m \neq n} |a_m - a_n|^{-(N-2)} \leq \text{const} \delta^{N-2}.$$

Hence, we can choose  $\delta$  small enough independently of  $n$  such that  $f(x) \geq 0$  a.e. in  $\mathcal{E}_{0, \tilde{h}_n}^{\sigma, 1}(a_n/\delta)$ . Hence we have constructed a supersolution  $\varphi \in L_{\text{loc}}^1(\mathbb{R}^N)$ ,  $\varphi > 0$  in  $\mathbb{R}^N \setminus \{a_n/\delta\}_{n \in \mathbb{N}}$  and  $\varphi$  continuous in  $\mathbb{R}^N \setminus \{a_n/\delta\}_{n \in \mathbb{N}}$ , such that

$$-\Delta\varphi - \sum_{n=1}^{\infty} \frac{h_n\left(\frac{x - \frac{a_n}{\delta}}{|x - \frac{a_n}{\delta}|}\right)}{|x - \frac{a_n}{\delta}|^2} \chi_{\mathcal{E}_{0, \tilde{h}_n}^{\sigma, 1}(a_n/\delta)} \varphi \geq \varepsilon \sum_{n=1}^{\infty} \frac{h_n\left(\frac{x - \frac{a_n}{\delta}}{|x - \frac{a_n}{\delta}|}\right)}{|x - \frac{a_n}{\delta}|^2} \chi_{\mathcal{E}_{0, \tilde{h}_n}^{\sigma, 1}(a_n/\delta)} \varphi \quad \text{in } (H_0^1(\Omega))^*,$$

for all  $\Omega \Subset \mathbb{R}^N$ . Therefore, arguing as in Lemma 3.1 and taking into account the scaling properties of the operator, we obtain

$$\tilde{\mu}(V) := \inf_{\substack{u \in C_c^\infty(\mathbb{R}^N \setminus \{a_n\}_{n \in \mathbb{N}}) \\ u \neq 0}} \frac{\int_{\mathbb{R}^N} (|\nabla u(x)|^2 - V(x)u^2(x)) dx}{\int_{\mathbb{R}^N} |\nabla u(x)|^2 dx} > 0,$$

i.e.

$$(27) \quad \int_{\mathbb{R}^N} V(x)u^2(x) dx \leq (1 - \tilde{\mu}(V)) \int_{\mathbb{R}^N} |\nabla u(x)|^2 dx$$

for all  $u \in C_c^\infty(\mathbb{R}^N \setminus \{a_n\}_{n \in \mathbb{N}})$ . By density of  $C_c^\infty(\mathbb{R}^N \setminus \{a_n\}_{n \in \mathbb{N}})$  in  $\mathcal{D}^{1,2}(\mathbb{R}^N)$  and the Fatou Lemma, we can easily prove that (27) holds for all  $u \in \mathcal{D}^{1,2}(\mathbb{R}^N)$ .  $\square$

**Remark 3.9.** Taking into account the characterization of  $\Lambda_N(h)$  given in [12, Lemma2.4], we can easily prove that, for any  $h \in L^\infty(\mathbb{S}^{N-1})$ ,

$$\mu_1(h) \leq -\left(\frac{N-2}{2}\right)^2 + [(\Lambda_N(h))^{-1} - 1] \|h^+\|_{L^\infty(\mathbb{S}^{N-1})},$$

which, together with (24), implies that, for  $\{h_n\}_{n \in \mathbb{N}}$  bounded in  $L^\infty(\mathbb{S}^{N-1})$ ,

$$\sup_{n \in \mathbb{N}} \Lambda_N(h_n) < 1 \quad \text{if and only if} \quad \inf_{n \in \mathbb{N}} \mu_1(h_n) > -\left(\frac{N-2}{2}\right)^2.$$

#### 4. SPECTRAL STABILITY UNDER PERTURBATION AT SINGULARITIES

In this section we discuss the stability of positivity with respect to perturbations of the potentials with a singularity sitting at dipolar-shaped neighborhoods either of a dipole or of infinity.

In order to analyze the stability of the sign of  $\mu(V)$  under perturbations at singularities, it is useful to investigate its attainability. Due to inverse square homogeneity, it is easy to verify that  $\mu\left(\frac{h(x/|x|)}{|x|^2}\right) = 1 - \Lambda_N(h)$  is not attained if  $h \in L^\infty(\mathbb{S}^{N-1})$  is positive somewhere, i.e. if  $\Lambda_N(h) > 0$ . If  $h \leq 0$  a.e. in  $\mathbb{R}^N$ , then  $\mu\left(\frac{h(x/|x|)}{|x|^2}\right) = 1$  could be achieved, as it happens for example if  $h$  vanishes in a nonempty open set.

The best constant in the Hardy-type inequality associated to a multisingular potential  $V \in \mathcal{V}$  is attained if  $\mu(V)$  stays strictly below the bound provided in Lemma 3.2.

**Proposition 4.1.** *Let  $V \in \mathcal{V}$  be as in (14). If*

$$(28) \quad \mu(V) < 1 - \max\{0, \Lambda_N(h_1), \dots, \Lambda_N(h_k), \Lambda_N(h_\infty)\}$$

*then  $\mu(V)$  is attained.*

PROOF. Let us assume that (28) holds. Hence there exists  $\alpha > 0$  such that  $\mu(V) = 1 - \bar{\Lambda} - \alpha$  where  $\bar{\Lambda} = \max\{0, \Lambda_N(h_1), \dots, \Lambda_N(h_k), \Lambda_N(h_\infty)\}$ . From Lemmas 3.3 and 3.7, there exist  $\tilde{V} \in \mathcal{V}$  and  $\tilde{W} \in L^{N/2}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$  such that  $V = \tilde{V} + \tilde{W}$  and

$$(29) \quad \mu(\tilde{V}) \geq 1 - \bar{\Lambda} - \frac{\alpha}{2}.$$

Let  $u_n \in \mathcal{D}^{1,2}(\mathbb{R}^N)$  a minimizing sequence for  $\mu(V)$ , namely

$$\int_{\mathbb{R}^N} |\nabla u_n(x)|^2 dx = 1 \quad \text{and} \quad \int_{\mathbb{R}^N} (|\nabla u_n(x)|^2 - V(x)u_n^2(x)) dx = \mu(V) + o(1) \quad \text{as } n \rightarrow \infty.$$

Being  $\{u_n\}_n$  bounded in  $\mathcal{D}^{1,2}(\mathbb{R}^N)$ , we can assume that, up to a subsequence still denoted as  $u_n$ ,  $u_n$  converges to some  $u$  a.e. and weakly in  $\mathcal{D}^{1,2}(\mathbb{R}^N)$ . Since

$$\begin{aligned} 1 - \bar{\Lambda} - \frac{\alpha}{2} &\leq \mu(\tilde{V}) \leq \int_{\mathbb{R}^N} (|\nabla u_n(x)|^2 - V(x)u_n^2(x)) dx + \int_{\mathbb{R}^N} \tilde{W}(x)u_n^2(x) dx \\ &= \mu(V) + \int_{\mathbb{R}^N} \tilde{W}(x)u^2(x) dx + o(1) = 1 - \bar{\Lambda} - \alpha + \int_{\mathbb{R}^N} \tilde{W}(x)u^2(x) dx \end{aligned}$$

as  $n \rightarrow \infty$ , we obtain that  $\int_{\mathbb{R}^N} \tilde{W}(x)u^2(x) dx \geq \frac{\alpha}{2} > 0$ , thus implying  $u \neq 0$ . From weak convergence of  $u_n$  to  $u$ , we deduce that

$$\begin{aligned} &\frac{\int_{\mathbb{R}^N} (|\nabla u(x)|^2 - V(x)u^2(x)) dx}{\int_{\mathbb{R}^N} |\nabla u(x)|^2 dx} \\ &= \frac{[\int_{\mathbb{R}^N} (|\nabla u_n(x)|^2 - V(x)u_n^2(x)) dx] - [\int_{\mathbb{R}^N} (|\nabla(u_n - u)(x)|^2 - V(x)(u_n - u)^2(x)) dx] + o(1)}{\int_{\mathbb{R}^N} |\nabla u_n(x)|^2 dx - \int_{\mathbb{R}^N} |\nabla(u_n - u)(x)|^2 dx + o(1)} \\ &\leq \mu(V) \frac{1 - \int_{\mathbb{R}^N} |\nabla(u_n - u)(x)|^2 dx + o(1)}{1 - \int_{\mathbb{R}^N} |\nabla(u_n - u)(x)|^2 dx + o(1)} = \mu(V) \left[ 1 + \frac{o(1)}{\int_{\mathbb{R}^N} |\nabla u(x)|^2 dx + o(1)} \right] \quad \text{as } n \rightarrow \infty. \end{aligned}$$

Letting  $n \rightarrow \infty$ , we obtain that  $u$  attains the infimum defining  $\mu(V)$ .  $\square$

Let us now study the stability of the sign of  $\mu(V)$  under perturbations at infinity.

**Theorem 4.2.** *For  $i = 1, \dots, k$ , let  $r_i, R \in \mathbb{R}^+$ ,  $a_i \in \mathbb{R}^N$ ,  $a_i \neq a_j$  for  $i \neq j$ ,  $h_i, h_\infty \in L^\infty(\mathbb{S}^{N-1})$  with  $\mu_1(h_i) > -(N-2)^2/4$ ,  $\mu_1(h_\infty) > -(N-2)^2/4$ , and*

$$V(x) = \sum_{i=1}^k \chi_{B(a_i, r_i)}(x) \frac{h_i\left(\frac{x-a_i}{|x-a_i|}\right)}{|x-a_i|^2} + \chi_{\mathbb{R}^N \setminus B(0, R)}(x) \frac{h_\infty\left(\frac{x}{|x|}\right)}{|x|^2} + W(x) \in \mathcal{V}$$

where  $W \in L^{N/2}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ . Assume that  $\mu(V) > 0$  and let  $h \in L^\infty(\mathbb{S}^{N-1})$  such that  $\mu_1(h + h_\infty) > -(N-2)^2/4$ ,  $\varepsilon > 0$  such that, setting  $H := h + h_\infty$ ,

$$\varepsilon < \begin{cases} \frac{1}{\max_{i=1, \dots, k, \infty} \{\Lambda_N(h_i), \Lambda_N(H)\}} - 1, & \text{if } \max_{i=1, \dots, k, \infty} \{\Lambda_N(h_i), \Lambda_N(H)\} > 0, \\ +\infty & \text{if } \max_{i=1, \dots, k, \infty} \{\Lambda_N(h_i), \Lambda_N(H)\} = 0, \end{cases}$$

and  $\sigma > 0$  such that  $0 < \frac{1}{\sigma} < \min_{i=1, \dots, k, \infty} \left\{ \sqrt{(N-2)^2/4 + \mu_1(\tilde{h}_i)}, \sqrt{(N-2)^2/4 + \mu_1(\tilde{H})} \right\}$ , where  $\tilde{h}_i := (1 + \varepsilon)h_i$  and  $\tilde{H} := (1 + \varepsilon)H$ . Then there exists  $\bar{R}$  such that

$$\mu\left(V + \frac{h(x/|x|)}{|x|^2} \chi_{\mathbb{R}^N \setminus \mathcal{E}_{\tilde{H}, \tilde{h}_\infty}^{\sigma, \bar{R}}}\right) > 0,$$

for all  $\tilde{R} > \bar{R}$ , where  $\mathcal{E}_{\tilde{H}, \tilde{h}_\infty}^{\sigma, \tilde{R}}$  is defined in (15).

PROOF. Let  $\sigma_1 > 0$  such that

$$\frac{N-2}{2} < \frac{1}{\sigma_1} < \frac{1}{\sigma_1} + \frac{1}{\sigma} < \frac{N-2}{2} + \min_{i=1, \dots, k, \infty} \left\{ \sqrt{\frac{(N-2)^2}{4} + \mu_1(\tilde{h}_i)}, \sqrt{\frac{(N-2)^2}{4} + \mu_1(\tilde{H})} \right\}.$$

Assume, by contradiction, that there exists a sequence  $R_n \rightarrow +\infty$  such that, setting

$$V_n = V + \frac{h(x/|x|)}{|x|^2} \chi_{\mathbb{R}^N \setminus \mathcal{E}_{\tilde{h}, \tilde{h}_\infty}^{\sigma, R_n}},$$

there holds  $\mu(V_n) \leq 0$ . By Lemmas 3.3 and 3.6,  $V_n = \tilde{V}_n + \tilde{W}$ , where  $\mu(\tilde{V}_n) \geq \frac{\varepsilon}{\varepsilon+1} > 0$  and

$$\tilde{W}(x) = \sum_{i=1}^k \chi_{B(a_i, r_i) \setminus \mathcal{E}_{0, \tilde{h}_i}^{\sigma_1, \delta}(a_i)}(x) \frac{h_i\left(\frac{x-a_i}{|x-a_i|}\right)}{|x-a_i|^2} + \chi_{\mathcal{E}_{\tilde{h}_\infty, 0}^{\sigma_1, R_0} \setminus B(0, R)}(x) \frac{h_\infty\left(\frac{x}{|x|}\right)}{|x|^2} + W(x)$$

for some  $\delta > 0$  and  $R_0 > 0$  independent of  $n$ . By Proposition 4.1,  $\mu(V_n)$  is attained by some  $\varphi_n \in \mathcal{D}^{1,2}(\mathbb{R}^N)$  such that

$$\int_{\mathbb{R}^N} |\nabla \varphi_n(x)|^2 dx = 1, \quad \int_{\mathbb{R}^N} |\nabla \varphi_n(x)|^2 dx - \int_{\mathbb{R}^N} V_n(x) \varphi_n^2(x) dx = \mu(V_n),$$

and

$$(30) \quad -\Delta \varphi_n - V_n \varphi_n = -\mu(V_n) \Delta \varphi_n \quad \text{in } \mathbb{R}^N.$$

Up to a subsequence,  $\varphi_n \rightharpoonup \varphi$  weakly in  $\mathcal{D}^{1,2}(\mathbb{R}^N)$  for some  $\varphi \in \mathcal{D}^{1,2}(\mathbb{R}^N)$ , and hence

$$\begin{aligned} \frac{\varepsilon}{\varepsilon+1} \leq \mu(\tilde{V}_n) &\leq \int_{\mathbb{R}^N} |\nabla \varphi_n(x)|^2 dx - \int_{\mathbb{R}^N} V_n(x) \varphi_n^2(x) dx + \int_{\mathbb{R}^N} \tilde{W}(x) \varphi_n^2(x) dx \\ &= \mu(V_n) + \int_{\mathbb{R}^N} \tilde{W}(x) \varphi_n^2(x) dx \leq \int_{\mathbb{R}^N} \tilde{W}(x) \varphi^2(x) dx + o(1) \quad \text{as } n \rightarrow +\infty. \end{aligned}$$

Therefore  $\int_{\mathbb{R}^N} \tilde{W}(x) \varphi^2(x) dx > 0$  and we conclude that  $\varphi \not\equiv 0$ . We claim that

$$(31) \quad \lim_{j \rightarrow \infty} \int_{\mathbb{R}^N} V_n(x) \varphi_n(x) \varphi(x) dx = \int_{\mathbb{R}^N} V(x) \varphi^2(x) dx.$$

Indeed for any  $\eta > 0$ , by density there exists  $\psi \in C_c^\infty(\mathbb{R}^N)$  such that  $\|\varphi - \psi\|_{\mathcal{D}^{1,2}(\mathbb{R}^N)} < \eta$ . Since  $\psi$  has compact support, from Hardy's inequality we have that, for large  $n$ ,

$$\begin{aligned} &\left| \int_{\mathbb{R}^N} V_n(x) \varphi_n(x) \varphi(x) dx - \int_{\mathbb{R}^N} V(x) \varphi^2(x) dx \right| \\ &\leq \left| \int_{\mathbb{R}^N} (V_n(x) - V(x)) \varphi_n(x) (\varphi(x) - \psi(x)) dx \right| + \left| \int_{\mathbb{R}^N} V(x) (\varphi_n(x) - \varphi(x)) \varphi(x) dx \right| \\ &\leq \text{const } \eta + \left| \int_{\mathbb{R}^N} V(x) (\varphi_n(x) - \varphi(x)) \varphi(x) dx \right| = \text{const } \eta + o(1) \quad \text{as } n \rightarrow \infty. \end{aligned}$$

(31) is thereby proved. From (31), multiplying (30) by  $\varphi$  and passing to limit as  $n \rightarrow \infty$ , we obtain

$$\mu(V) \int_{\mathbb{R}^N} |\nabla \varphi(x)|^2 dx \leq \int_{\mathbb{R}^N} (|\nabla \varphi(x)|^2 - V(x) \varphi^2(x)) dx \leq 0,$$

a contradiction.  $\square$

The following theorem is the counterpart of Theorem 4.2 as far as the possibility of perturbing singularities at dipoles is concerned.

**Theorem 4.3.** For  $i = 1, \dots, k$ , let  $r_i, R \in \mathbb{R}^+$ ,  $a_i \in \mathbb{R}^N$ ,  $a_i \neq a_j$  for  $i \neq j$ ,  $h_i, h_\infty \in L^\infty(\mathbb{S}^{N-1})$  with  $\mu_1(h_i) > -(N-2)^2/4$ ,  $\mu_1(h_\infty) > -(N-2)^2/4$ , and

$$V(x) = \sum_{i=1}^k \chi_{B(a_i, r_i)}(x) \frac{h_i\left(\frac{x-a_i}{|x-a_i|}\right)}{|x-a_i|^2} + \chi_{\mathbb{R}^N \setminus B(0, R)}(x) \frac{h_\infty\left(\frac{x}{|x|}\right)}{|x|^2} + W(x) \in \mathcal{V}$$

where  $W \in L^{N/2}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ . Assume that  $\mu(V) > 0$  and let  $h \in L^\infty(\mathbb{S}^{N-1})$  such that, for some  $i_0 \in \{1, \dots, k\}$ ,  $\mu_1(h + h_{i_0}) > -(N-2)^2/4$ ,  $\varepsilon > 0$  such that, setting  $H := h + h_{i_0}$ ,

$$\varepsilon < \begin{cases} \frac{1}{\max_{i=1, \dots, k, \infty} \{\Lambda_N(h_i), \Lambda_N(H)\}} - 1, & \text{if } \max_{i=1, \dots, k, \infty} \{\Lambda_N(h_i), \Lambda_N(H)\} > 0, \\ +\infty & \text{if } \max_{i=1, \dots, k, \infty} \{\Lambda_N(h_i), \Lambda_N(H)\} = 0, \end{cases}$$

and  $\sigma > 0$  such that  $0 < \frac{1}{\sigma} < \min_{i=1, \dots, k, \infty} \left\{ \sqrt{(N-2)^2/4 + \mu_1(\tilde{h}_i)}, \sqrt{(N-2)^2/4 + \mu_1(\tilde{H})} \right\}$ , where  $\tilde{h}_i := (1 + \varepsilon)h_i$  and  $\tilde{H} := (1 + \varepsilon)H$ . Then, for every  $i \in \{1, \dots, k\}$ , there exists  $\bar{r}$  such that

$$\mu\left(V + \frac{h\left(\frac{x-a_{i_0}}{|x-a_{i_0}|}\right)}{|x-a_{i_0}|^2} \chi_{\mathcal{E}_{\tilde{h}_{i_0}, \tilde{H}}(a_{i_0})}^{\sigma, r}\right) > 0,$$

for all  $0 < r < \bar{r}$ .

PROOF. Let  $\sigma_1 > 0$  such that

$$\frac{N-2}{2} < \frac{1}{\sigma_1} < \frac{1}{\sigma_1} + \frac{1}{\sigma} < \frac{N-2}{2} + \min_{i=1, \dots, k, \infty} \left\{ \sqrt{\frac{(N-2)^2}{4} + \mu_1(\tilde{h}_i)}, \sqrt{\frac{(N-2)^2}{4} + \mu_1(\tilde{H})} \right\}.$$

Assume, by contradiction, that there exists a sequence  $r_n \rightarrow 0^+$  such that, setting

$$V_n = V + \frac{h\left(\frac{x-a_{i_0}}{|x-a_{i_0}|}\right)}{|x-a_{i_0}|^2} \chi_{\mathcal{E}_{\tilde{h}_{i_0}, \tilde{H}}(a_{i_0})}^{\sigma, r_n},$$

there holds  $\mu(V_n) \leq 0$ . By Lemmas 3.3 and 3.6,  $V_n = \tilde{V}_n + \tilde{W}$ , where  $\mu(\tilde{V}_n) \geq \frac{\varepsilon}{\varepsilon+1} > 0$  and

$$\tilde{W}(x) = \sum_{i=1}^k \chi_{B(a_i, r_i) \setminus \mathcal{E}_{\tilde{h}_i}^{\sigma_1, \delta}(a_i)}(x) \frac{h_i\left(\frac{x-a_i}{|x-a_i|}\right)}{|x-a_i|^2} + \chi_{\mathcal{E}_{\tilde{h}_\infty, 0}^{\sigma_1, R_0} \setminus B(0, R)}(x) \frac{h_\infty\left(\frac{x}{|x|}\right)}{|x|^2} + W(x)$$

for some  $\delta > 0$  and  $R_0 > 0$  independent of  $n$ . By Proposition 4.1,  $\mu(V_n)$  is attained by some  $\varphi_n \in \mathcal{D}^{1,2}(\mathbb{R}^N)$  such that  $\int_{\mathbb{R}^N} |\nabla \varphi_n(x)|^2 dx = 1$ ,  $\int_{\mathbb{R}^N} |\nabla \varphi_n(x)|^2 dx - \int_{\mathbb{R}^N} V_n(x) \varphi_n^2(x) dx = \mu(V_n)$ , and (30) is satisfied. Up to a subsequence,  $\varphi_n \rightharpoonup \varphi$  weakly in  $\mathcal{D}^{1,2}(\mathbb{R}^N)$  for some  $\varphi \in \mathcal{D}^{1,2}(\mathbb{R}^N)$ , and hence

$$\frac{\varepsilon}{\varepsilon+1} \leq \mu(V_n) + \int_{\mathbb{R}^N} \tilde{W}(x) \varphi_n^2(x) dx \leq \int_{\mathbb{R}^N} \tilde{W}(x) \varphi^2(x) dx + o(1) \quad \text{as } n \rightarrow +\infty.$$

Therefore  $\int_{\mathbb{R}^N} \tilde{W}(x) \varphi^2(x) dx > 0$  and we conclude that  $\varphi \not\equiv 0$ . We claim that

$$(32) \quad \lim_{j \rightarrow \infty} \int_{\mathbb{R}^N} V_n(x) \varphi_n(x) \varphi(x) dx = \int_{\mathbb{R}^N} V(x) \varphi^2(x) dx.$$

Indeed for any  $\eta > 0$ , by density there exists  $\psi \in C_c^\infty(\mathbb{R}^N \setminus \{a_{i_0}\})$  such that  $\|\varphi - \psi\|_{\mathcal{D}^{1,2}(\mathbb{R}^N)} < \eta$ . Since the support of  $\psi$  is detouched from  $a_{i_0}$ , from Hardy's inequality we have that, for large  $n$ ,

$$\begin{aligned} & \left| \int_{\mathbb{R}^N} V_n(x) \varphi_n(x) \varphi(x) dx - \int_{\mathbb{R}^N} V(x) \varphi^2(x) dx \right| \\ & \leq \left| \int_{\mathbb{R}^N} (V_n(x) - V(x)) \varphi_n(x) (\varphi(x) - \psi(x)) dx \right| + \left| \int_{\mathbb{R}^N} V(x) (\varphi_n(x) - \varphi(x)) \varphi(x) dx \right| \\ & \leq \text{const } \eta + \left| \int_{\mathbb{R}^N} V(x) (\varphi_n(x) - \varphi(x)) \varphi(x) dx \right| = \text{const } \eta + o(1) \quad \text{as } n \rightarrow \infty. \end{aligned}$$

(32) is thereby proved. From (32), multiplying (30) by  $\varphi$  and passing to limit as  $n \rightarrow \infty$ , we obtain

$$\mu(V) \int_{\mathbb{R}^N} |\nabla \varphi(x)|^2 dx \leq \int_{\mathbb{R}^N} (|\nabla \varphi(x)|^2 - V(x) \varphi^2(x)) dx \leq 0,$$

a contradiction.  $\square$

## 5. LOCALIZATION OF BINDING

This section deals with the *localization of binding* of Schrödinger operators with potentials in the class  $\mathcal{V}$ . The theory of localization of binding was first developed by Simon [29] who proved that if  $V_1$  and  $V_2$  are compactly supported potentials such that the corresponding Schrödinger operators are positive, then also the operator  $-\Delta - V_1 - V_2(\cdot - y)$  is positive definite provided  $|y|$  is sufficiently large. Pinchover [24] extended the above result to the case of potentials belonging to the Kato class.

We notice that both Simon and Pinchover consider potentials which are lower order perturbations of the Laplacian, thus excluding the case of potentials with inverse square type singularities. Such a case presents an additional difficulty due to the interaction of singularities at infinity. In [11], the following localization of binding result was proved for locally isotropic inverse square potentials under some additional assumptions to control the singularity at infinity:

**Theorem 5.1.** [11, Theorem 1.5] *Let*

$$\begin{aligned} V_1(x) &= \sum_{i=1}^{k_1} \chi_{B(a_i^1, r_i^1)}(x) \frac{h_i^1 \left( \frac{x - a_i^1}{|x - a_i^1|} \right)}{|x - a_i^1|^2} + \chi_{\mathbb{R}^N \setminus B(0, R_1)}(x) \frac{h_\infty^1 \left( \frac{x}{|x|} \right)}{|x|^2} + W_1(x) \in \mathcal{V}, \\ V_2(x) &= \sum_{i=1}^{k_2} \chi_{B(a_i^2, r_i^2)}(x) \frac{h_i^2 \left( \frac{x - a_i^2}{|x - a_i^2|} \right)}{|x - a_i^2|^2} + \chi_{\mathbb{R}^N \setminus B(0, R_2)}(x) \frac{h_\infty^2 \left( \frac{x}{|x|} \right)}{|x|^2} + W_2(x) \in \mathcal{V}. \end{aligned}$$

*Assume that the functions  $h_i^j$  are constant for all  $j = 1, 2, i = 1, \dots, k_j$ ,  $\mu(V_1), \mu(V_2) > 0$ , and that (6) is satisfied. Then, there exists  $R > 0$  such that, for every  $y \in \mathbb{R}^N$  with  $|y| \geq R$ , the quadratic form associated to the operator  $-\Delta - (V_1 + V_2(\cdot - y))$  is positive definite.*

For general anisotropic inverse-square potentials, condition (6) is necessary to have a localization of binding type result, as the following proposition states.

**Proposition 5.2.** *For  $j = 1, 2$ , let*

$$V_j = \sum_{i=1}^{k_j} \chi_{B(a_i^j, r_i^j)}(x) \frac{h_i^j \left( \frac{x-a_i^j}{|x-a_i^j|} \right)}{|x-a_i^j|^2} + \chi_{\mathbb{R}^N \setminus B(0, R_j)}(x) \frac{h_\infty^j \left( \frac{x}{|x|} \right)}{|x|^2} + W_j(x) \in \mathcal{V}.$$

*If there exists  $y \in \mathbb{R}^N$  such that  $\mu(V_1 + V_2(\cdot - y)) > 0$ , then*

$$\mu_1(h_\infty^1 + h_\infty^2) > -\left(\frac{N-2}{2}\right)^2.$$

PROOF. Assume that, for some  $y \in \mathbb{R}^N$ , for some  $\varepsilon > 0$ , and for any  $u \in \mathcal{D}^{1,2}(\mathbb{R}^N)$ ,

$$(33) \quad \int_{\mathbb{R}^N} (|\nabla u(x)|^2 - V_1(x)u^2(x) - V_2(x-y)u^2(x)) dx \geq \varepsilon \int_{\mathbb{R}^N} |\nabla u(x)|^2 dx.$$

Arguing by contradiction, suppose that  $\mu_1(h_\infty^1 + h_\infty^2) \leq -(N-2)^2/4$ , or, equivalently, that  $\Lambda_N(h_\infty^1 + h_\infty^2) \geq 1$ . Let  $0 < \delta < \varepsilon \Lambda_N(h_\infty^1 + h_\infty^2)^{-1}$ . By (4) and density of  $C_c^\infty(\mathbb{R}^N \setminus \{0\})$  in  $\mathcal{D}^{1,2}(\mathbb{R}^N)$ , there exists  $\phi \in C_c^\infty(\mathbb{R}^N \setminus \{0\})$  such that

$$\int_{\mathbb{R}^N} |\nabla \phi(x)|^2 dx - \int_{\mathbb{R}^N} \frac{(h_\infty^1 + h_\infty^2) \left( \frac{x}{|x|} \right)}{|x|^2} \phi^2(x) dx < \delta \int_{\mathbb{R}^N} \frac{(h_\infty^1 + h_\infty^2) \left( \frac{x}{|x|} \right)}{|x|^2} \phi^2(x) dx.$$

Let  $\phi_\mu(x) = \mu^{-(N-2)/2} \phi(x/\mu)$ . A direct calculation (similar to that performed in the proof of Proposition 1.2) yields

$$\begin{aligned} & \int_{\mathbb{R}^N} |\nabla \phi_\mu(x)|^2 dx - \int_{\mathbb{R}^N} V_1(x) \phi_\mu^2(x) dx - \int_{\mathbb{R}^N} V_2(x-y) \phi_\mu^2(x) dx \\ &= \int_{\mathbb{R}^N} |\nabla \phi(x)|^2 dx - \int_{\mathbb{R}^N} \frac{h_\infty^1 \left( \frac{x}{|x|} \right)}{|x|^2} \phi^2(x) dx - \int_{\mathbb{R}^N \setminus B\left(\frac{y}{\mu}, \frac{R_2}{\mu}\right)} \frac{h_\infty^2 \left( \frac{x-(y/\mu)}{|x-(y/\mu)|} \right)}{|x-(y/\mu)|^2} \phi^2(x) dx + o(1), \end{aligned}$$

as  $\mu \rightarrow \infty$ . From continuity of  $\varphi$  and the Dominated Convergence Theorem, we deduce that

$$\begin{aligned} & \int_{\mathbb{R}^N \setminus B\left(\frac{y}{\mu}, \frac{R_2}{\mu}\right)} \frac{h_\infty^2 \left( \frac{x-(y/\mu)}{|x-(y/\mu)|} \right)}{|x-(y/\mu)|^2} \phi^2(x) dx = \int_{\mathbb{R}^N \setminus B\left(0, \frac{R_2}{\mu}\right)} \frac{h_\infty^2 \left( \frac{x}{|x|} \right)}{|x|^2} \phi^2(x + \frac{y}{\mu}) dx \\ &= \int_{\mathbb{R}^N} \frac{h_\infty^2 \left( \frac{x}{|x|} \right)}{|x|^2} \phi^2(x) dx + o(1), \quad \text{as } \mu \rightarrow \infty, \end{aligned}$$

hence

$$\begin{aligned} & \int_{\mathbb{R}^N} |\nabla \phi_\mu(x)|^2 dx - \int_{\mathbb{R}^N} V_1(x) \phi_\mu^2(x) dx - \int_{\mathbb{R}^N} V_2(x-y) \phi_\mu^2(x) dx \\ &= \int_{\mathbb{R}^N} |\nabla \phi(x)|^2 dx - \int_{\mathbb{R}^N} \frac{(h_\infty^1 + h_\infty^2) \left( \frac{x}{|x|} \right)}{|x|^2} \phi^2(x) dx + o(1), \end{aligned}$$

as  $\mu \rightarrow \infty$ . Letting  $\mu \rightarrow \infty$  we obtain that

$$\begin{aligned} \varepsilon \int_{\mathbb{R}^N} |\nabla \phi(x)|^2 dx &\leq \int_{\mathbb{R}^N} |\nabla \phi(x)|^2 dx - \int_{\mathbb{R}^N} \frac{(h_\infty^1 + h_\infty^2) \left( \frac{x}{|x|} \right)}{|x|^2} \phi^2(x) dx \\ &< \delta \Lambda_N(h_\infty^1 + h_\infty^2) \int_{\mathbb{R}^N} |\nabla \phi(x)|^2 dx < \varepsilon \int_{\mathbb{R}^N} |\nabla \phi(x)|^2 dx, \end{aligned}$$

thus giving a contradiction.  $\square$

On the other hand, the control at infinity required in (6) is not sufficient to obtain a localization of binding result for potentials which are anisotropic at infinity, as the following example shows.

**Example 5.3.** For  $N \geq 4$ , there exist  $V_1, V_2 \in \mathcal{V}$  such that  $\mu(V_1), \mu(V_2) > 0$ , (6) holds, and for every  $R > 0$  there exists  $y_R \in \mathbb{R}^N$  such that  $|y_R| > R$  and the quadratic form associated to the operator  $-\Delta - (V_1 + V_2(\cdot - y_R))$  is not positive semidefinite, i.e.  $\mu(V_1 + V_2(\cdot - y_R)) < 0$ .

For  $N \geq 4$ , let  $\bar{y} = (0, \dots, 0, 1) \in \mathbb{R}^N$  and  $\frac{1}{2} \left(\frac{N-3}{2}\right)^2 < \lambda < \left(\frac{N-3}{2}\right)^2$ . From [28], scaling invariance, translation invariance in the  $\bar{y}$ -direction, and density of  $C_c^\infty((\mathbb{R}^{N-1} \setminus \{0\}) \times \mathbb{R})$  in  $\mathcal{D}^{1,2}(\mathbb{R}^N)$ , it follows that, for any  $0 < \varepsilon < 2\lambda - \left(\frac{N-3}{2}\right)^2$ , there exists  $\psi \in C_c^\infty((\mathbb{R}^{N-1} \setminus \{0\}) \times \mathbb{R})$  such that

$$\text{supp } \psi \subset Q := \left\{ x \in \mathbb{R}^N : \frac{x}{|x|} \cdot \bar{y} > \frac{\sqrt{3}}{2} \text{ and } \frac{x - \bar{y}}{|x - \bar{y}|} \cdot \bar{y} < -\frac{\sqrt{3}}{2} \right\}$$

and

$$\left(\frac{N-3}{2}\right)^2 < \frac{\int_{\mathbb{R}^N} |\nabla \psi(x)|^2 dx}{\int_{\mathbb{R}^N} \frac{\psi^2(x)}{|x'|^2} dx} < \left(\frac{N-3}{2}\right)^2 + \varepsilon,$$

where a generic point  $x \in \mathbb{R}^N$  is denoted, from now on, as  $x = (x', x_N) \in \mathbb{R}^{N-1} \times \mathbb{R}$ . Since  $\psi \in C_c^\infty(\mathbb{R}^N \setminus \{x' = 0\})$ ,

$$\delta = \min \left\{ \sqrt{1 - \left(\frac{x}{|x|} \cdot \bar{y}\right)^2}, \sqrt{1 - \left(\frac{x - \bar{y}}{|x - \bar{y}|} \cdot \bar{y}\right)^2} : x \in \text{supp } \psi \right\} > 0.$$

Hence

$$\text{supp } \psi \subset Q' := \left\{ x \in \mathbb{R}^N : \frac{\sqrt{3}}{2} < \frac{x}{|x|} \cdot \bar{y} < \sqrt{1 - \delta^2} \text{ and } -\sqrt{1 - \delta^2} < \frac{x - \bar{y}}{|x - \bar{y}|} \cdot \bar{y} < -\frac{\sqrt{3}}{2} \right\}$$

Let

$$V_1(x) = \frac{\lambda \chi_{C^+}(x)}{|x|^2 \left(1 - \left(\frac{x}{|x|} \cdot \bar{y}\right)^2\right)} = \frac{\lambda \chi_{C^+}(x)}{|x'|^2} \quad \text{and} \quad V_2(x) = \frac{\lambda \chi_{C^-}(x)}{|x|^2 \left(1 - \left(\frac{x}{|x|} \cdot \bar{y}\right)^2\right)} = \frac{\lambda \chi_{C^-}(x)}{|x'|^2},$$

where

$$C^+ = \left\{ x \in \mathbb{R}^N : \frac{\sqrt{3}}{2} < \frac{x}{|x|} \cdot \bar{y} < \sqrt{1 - \delta^2} \right\} \quad \text{and} \\ C^- = \left\{ x \in \mathbb{R}^N : -\sqrt{1 - \delta^2} < \frac{x}{|x|} \cdot \bar{y} < -\frac{\sqrt{3}}{2} \right\} = -C^+.$$

We notice that  $C^+ \cap (\bar{y} + C^-) = Q'$ . Moreover, we can write  $V_1, V_2$  as

$$V_1(x) = \frac{h_1\left(\frac{x}{|x|}\right)}{|x|^2} \quad \text{with} \quad h_1(\theta) = \frac{\lambda \chi_{\{\theta \in \mathbb{S}^{N-1} : \sqrt{3}/2 < \theta \cdot \bar{y} < \sqrt{1 - \delta^2}\}}(\theta)}{(1 - (\theta \cdot \bar{y})^2)}, \\ V_2(x) = \frac{h_2\left(\frac{x}{|x|}\right)}{|x|^2} \quad \text{with} \quad h_2(\theta) = \frac{\lambda \chi_{\{\theta \in \mathbb{S}^{N-1} : -\sqrt{1 - \delta^2} < \theta \cdot \bar{y} < -\sqrt{3}/2\}}(\theta)}{(1 - (\theta \cdot \bar{y})^2)}.$$

Being  $\lambda < \left(\frac{N-3}{2}\right)^2$ , from [3, 28] it follows easily that  $\mu(V_1), \mu(V_2) > 0$ . Moreover,  $C^+ \cap C^- = \emptyset$  and

$$V_1(x) + V_2(x) = \frac{h_1\left(\frac{x}{|x|}\right) + h_2\left(\frac{x}{|x|}\right)}{|x|^2} = \frac{\lambda(\chi_{C^+}(x) + \chi_{C^-}(x))}{|x'|^2} = \frac{\lambda\chi_{C^+ \cup C^-}(x)}{|x'|^2} \leq \frac{\lambda}{|x'|^2},$$

hence, from  $\lambda < \left(\frac{N-3}{2}\right)^2$  and [3, 28], it follows that  $\mu(V_1 + V_2) > 0$ , and thus  $\mu_1(h_1 + h_2) > -\left(\frac{N-2}{2}\right)^2$ .

For any  $\mu > 0$ , let  $\psi_\mu(x) := \mu^{-\frac{N-2}{2}} \psi(x/\mu)$ . Since

$$V_1(x) + V_2(x - \bar{y}) = \frac{\lambda(\chi_{C^+}(x) + \chi_{\bar{y} + C^-}(x))}{|x'|^2},$$

a direct calculation yields, for all  $\mu > 0$ ,

$$\begin{aligned} \frac{\int_{\mathbb{R}^N} (V_1(x) + V_2(x - \mu \bar{y})) \psi_\mu^2(x) dx}{\int_{\mathbb{R}^N} |\nabla \psi_\mu(x)|^2 dx} &= \frac{\int_{\mathbb{R}^N} (V_1(x) + V_2(x - \bar{y})) \psi^2(x) dx}{\int_{\mathbb{R}^N} |\nabla \psi(x)|^2 dx} \\ &\geq 2\lambda \frac{\int_{Q'} \frac{\psi^2(x)}{|x'|^2} dx}{\int_{\mathbb{R}^N} |\nabla \psi(x)|^2 dx} = 2\lambda \frac{\int_{\mathbb{R}^N} \frac{\psi^2(x)}{|x'|^2} dx}{\int_{\mathbb{R}^N} |\nabla \psi(x)|^2 dx} > \frac{2\lambda}{\left(\frac{N-3}{2}\right)^2 + \varepsilon} > 1, \end{aligned}$$

thus implying that  $\mu(V_1 + V_2(\cdot - \mu \bar{y})) < 0$  for all  $\mu > 0$ . Hence, for every  $R > 0$ , it is enough to choose  $\mu > R$  and  $y_R = \mu \bar{y}$  to obtain the example we are looking for.

**Theorem 5.4.** *Let*

$$\begin{aligned} V_1(x) &= \sum_{i=1}^{k_1} \chi_{B(a_i^1, r_i^1)}(x) \frac{h_i^1\left(\frac{x - a_i^1}{|x - a_i^1|}\right)}{|x - a_i^1|^2} + \chi_{\mathbb{R}^N \setminus B(0, R_1)}(x) \frac{h_\infty^1\left(\frac{x}{|x|}\right)}{|x|^2} + W_1(x) \in \mathcal{V}, \\ V_2(x) &= \sum_{i=1}^{k_2} \chi_{B(a_i^2, r_i^2)}(x) \frac{h_i^2\left(\frac{x - a_i^2}{|x - a_i^2|}\right)}{|x - a_i^2|^2} + \chi_{\mathbb{R}^N \setminus B(0, R_2)}(x) \frac{h_\infty^2\left(\frac{x}{|x|}\right)}{|x|^2} + W_2(x) \in \mathcal{V}, \end{aligned}$$

where  $W_i \in L^\infty(\mathbb{R}^N)$ ,  $W_i(x) = O(|x|^{-2-\delta})$ ,  $i = 1, 2$ , with  $\delta > 0$ , as  $|x| \rightarrow \infty$ . Assume that  $\mu(V_1), \mu(V_2) > 0$ , and that

$$(34) \quad \operatorname{ess\,sup}_{\mathbb{S}^{N-1}} (h_1^\infty)^+ + \operatorname{ess\,sup}_{\mathbb{S}^{N-1}} (h_2^\infty)^+ < \frac{(N-2)^2}{4}.$$

Then, there exists  $R > 0$  such that, for every  $y \in \mathbb{R}^N$  with  $|y| \geq R$ , there exists  $\Phi_y \in \mathcal{D}^{1,2}(\mathbb{R}^N)$ ,  $\Phi_y > 0$  and continuous in  $\mathbb{R}^N \setminus \{a_i^1, a_i^2 + y\}_{i=1, \dots, k_j, j=1, 2}$ , such that

$$-\Delta \Phi_y - (V_1 + V_2(\cdot - y)) \Phi_y > 0 \quad \text{in } (\mathcal{D}^{1,2}(\mathbb{R}^N))^*.$$

**PROOF.** Let us set  $h_j = (\operatorname{ess\,sup}_{\mathbb{S}^{N-1}} h_j^\infty) - h_j^\infty$ ,  $j = 1, 2$  and, for any  $j = 1, 2$  and  $i = 1, \dots, k_j$ , let us choose  $\max\{0, \mu_1(h_i^j)\} < c_i^j < \mu_1(h_i^j) + \left(\frac{N-2}{2}\right)^2$ .

In view of Theorems 4.2 and 4.3 and by assumption (34), there exist  $\mathcal{N}_1$  and  $\mathcal{N}_2$  neighborhoods of infinity and  $\mathcal{N}_i^j$  neighborhoods of  $a_i^j$ ,  $j = 1, 2$  and  $i = 1, \dots, k_j$ , such that  $\mu(\tilde{V}_j) > 0$ ,  $j = 1, 2$ , where  $\tilde{V}_j(x) := V_j(x) + \sum_{i=1}^{k_j} \chi_{\mathcal{N}_i^j}(x) \frac{c_i^j}{|x - a_i^j|^2} + |x|^{-2} h_j(x/|x|) \chi_{\mathcal{N}_j}(x)$ . We notice that  $\tilde{V}_j \geq V_j$  a.e. in  $\mathbb{R}^N$ .

Let  $0 < \varepsilon < \left(\frac{N-2}{2}\right)^2 - \text{ess sup}_{\mathbb{S}^{N-1}}(h_1^\infty)^+ - \text{ess sup}_{\mathbb{S}^{N-1}}(h_2^\infty)^+$  and, for  $j = 1, 2$ , set

$$\Lambda = \left(\frac{N-2}{2}\right)^2 - \varepsilon \quad \text{and} \quad \gamma_j = \Lambda - \text{ess sup}_{\mathbb{S}^{N-1}} h_j^\infty.$$

Let us also choose  $0 < \eta \ll 1$  such that

$$(35) \quad \text{ess sup}_{\mathbb{S}^{N-1}} h_2^\infty < \gamma_1(1-2\eta) \quad \text{and} \quad \text{ess sup}_{\mathbb{S}^{N-1}} h_1^\infty < \gamma_2(1-2\eta).$$

We can choose  $\bar{R} > 0$  such that, for  $j = 1, 2$ ,  $\bigcup_{i=1}^{k_j} B(a_i^j, r_i^j) \subset B(0, \bar{R})$ , and define

$$(36) \quad p_j(x) := \begin{cases} |x - a_i^j|^{-2+\tau} & \text{in } B(a_i^j, r_i^j), \\ 1 & \text{in } B(0, \bar{R}) \setminus \bigcup_{i=1}^{k_j} B(a_i^j, r_i^j), \\ 0 & \text{in } \mathbb{R}^N \setminus B(0, \bar{R}), \end{cases}$$

with  $0 < \tau < \min_{i,j} \left\{1, \frac{N-2}{2} - \sqrt{\left(\frac{N-2}{2}\right)^2 + \mu_1(h_i^j) - c_i^j}\right\}$ . In view of Theorem 4.2, there exist  $\tilde{R}_j$  such that the quadratic forms associated to the operators  $-\Delta - \tilde{V}_j - \frac{\gamma_j}{|x|^2} \chi_{\mathbb{R}^N \setminus B(0, \tilde{R}_j)}$  are positive definite. Therefore, since  $p_j \in L^{N/2}$ , the infima

$$\mu_j = \inf_{u \in \mathcal{D}^{1,2}(\mathbb{R}^N) \setminus \{0\}} \frac{\int_{\mathbb{R}^N} [|\nabla u(x)|^2 - \tilde{V}_j(x)u^2(x) - \gamma_j|x|^{-2}\chi_{\mathbb{R}^N \setminus B(0, \tilde{R}_j)}u^2(x)] dx}{\int_{\mathbb{R}^N} p_j(x)u^2(x) dx}$$

are achieved by some  $\psi_j \in \mathcal{D}^{1,2}(\mathbb{R}^N)$ ,  $\psi_j > 0$  and continuous in  $\mathbb{R}^N \setminus \{a_1^j, \dots, a_{k_j}^j\}$ ,  $\mathcal{D}^{1,2}(\mathbb{R}^N)$ -weakly solving

$$(37) \quad -\Delta \psi_j(x) - \tilde{V}_j(x)\psi_j(x) = \mu_j p_j(x)\psi_j(x) + \frac{\gamma_j}{|x|^2} \chi_{\mathbb{R}^N \setminus B(0, \tilde{R}_j)} \psi_j(x).$$

From the asymptotic analysis of the exact behavior near the singularity of solutions to Schrödinger equations with inverse square potentials proved in [13] (see also [22] and [12]), there holds

$$\lim_{|x| \rightarrow +\infty} \psi_j(x)|x|^{N-2+\sigma_\Lambda} = \ell_j > 0, \quad \text{where} \quad \sigma_\Lambda := -\frac{N-2}{2} + \sqrt{\left(\frac{N-2}{2}\right)^2 - \Lambda},$$

hence the function  $\varphi_j := \frac{\psi_j}{\ell_j}$  solves (37) and  $\varphi_j(x) \sim |x|^{-(N-2+\sigma_\Lambda)}$  at  $\infty$ . Then there exists  $\rho > \max\{\tilde{R}_1, \tilde{R}_2, \bar{R}\}$  such that, in  $\mathbb{R}^N \setminus B(0, \rho)$ ,

$$(38) \quad (1 - \eta^2)|x|^{-(N-2+\sigma_\Lambda)} \leq \varphi_j(x) \leq (1 + \eta)|x|^{-(N-2+\sigma_\Lambda)}$$

and that

$$(39) \quad |W_1(x)| \leq \eta\gamma_2|x|^{-2} \quad \text{and} \quad |W_2(x)| \leq \eta\gamma_1|x|^{-2}.$$

Moreover, from [12, Theorem 1.1], we can deduce that for some positive constant  $C$

$$(40) \quad \frac{1}{C}|x - a_i^j|^{\sigma_i^j} \leq \varphi_j(x) \leq C|x - a_i^j|^{\sigma_i^j} \quad \text{in } B(a_i^j, r_i^j), \quad i = 1, \dots, k_j,$$

where  $\sigma_i^j = -\frac{N-2}{2} + \sqrt{\left(\frac{N-2}{2}\right)^2 + \mu_1(h_i^j) - c_i^j}$ .

For any  $y \in \mathbb{R}^N$ , let us consider the function

$$\Phi_y(x) := \gamma_2\varphi_1(x) + \gamma_1\varphi_2(x-y) \in \mathcal{D}^{1,2}(\mathbb{R}^N).$$

Then  $\Phi_y$  satisfies, in the weak  $\mathcal{D}^{1,2}(\mathbb{R}^N)$ -sense,

$$-\Delta \Phi_y - (\tilde{V}_1 + \tilde{V}_2(\cdot - y)) \Phi_y = f$$

where

$$\begin{aligned} f(x) &= \mu_1 \gamma_2 p_1(x) \varphi_1(x) + \frac{\gamma_1 \gamma_2}{|x|^2} \chi_{\mathbb{R}^N \setminus B(0, \tilde{R}_1)} \varphi_1(x) + \mu_2 \gamma_1 p_2(x - y) \varphi_2(x - y) \\ &\quad + \frac{\gamma_1 \gamma_2}{|x - y|^2} \chi_{\mathbb{R}^N \setminus B(y, \tilde{R}_2)} \varphi_2(x - y) - \gamma_1 \tilde{V}_1(x) \varphi_2(x - y) - \gamma_2 \tilde{V}_2(x - y) \varphi_1(x). \end{aligned}$$

From (35), (38), and (39), it follows that in  $\mathbb{R}^N \setminus (B(0, \rho) \cup B(y, \rho))$

$$\begin{aligned} f(x) &\geq \frac{\gamma_1 \gamma_2}{|x|^2} \varphi_1(x) + \frac{\gamma_1 \gamma_2}{|x - y|^2} \varphi_2(x - y) \\ &\quad - \gamma_1 \left( \frac{\text{ess sup}_{\mathbb{S}^{N-1}} h_1^\infty}{|x|^2} + W_1(x) \right) \varphi_2(x - y) - \gamma_2 \left( \frac{\text{ess sup}_{\mathbb{S}^{N-1}} h_2^\infty}{|x - y|^2} + W_2(x - y) \right) \varphi_1(x) \\ &\geq \frac{\gamma_1 \gamma_2 (1 - \eta^2)}{|x|^{N+\sigma_\Lambda}} + \frac{\gamma_1 \gamma_2 (1 - \eta^2)}{|x - y|^{N+\sigma_\Lambda}} - \frac{\gamma_1 \gamma_2 (1 - \eta^2)}{|x|^2 |x - y|^{N-2+\sigma_\Lambda}} - \frac{\gamma_1 \gamma_2 (1 - \eta^2)}{|x - y|^2 |x|^{N-2+\sigma_\Lambda}} \\ &= \gamma_1 \gamma_2 (1 - \eta^2) \left( \frac{1}{|x|^{N+\sigma_\Lambda-2}} - \frac{1}{|x - y|^{N+\sigma_\Lambda-2}} \right) \left( \frac{1}{|x|^2} - \frac{1}{|x - y|^2} \right) \geq 0. \end{aligned}$$

For  $|y|$  sufficiently large,  $B(0, \rho) \cap B(y, \rho) = \emptyset$ . In  $B(a_i^1, r_i^1)$ , for  $i = 1, \dots, k_1$ , from (36), (38), (40) and the fact that  $\sigma_i^j < -\tau$  for all  $j = 1, 2$  and  $i = 1, \dots, k_j$ , we have that

$$\begin{aligned} f(x) &\geq \mu_1 \gamma_2 p_1(x) \varphi_1(x) + \frac{\gamma_1 \gamma_2}{|x - y|^2} \varphi_2(x - y) - \gamma_1 \tilde{V}_1(x) \varphi_2(x - y) - \gamma_2 \tilde{V}_2(x - y) \varphi_1(x) \\ &\geq |x - a_i^1|^{-2+\tau+\sigma_i^j} \left[ \frac{\mu_1 \gamma_2}{C} + o(1) \right], \quad \text{as } |y| \rightarrow \infty. \end{aligned}$$

In  $B(0, \rho) \setminus \bigcup_{i=1}^{k_1} B(a_i^1, r_i^1)$ , from (36), (38), (39) and since  $\varphi_1 > c > 0$ , we obtain that

$$f(x) \geq \mu_1 \gamma_c + o(1), \quad \text{as } |y| \rightarrow \infty.$$

In a similar way we can prove that, if  $|y|$  is large enough,  $f(x) > 0$ , a.e. in  $B(y, \rho)$ . Since  $\tilde{V}_1(x) + \tilde{V}_2(x - y) \geq V_1(x) + V_2(x - y)$ , we conclude that

$$-\Delta \Phi_y - (V_1 + V_2(\cdot - y)) \Phi_y > 0 \quad \text{in } (\mathcal{D}^{1,2}(\mathbb{R}^N))^*.$$

□

**Proof of Theorem 1.5.** Let us fix  $\varepsilon \in (0, 1)$  such that, for  $j = 1, 2$ ,

$$(41) \quad \varepsilon < \min \left\{ 2S \mu(V_j), \frac{\mu(V_j)}{4} \left[ \frac{4 \left( \sum_{i=1}^{k_j} \|h_i^j\|_{L^\infty(\mathbb{S}^{N-1})} + \|h_\infty^j\|_{L^\infty(\mathbb{S}^{N-1})} \right)}{(N-2)^2} + \frac{\|W_j\|_{L^{N/2}(\mathbb{R}^N)}}{S} \right]^{-1} \right\}$$

and

$$\mu((1 + \varepsilon)V_j) > 0, \quad (1 + \varepsilon) \left[ \text{ess sup}_{\mathbb{S}^{N-1}} (h_1^\infty)^+ + \text{ess sup}_{\mathbb{S}^{N-1}} (h_2^\infty)^+ \right] < \frac{(N-2)^2}{4},$$

see the proof of Lemma 3.1. Fix  $\tilde{R} > 0$  such that

$$(42) \quad \|W_j \chi_{\mathbb{R}^N \setminus B(0, \tilde{R})}\|_{L^{N/2}(\mathbb{R}^N)} < \min \left\{ \frac{\varepsilon}{4}, \frac{\varepsilon}{4} S \right\}.$$

Denoting  $V_{j,\tilde{R}} := V_j - W_j \chi_{\mathbb{R}^N \setminus B(0,\tilde{R})}$ , from (41) and (42), there results

$$\begin{aligned} & \int_{\mathbb{R}^N} (|\nabla u(x)|^2 - V_{j,\tilde{R}}(x)u^2(x)) dx \\ & \geq \left[ \mu(V_j) - \|W_j \chi_{\mathbb{R}^N \setminus B(0,\tilde{R})}\|_{L^{N/2}(\mathbb{R}^N)} S^{-1} \right] \int_{\mathbb{R}^N} |\nabla u(x)|^2 dx \geq \frac{\mu(V_j)}{2} \int_{\mathbb{R}^N} |\nabla u(x)|^2 dx, \end{aligned}$$

therefore, from (41), it follows

$$\begin{aligned} & \int_{\mathbb{R}^N} (|\nabla u(x)|^2 - (1+\varepsilon)V_{j,\tilde{R}}(x)u^2(x)) dx \\ & \geq \left[ \frac{\mu(V_j)}{2} - \varepsilon \left( \frac{4 \left( \sum_{i=1}^{k_j} \|h_i^j\|_{L^\infty(\mathbb{S}^{N-1})} + \|h_\infty^j\|_{L^\infty(\mathbb{S}^{N-1})} \right)}{(N-2)^2} + \frac{\|W_j\|_{L^{N/2}(\mathbb{R}^N)}}{S} \right) \right] \int_{\mathbb{R}^N} |\nabla u(x)|^2 dx \\ & \geq \frac{\mu(V_j)}{4} \int_{\mathbb{R}^N} |\nabla u(x)|^2 dx. \end{aligned}$$

Hence the potentials  $(1+\varepsilon)V_{j,\tilde{R}}$  satisfy the assumptions of Lemma 5.4, which yields, for  $|y|$  sufficiently large, the existence of  $\Phi_y \in \mathcal{D}^{1,2}(\mathbb{R}^N)$ ,  $\Phi_y > 0$  and continuous in  $\mathbb{R}^N \setminus \{a_i^1, a_i^2 + y\}_{i=1,\dots,k_j, j=1,2}$ , such that

$$-\Delta \Phi_y - V_{\tilde{R},y} \Phi_y > \varepsilon V_{\tilde{R},y} \Phi_y(x) \quad \text{in } (\mathcal{D}^{1,2}(\mathbb{R}^N))^*,$$

where  $V_{\tilde{R},y}(x) := V_{1,\tilde{R}}(x) + V_{2,\tilde{R}}(x-y)$ . From Lemma 3.1, we deduce that  $\mu(V_{\tilde{R},y}) \geq \frac{\varepsilon}{\varepsilon+1}$ . Hence, from (42), for any  $u \in \mathcal{D}^{1,2}(\mathbb{R}^N)$ , there holds

$$\begin{aligned} & \int_{\mathbb{R}^N} (|\nabla u(x)|^2 - (V_1(x) + V_2(x-y))u^2(x)) dx \\ & = \int_{\mathbb{R}^N} (|\nabla u(x)|^2 - V_{\tilde{R},y}(x)u^2(x)) dx - \int_{\mathbb{R}^N \setminus B(0,\tilde{R})} W_1(x)u^2(x) dx - \int_{\mathbb{R}^N \setminus B(y,\tilde{R})} W_2(x-y)u^2(x) dx \\ & \geq \left[ \frac{\varepsilon}{\varepsilon+1} - \|W_1 \chi_{\mathbb{R}^N \setminus B(0,\tilde{R})}\|_{L^{N/2}(\mathbb{R}^N)} S^{-1} - \|W_2 \chi_{\mathbb{R}^N \setminus B(0,\tilde{R})}\|_{L^{N/2}(\mathbb{R}^N)} S^{-1} \right] \int_{\mathbb{R}^N} |\nabla u(x)|^2 dx \\ & \geq \left[ \frac{\varepsilon}{\varepsilon+1} - \frac{\varepsilon}{2} \right] \int_{\mathbb{R}^N} |\nabla u(x)|^2 dx = \frac{\varepsilon(1-\varepsilon)}{2(\varepsilon+1)} \int_{\mathbb{R}^N} |\nabla u(x)|^2 dx. \end{aligned}$$

Therefore  $\mu(V_1 + V_2(\cdot - y)) \geq \frac{\varepsilon(1-\varepsilon)}{2(\varepsilon+1)} > 0$  for  $|y|$  sufficiently large. The theorem is thereby proved.  $\square$

## 6. ESSENTIAL SELF-ADJOINTNESS

By the Shattering Lemma 1.3, Schrödinger operators with potentials in  $\mathcal{V}$  are compact perturbations of positive operators, see Corollary 1.4. As a consequence, they are *semi-bounded symmetric operators* and their  $L^2(\mathbb{R}^N)$ -spectrum is bounded from below.

In the present section, we discuss essential self-adjointness of operators  $-\Delta - V$  on the domain  $C_c^\infty(\mathbb{R}^N \setminus \{a_1, \dots, a_k\})$ , for every  $V \in \mathcal{V}$ .

From Corollary 1.4 (see also Lemmas 3.3 and 3.6), we can split any  $V \in \mathcal{V}$  as  $V(x) = \tilde{V}(x) + \tilde{W}(x)$  where

$$(43) \quad \tilde{V}(x) = \sum_{i=1}^k \chi_{\mathcal{U}_i}(x) \frac{h_i\left(\frac{x-a_i}{|x-a_i|}\right)}{|x-a_i|^2} + \chi_{\mathcal{U}_\infty}(x) \frac{h_\infty\left(\frac{x}{|x|}\right)}{|x|^2}, \quad \mu(\tilde{V}) > 0,$$

$\mathcal{U}_i \subset B(a_i, 1/2)$  is a neighborhood of  $a_i$  for every  $i = 1, \dots, k$ ,  $\mathcal{U}_\infty$  is a neighborhood of  $\infty$ , and

$$\tilde{W} \in L^{N/2}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N).$$

The following self-adjointness criterion in  $\mathcal{V}$  is an easy consequence of the Kato-Rellich Theorem (see e.g. [20, Theorem 4.4]) and well-known self-adjointness criteria for positive operators.

**Lemma 6.1. [Self-adjointness criterion in  $\mathcal{V}$ ]** *Let  $V \in \mathcal{V}$  and  $V = \tilde{V} + \tilde{W}$ , with  $\tilde{V}$  as in (43) and  $\tilde{W} \in L^{N/2}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ . Then the operator  $-\Delta - V$  is essentially self-adjoint in  $C_c^\infty(\mathbb{R}^N \setminus \{a_1, \dots, a_k\})$  if and only if  $\text{Range}(-\Delta - \tilde{V} + b)$  is dense in  $L^2(\mathbb{R}^N)$  for some  $b \in L^\infty(\mathbb{R}^N)$  with  $\text{ess inf}_{\mathbb{R}^N} b > 0$ .*

As a consequence of the above criterion, the following *non self-adjointness condition* in  $\mathcal{V}$  holds. We refer to [11] for more details.

**Corollary 6.2.** *Let  $V \in \mathcal{V}$  and  $V = \tilde{V} + \tilde{W}$ , with  $\tilde{V}$  as in (43) and  $\tilde{W} \in L^{N/2}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ . Assume that there exist  $v \in L^2(\mathbb{R}^N)$ ,  $v(x) \geq 0$  a.e. in  $\mathbb{R}^N$ ,  $\int_{\mathbb{R}^N} v^2 > 0$ , a distribution  $h \in H^{-1}(\mathbb{R}^N)$ , and  $b > 0$  such that*

$$(44) \quad H^{-1}(\mathbb{R}^N) \langle h, u \rangle_{H^1(\mathbb{R}^N)} \leq 0 \quad \text{for all } u \in H^1(\mathbb{R}^N), \quad u \geq 0 \text{ a.e. in } \mathbb{R}^N,$$

and

$$-\Delta v - \tilde{V} v + b v = h \quad \text{in } \mathcal{D}'(\mathbb{R}^N \setminus \{a_1, \dots, a_k\}).$$

Then the operator  $-\Delta - V$  is not essentially self-adjoint in  $C_c^\infty(\mathbb{R}^N \setminus \{a_1, \dots, a_k\})$ .

We now prove Theorem 1.6.)

### Proof of Theorem 1.6.

**Step 1:** if  $\mu_1(h_i) \geq -\left(\frac{N-2}{2}\right)^2 + 1$  for all  $i \in \{1, \dots, k\}$ , then  $-\Delta - V$  is essentially self-adjoint.

For  $i = 1, \dots, k$ , let  $\phi_i \in C_c^\infty(\mathbb{R}^N)$  such that  $\phi_i \equiv 1$  in  $\mathcal{U}_i$ ,  $\phi_i \equiv 0$  in  $\mathbb{R}^N \setminus B(a_i, 1)$ , and  $0 \leq \phi_i \leq 1$ , and let  $\phi_\infty \in C_c^\infty(\mathbb{R}^N)$  such that  $\text{supp } \phi_\infty \subset \mathbb{R}^N \setminus \{0\}$ ,  $\phi_\infty \equiv 1$  in  $\mathcal{U}_\infty$ , and  $0 \leq \phi_\infty \leq 1$ . Then, setting

$$\bar{V}(x) = \sum_{i=1}^k \frac{h_i\left(\frac{x-a_i}{|x-a_i|}\right)}{|x-a_i|^2} \phi_i(x) + \frac{h_\infty\left(\frac{x}{|x|}\right)}{|x|^2} \phi_\infty(x),$$

we have that  $\tilde{V} - \bar{V} \in L^\infty(\mathbb{R}^N)$ .

For  $\alpha > \max\{0, -\text{ess inf}_{\mathbb{R}^N}(\tilde{V} - \bar{V})\}$  and  $b(x) := \tilde{V} - \bar{V} + \alpha$ , there holds  $b \in L^\infty(\mathbb{R}^N)$  and  $\text{ess inf}_{\mathbb{R}^N} b > 0$ . Let us fix  $f \in C_c^\infty(\mathbb{R}^N \setminus \{a_1, \dots, a_k\})$ . Since  $C_c^\infty(\mathbb{R}^N \setminus \{a_1, \dots, a_k\})$  is dense in  $L^2(\mathbb{R}^N)$ , in view of Lemma 6.1, to prove essential self-adjointness it is enough to find some  $g \in \text{Range}(-\Delta - \tilde{V} + b) = \text{Range}(-\Delta - \bar{V} + \alpha)$  such that  $g$  is arbitrarily closed to  $f$  in  $L^2(\mathbb{R}^N)$ . To

this aim, we fix  $\varepsilon > 0$  and claim that there exist  $\tilde{h}_i \in C^\infty(\mathbb{S}^{N-1})$ ,  $i = 1, \dots, k, \infty$ , and  $\gamma_i \in \mathbb{R}$ ,  $i = 1, \dots, k$ , such that, setting

$$\widehat{V}(x) := \sum_{i=1}^k \frac{\tilde{h}_i\left(\frac{x-a_i}{|x-a_i|}\right) - \gamma_i}{|x-a_i|^2} \phi_i(x) + \phi_\infty(x) \frac{\tilde{h}_\infty\left(\frac{x}{|x|}\right)}{|x|^2},$$

there holds

$$(45) \quad \int_{\mathbb{R}^N} (|\nabla u|^2 - \widehat{V} u^2 + \alpha u^2) dx \geq \min \left\{ \frac{1}{2} \mu(\tilde{V}), \operatorname{ess\,inf}_{\mathbb{R}^N} b \right\} \|u\|_{H^1(\mathbb{R}^N)}^2, \quad \text{for all } u \in H^1(\mathbb{R}^N),$$

$$(46) \quad \mu_1(\tilde{h}_i) + \gamma_i > -\left(\frac{N-2}{2}\right)^2 + 1, \quad \text{for all } i = 1, \dots, k,$$

$$(47) \quad \|(\widehat{V} - \bar{V})u\|_{L^2(\mathbb{R}^N)} < \varepsilon,$$

for every  $u \in H^1(\mathbb{R}^N)$  solving

$$(48) \quad -\Delta u(x) - \widehat{V}(x) u(x) + \alpha u(x) = f(x).$$

In order to prove the claim, for every  $i = 1, \dots, k, \infty$  let  $h_n^i \in C^\infty(\mathbb{S}^{N-1})$  such that  $h_n^i \rightarrow h_i$  a.e. in  $\mathbb{S}^{N-1}$ , and  $|h_n^i(\theta)| \leq \|h_i\|_{L^\infty(\mathbb{S}^{N-1})}$  for a.e.  $\theta \in \mathbb{S}^{N-1}$ . We notice that the existence of such approximating sequences can be proved using convolution methods in local charts. From Lemma A.1, it follows that  $\lim_{n \rightarrow +\infty} \mu_1(h_n^i) = \mu_1(h_i)$ , while, from the Dominated Convergence Theorem,  $\lim_{n \rightarrow +\infty} \int_{\mathbb{S}^{N-1}} (h_n^i - h_i)^2 = 0$ , hence, for any  $i \in \{1, \dots, k\}$  such that  $\mu_1(h_i) = -\left(\frac{N-2}{2}\right)^2 + 1$ , there exists a sequence  $\{\delta_n^i\}_{n \in \mathbb{N}} \subset (0, \infty)$  such that

$$\delta_n^i > 0, \quad \lim_{n \rightarrow +\infty} \delta_n^i = 0, \quad \text{and } \delta_n^i > \max \left\{ 0, \left| 1 - \left(\frac{N-2}{2}\right)^2 - \mu_1(h_i) \right|, \sqrt{\int_{\mathbb{S}^{N-1}} (h_n^i - h_i)^2} \right\}.$$

Hence, setting, for  $i = 1, \dots, k$ ,

$$\gamma_n^i := \begin{cases} 0, & \text{if } \mu_1(h_i) > -\left(\frac{N-2}{2}\right)^2 + 1, \\ -\mu_1(h_n^i) - \left(\frac{N-2}{2}\right)^2 + 1 + \delta_n^i, & \text{if } \mu_1(h_i) = -\left(\frac{N-2}{2}\right)^2 + 1, \end{cases}$$

and

$$\begin{aligned} \sigma_n^i &= -\frac{N-2}{2} + \sqrt{\frac{(N-2)^2}{4} + \mu_1(h_n^i) + \gamma_n^i} \\ &= \begin{cases} -\frac{N-2}{2} + \sqrt{1 + \delta_n^i}, & \text{if } \mu_1(h_i) = -\left(\frac{N-2}{2}\right)^2 + 1, \\ -\frac{N-2}{2} + \sqrt{\frac{(N-2)^2}{4} + \mu_1(h_n^i)}, & \text{if } \mu_1(h_i) > -\left(\frac{N-2}{2}\right)^2 + 1, \end{cases} \end{aligned}$$

there holds  $\sigma_n^i + \frac{N-2}{2} > 1$  for  $n$  sufficiently large. Let us set

$$(49) \quad \tilde{V}_n(x) := \sum_{i=1}^k \frac{\tilde{h}_n^i\left(\frac{x-a_i}{|x-a_i|}\right) - \gamma_n^i}{|x-a_i|^2} \phi_i(x) + \phi_\infty(x) \frac{\tilde{h}_n^\infty\left(\frac{x}{|x|}\right)}{|x|^2}.$$

From [12, Theorem 1.2] and taking into account i-ii) of Lemma A.1, there exists a positive constant  $C$  independent on  $n$ , such that every solution  $u \in H^1(\mathbb{R}^N)$  of equation

$$(50) \quad -\Delta u(x) - \tilde{V}_n(x) u(x) + \alpha u(x) = f(x)$$

can be estimated as

$$|u(x)| \leq C |x - a_i|^{\sigma_n^i} \|u\|_{H^1(\mathbb{R}^N)} \quad \text{in } B(a_i, 1), \quad \text{for all } i = 1, \dots, k.$$

From iii) of Lemma A.1, it follows that  $\tilde{V}_n \rightarrow \bar{V}$  in  $L^{N/2, \infty}$ , hence it follows that

$$\begin{aligned} \int_{\mathbb{R}^N} (|\nabla u|^2 - \tilde{V}_n u^2 + \alpha u^2) dx &\geq \left[ \min \left\{ \mu(\tilde{V}), \operatorname{ess\,inf}_{\mathbb{R}^N} b \right\} + o(1) \right] \|u\|_{H^1(\mathbb{R}^N)}^2 \\ &\geq \frac{1}{2} \min \left\{ \mu(\tilde{V}), \operatorname{ess\,inf}_{\mathbb{R}^N} b \right\} \|u\|_{H^1(\mathbb{R}^N)}^2 \end{aligned}$$

for all  $u \in H^1(\mathbb{R}^N)$ . Moreover, by testing (50) with  $u$ , every solution  $u \in H^1(\mathbb{R}^N)$  of (50) satisfies

$$\|u\|_{H^1(\mathbb{R}^N)} \leq \frac{2\|f\|_{L^2(\mathbb{R}^N)}}{\min\{\mu(\tilde{V}), \operatorname{ess\,inf}_{\mathbb{R}^N} b\}}.$$

Then for every solution  $u \in H^1(\mathbb{R}^N)$  of (50), there holds

$$\begin{aligned} \left\| \frac{(h_n^\infty - h_\infty)\left(\frac{x}{|x|}\right)\phi_\infty u}{|x|^2} \right\|_{L^2(\mathbb{R}^N)}^2 &\leq \left( \int_{\mathbb{R}^N} |u(x)|^{2^*} dx \right)^{2/2^*} \left( \int_{\mathbb{R}^N} \frac{|h_n^\infty - h_\infty|^N \left(\frac{x}{|x|}\right)\phi_\infty^N(x)}{|x|^{2N}} dx \right)^{2/N} \\ &\leq S^{-1} \frac{4\|f\|_{L^2(\mathbb{R}^N)}^2}{(\min\{\mu(\tilde{V}), \operatorname{ess\,inf}_{\mathbb{R}^N} b\})^2} \left( \int_{\mathbb{R}^N} \frac{|h_n^\infty - h_\infty|^N \left(\frac{x}{|x|}\right)\phi_\infty^N(x)}{|x|^{2N}} dx \right)^{2/N} = o(1) \end{aligned}$$

as  $n \rightarrow +\infty$ . Furthermore for all  $i = 1, \dots, k, \infty$  and for any solution  $u \in H^1(\mathbb{R}^N)$  of (50), there holds

$$\left\| \frac{(h_n^i - h_i)\left(\frac{x-a_i}{|x-a_i|}\right)\phi_i u}{|x-a_i|^2} \right\|_{L^2(\mathbb{R}^N)}^2 \leq C^2 \|u\|_{H^1(\mathbb{R}^N)}^2 \left( \int_{\mathbb{S}^{N-1}} (h_n^i(\theta) - h_i(\theta))^2 dV(\theta) \right) \int_0^1 r^{N-5+2\sigma_n^i} dr.$$

Hence, if  $\mu_1(h_i) > -\left(\frac{N-2}{2}\right)^2 + 1$ , we have that

$$\begin{aligned} \left\| \frac{(h_n^i - h_i)\left(\frac{x-a_i}{|x-a_i|}\right)\phi_i u}{|x-a_i|^2} \right\|_{L^2(\mathbb{R}^N)}^2 &\leq C^2 \frac{4\|f\|_{L^2(\mathbb{R}^N)}^2}{(\min\{\mu(\tilde{V}), \operatorname{ess\,inf}_{\mathbb{R}^N} b\})^2} \times \\ &\quad \times \left( \int_{\mathbb{S}^{N-1}} (h_n^i(\theta) - h_i(\theta))^2 dV(\theta) \right) \left[ \frac{1}{2\left(\sqrt{\frac{(N-2)^2}{4} + \mu_1(h_i)} - 1\right)} + o(1) \right] = o(1), \end{aligned}$$

as  $n \rightarrow +\infty$ , while, if  $\mu_1(h_i) = -\left(\frac{N-2}{2}\right)^2 + 1$ , by the choice of  $\delta_n^i$ , there holds

$$\begin{aligned} \left\| \frac{(h_n^i - h_i)\left(\frac{x-a_i}{|x-a_i|}\right)\phi_i u}{|x-a_i|^2} \right\|_{L^2(\mathbb{R}^N)}^2 &\leq C^2 \|u\|_{H^1(\mathbb{R}^N)}^2 \left( \int_{\mathbb{S}^{N-1}} (h_n^i(\theta) - h_i(\theta))^2 dV(\theta) \right) \frac{\sqrt{1 + \delta_n^i} + 1}{2\delta_n^i} \\ &\leq \frac{1}{2} C^2 \frac{4\|f\|_{L^2(\mathbb{R}^N)}^2}{(\min\{\mu(\tilde{V}), \operatorname{ess\,inf}_{\mathbb{R}^N} b\})^2} \|h_n^i - h_i\|_{L^2(\mathbb{S}^{N-1})} (\sqrt{1 + \delta_n^i} + 1) = o(1) \end{aligned}$$

as  $n \rightarrow +\infty$ . Furthermore, for all  $i \in \{1, \dots, k\}$  such that  $\mu(h_i) = -\left(\frac{N-2}{2}\right)^2 + 1$ , we have that

$$\begin{aligned} \left\| \frac{\gamma_n^i \phi_i u}{|x - a_i|^2} \right\|_{L^2(\mathbb{R}^N)}^2 &\leq C^2 (\gamma_n^i)^2 \|u\|_{H^1(\mathbb{R}^N)}^2 \int_0^1 r^{N-5+2\sigma_n^i} dr \\ &\leq \frac{2C^2 \|f\|_{L^2(\mathbb{R}^N)}^2}{(\min\{\mu(\tilde{V}), \text{ess inf}_{\mathbb{R}^N} b\})^2} \frac{(\gamma_n^i)^2 (\sqrt{1 + \delta_n^i} + 1)}{\delta_n^i} = o(1) \end{aligned}$$

as  $n \rightarrow +\infty$ . Therefore, it is possible to choose  $n$  large enough in order to ensure that every solution  $u$  of (50) satisfies

$$\|(\tilde{V}_n - \bar{V})u\|_{L^2(\mathbb{R}^N)} < \varepsilon.$$

For such an  $n$ , let us set  $\hat{V} = \tilde{V}_n$ ,  $\tilde{h}_i = \tilde{h}_n^i$ ,  $i = 1, \dots, k, \infty$ , and  $\gamma_i = \gamma_n^i$ ,  $i = 1, \dots, k$ , so that conditions (45–47) are satisfied and the claim is proved.

Hence, by the Lax-Milgram Theorem, there exists  $w \in H^1(\mathbb{R}^N)$  satisfying

$$(51) \quad -\Delta w(x) - \hat{V}(x)w(x) + \alpha w(x) = f(x).$$

Since  $\hat{V}$  is smooth outside poles, by classical regularity theory  $w \in C^\infty(\mathbb{R}^N \setminus \{a_1, \dots, a_k\})$ . From [12, Theorem 1.1] we deduce the following asymptotic behavior of  $w$  at poles

$$(52) \quad w(x) \sim |x - a_i|^{\sigma_i} \psi_1^{\hat{h}_i} \left( \frac{x - a_i}{|x - a_i|} \right), \text{ as } x \rightarrow a_i,$$

where

$$\hat{h}_i := \tilde{h}_i - \gamma_i \quad \text{and} \quad \sigma_i = -\frac{N-2}{2} + \sqrt{\frac{(N-2)^2}{4} + \mu_1(\tilde{h}_i) + \gamma_i}.$$

Hence the function  $g(x) := \hat{V}(x)w(x) - \alpha w(x) + f(x)$  satisfies

$$|g(x)| \leq \text{const} \frac{\psi_1^{\hat{h}_i} \left( \frac{x - a_i}{|x - a_i|} \right)}{|x - a_i|^{2-\sigma_i}} \quad \text{in } \mathcal{U}_i \quad \text{for all } i = 1, \dots, k.$$

Since, by (46),  $\sigma_i + \frac{N-2}{2} > 1$ , it turns out that  $g \in L^2(\mathbb{R}^N)$ . Writing  $w$  by its Green's representation formula and using well known properties of differentiability of *Newtonian potentials*, the following asymptotic estimate for the gradient of  $w$  near the poles can be deduced

$$(53) \quad \nabla w(x) = \begin{cases} O(|x - a_i|^{\sigma_i - 1}), & \text{if } \mu_1(\hat{h}_i) < N - 1, \\ O(|x - a_i|^{-\tau}), & \text{if } \mu_1(\hat{h}_i) \geq N - 1, \end{cases} \quad \text{as } x \rightarrow a_i,$$

for some  $0 < \tau < \frac{N-2}{2}$ , see [11] for more details in the Hardy case.

For all  $n \in \mathbb{N}$  let  $\eta_n$  be a cut-off function such that  $\eta_n \in C_c^\infty(\mathbb{R}^N \setminus \{a_1, \dots, a_k\})$ ,  $0 \leq \eta_n \leq 1$ , and

$$\begin{aligned} \eta_n(x) &\equiv 0 \text{ in } \bigcup_{i=1}^k B\left(a_i, \frac{1}{2n}\right) \cup (\mathbb{R}^N \setminus B(0, 2n)), \quad \eta_n(x) \equiv 1 \text{ in } B(0, n) \setminus \bigcup_{i=1}^k B\left(a_i, \frac{1}{n}\right), \\ |\nabla \eta_n(x)| &\leq Cn \text{ in } \bigcup_{i=1}^k \left(B\left(a_i, \frac{1}{n}\right) \setminus B\left(a_i, \frac{1}{2n}\right)\right), \quad |\nabla \eta_n(x)| \leq \frac{C}{n} \text{ in } B(0, 2n) \setminus B(0, n), \\ |\Delta \eta_n(x)| &\leq Cn^2 \text{ in } \bigcup_{i=1}^k \left(B\left(a_i, \frac{1}{n}\right) \setminus B\left(a_i, \frac{1}{2n}\right)\right), \quad |\Delta \eta_n(x)| \leq \frac{C}{n^2} \text{ in } B(0, 2n) \setminus B(0, n), \end{aligned}$$

for some positive constant  $C$  independent of  $n$ . Setting  $f_n := \eta_n f - 2\nabla \eta_n \cdot \nabla w - w \Delta \eta_n$ , we notice that  $\eta_n f \rightarrow f$  in  $L^2(\mathbb{R}^N)$ , while (52) and (53) yield

$$\begin{aligned} &\int_{\mathbb{R}^N} |\nabla \eta_n(x)|^2 |\nabla w(x)|^2 dx \\ &\leq \text{const } n^2 \sum_{i=1}^k \int_{B(a_i, \frac{1}{n}) \setminus B(a_i, \frac{1}{2n})} |\nabla w(x)|^2 dx + \frac{\text{const}}{n^2} \int_{B(0, 2n) \setminus B(0, n)} |\nabla w(x)|^2 dx \\ &\leq \text{const } n^2 \left[ \sum_{\mu_1(\hat{h}_i) < N-1} \int_{B(0, \frac{1}{n})} |x|^{2\sigma_i-2} dx + \sum_{\mu_1(\hat{h}_i) \geq N-1} \int_{B(0, \frac{1}{n})} |x|^{-2\tau} dx \right] + \frac{\text{const}}{n^2} \|w\|_{H^1(\mathbb{R}^N)} \\ &\leq \text{const} \left[ \sum_{\mu_1(\hat{h}_i) < N-1} n^{-2\sigma_i+4-N} + \sum_{\mu_1(\hat{h}_i) \geq N-1} n^{2+2\tau-N} + n^{-2} \right] \end{aligned}$$

and

$$\begin{aligned} &\int_{\mathbb{R}^N} |\Delta \eta_n(x)|^2 |w(x)|^2 dx \\ &\leq \text{const } n^4 \sum_{i=1}^k \int_{B(a_i, \frac{1}{n}) \setminus B(a_i, \frac{1}{2n})} |w(x)|^2 dx + \frac{\text{const}}{n^4} \int_{B(0, 2n) \setminus B(0, n)} |w(x)|^2 dx \\ &\leq \text{const } n^4 \sum_{i=1}^k \int_{B(0, 1/n)} |x|^{2\sigma_i} dx + \frac{\text{const}}{n^4} \|w\|_{H^1(\mathbb{R}^N)} \leq \text{const} \left[ \sum_{i=1}^k n^{-2\sigma_i+4-N} + n^{-4} \right]. \end{aligned}$$

Since  $-2\sigma_i + 4 - N < 0$ , we conclude that  $f_n \rightarrow f$  in  $L^2(\mathbb{R}^N)$ . Hence, for  $n$  large enough,  $\|f_n - f\|_{L^2(\mathbb{R}^N)} < \varepsilon$ . The functions  $w_n := \eta_n w \in C_c^\infty(\mathbb{R}^N \setminus \{a_1, \dots, a_k\})$  solve

$$-\Delta w_n(x) - \tilde{V}(x)w_n(x) + b w_n(x) = g_n(x),$$

where  $g_n(x) := f_n(x) + (\hat{V}(x) - \bar{V}(x))w_n(x)$ , i.e.  $g_n \in \text{Range}(-\Delta - \tilde{V} + b)$ . Moreover, from (47) and  $|\eta_n| \leq 1$ , we deduce that

$$\|g_n - f_n\|_{L^2(\mathbb{R}^N)} = \|(\hat{V} - \bar{V})w_n\|_{L^2(\mathbb{R}^N)} \leq \|(\hat{V} - \bar{V})w\|_{L^2(\mathbb{R}^N)} < \varepsilon,$$

hence  $\|g_n - f\|_{L^2(\mathbb{R}^N)} < 2\varepsilon$  for large  $n$ . The proof of step 1 is thereby complete.

**Step 2:** if  $\mu(h_i) < -\left(\frac{N-2}{2}\right)^2 + 1$  for some  $i \in \{1, \dots, k\}$ , then  $-\Delta - V$  is not essentially self-adjoint.

Let  $V = \tilde{V} + \tilde{W}$ , with  $\tilde{V}$  as in (43) and  $\tilde{W} \in L^{N/2}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ . Let us fix  $i \in \{1, \dots, k\}$  such that  $\mu(h_i) < -\left(\frac{N-2}{2}\right)^2 + 1$  and  $\alpha < 0$ , and consider the solution  $\varphi \in C^1((-\infty, \ln \delta])$  of the Cauchy problem

$$\begin{cases} \varphi''(s) - \omega_{h_i}^2 \varphi(s) = b e^{2s} \varphi(s), \\ \varphi(\ln \delta) = 0, \quad \varphi'(\ln \delta) = \alpha, \end{cases}$$

where  $\omega_{h_i} := \sqrt{\left(\frac{N-2}{2}\right)^2 + \mu_1(h_i)}$  and  $\delta$  is such that  $B(a_i, \delta) \subset \mathcal{U}_i$ . From the Gronwall's inequality (see [11, Lemma A.2] for more details), we can estimate  $\varphi$  as

$$(54) \quad 0 \leq \varphi(s) \leq C e^{-\omega_{h_i} s} \quad \text{for all } s \leq \ln \delta,$$

for some positive constant  $C = C(h_i, \delta, \alpha, b)$ . Let us set

$$v(x) := \begin{cases} |x - a_i|^{-\frac{N-2}{2}} \varphi(\ln |x - a_i|) \psi_1^{h_i}\left(\frac{x - a_i}{|x - a_i|}\right), & \text{if } x \in B(a_i, \delta) \setminus \{a_i\}, \\ 0, & \text{if } x \in \mathbb{R}^N \setminus B(a_i, \delta). \end{cases}$$

From (54) we infer that

$$(55) \quad 0 \leq v(x) \leq C |x - a_i|^{-\frac{N-2}{2} - \sqrt{\left(\frac{N-2}{2}\right)^2 + \mu(h_i)}} \quad \text{in } B(a_i, \delta).$$

The assumption  $\mu(h_i) < -\left(\frac{N-2}{2}\right)^2 + 1$  and estimate (55) ensure that  $v \in L^2(\mathbb{R}^N)$ . Moreover the restriction of  $v$  to  $B(a_i, \delta)$  satisfies

$$\begin{cases} -\Delta v(x) - \frac{h_i\left(\frac{x - a_i}{|x - a_i|}\right)}{|x - a_i|^2} v(x) + b v(x) = 0, & \text{in } B(a_i, \delta), \\ v = 0 \quad \text{and} \quad \frac{\partial v}{\partial \nu} = \alpha \delta^{-\frac{N}{2}} \psi_1^{h_i}\left(\frac{x - a_i}{|x - a_i|}\right), & \text{on } \partial B(a_i, \delta). \end{cases}$$

As a consequence the distribution  $-\Delta v - \tilde{V} v + b v \in \mathcal{D}'(\mathbb{R}^N \setminus \{a_1, \dots, a_k\})$  acts as follows:

$$\mathcal{D}'(\mathbb{R}^N \setminus \{a_1, \dots, a_k\}) \langle -\Delta v - \tilde{V} v + b v, \varphi \rangle_{C_c^\infty(\mathbb{R}^N \setminus \{a_1, \dots, a_k\})} = \delta^{-\frac{N}{2}} \alpha \int_{\partial B(a_i, \delta)} \psi_1^{h_i}\left(\frac{x - a_i}{|x - a_i|}\right) \varphi(x) ds.$$

Hence  $h = -\Delta v - \tilde{V} v + b v \in H^{-1}(\mathbb{R}^N)$  and satisfies (44) as  $\alpha < 0$ . From Corollary 6.2, we finally deduce that the operator  $-\Delta - V$  is not essentially self-adjoint in  $C_c^\infty(\mathbb{R}^N \setminus \{a_1, \dots, a_k\})$ .  $\square$

The following theorem analyzes essential self-adjointness of anisotropic Schrödinger operators with potentials carrying infinitely many singularities on reticular structures.

**Theorem 6.3.** *Assume that  $\{h_n\}_{n \in \mathbb{N}} \subset C^\infty(\mathbb{S}^{N-1})$  satisfy (22) and  $\{a_n\}_n \subset \mathbb{R}^N$  satisfy (23) and  $|a_n - a_m| \geq 1$  for all  $n \neq m$ . Let  $\varepsilon > 0$ ,  $\sigma > 0$ , and  $\bar{\delta} > 0$  as in Lemma 3.8,  $0 < \delta < \min\{1/4, \bar{\delta}\}$ , and*

$$V(x) = \sum_{n=1}^{\infty} \frac{h_n\left(\frac{x - a_n}{|x - a_n|}\right)}{|x - a_n|^2} \chi_{\mathcal{E}_{0, \tilde{h}_n}^{\sigma, \delta}(a_n)}, \quad \text{where } \tilde{h}_n := (1 + \varepsilon)h_n.$$

*Then  $-\Delta - V$  is essentially self-adjoint in  $C_c^\infty(\mathbb{R}^N \setminus \{a_n\}_{n \in \mathbb{N}})$  if and only if*

$$\mu_1(h_n) \geq -\left(\frac{N-2}{2}\right)^2 + 1 \quad \text{for all } n \in \mathbb{N}.$$

PROOF. For  $n \in \mathbb{N}$ , let  $\phi_n \in C_c^\infty(\mathbb{R}^N)$  such that  $\phi_n \equiv 1$  in  $\mathcal{E}_{0, \tilde{h}_n}^{\sigma, \delta}(a_n)$ ,  $\phi_n \equiv 0$  in  $\mathbb{R}^N \setminus \mathcal{E}_{0, \tilde{h}_n}^{\sigma, 2\delta}(a_n)$ , and  $0 \leq \phi_n \leq 1$ . Then, setting

$$\bar{V}(x) = \sum_{n=1}^{\infty} \frac{h_n \left( \frac{x-a_n}{|x-a_n|} \right)}{|x-a_n|^2} \phi_n(x),$$

we have that  $\bar{V} \in C^\infty(\mathbb{R}^N \setminus \{a_n\}_{n \in \mathbb{N}})$  and  $V - \bar{V} \in L^\infty(\mathbb{R}^N)$ .

For  $\alpha > \max\{0, -\text{ess inf}_{\mathbb{R}^N}(V - \bar{V})\}$  and  $b(x) := V - \bar{V} + \alpha$ , there holds  $b \in L^\infty(\mathbb{R}^N)$  and  $\text{ess inf}_{\mathbb{R}^N} b > 0$ . From the Kato-Rellich Theorem the operator  $-\Delta - V$  is essentially self-adjoint in  $C_c^\infty(\mathbb{R}^N \setminus \{a_n\}_{n \in \mathbb{N}})$  if and only if  $-\Delta - V + b = -\Delta - \bar{V} + \alpha$  is essentially self-adjoint. In view of Lemma 3.8,  $-\Delta - V + b$  is positive. Hence essential self-adjointness is equivalent to density of  $\text{Range}(-\Delta - V + b)$  in  $L^2(\mathbb{R}^N)$ .

Let us first prove that, if  $\inf_{n \in \mathbb{N}} \mu_1(h_n) \geq -\left(\frac{N-2}{2}\right)^2 + 1$ , then  $-\Delta - V$  is essentially self-adjoint. Let  $f \in C_c^\infty(\mathbb{R}^N \setminus \{a_n\}_{n \in \mathbb{N}})$  and  $\varepsilon > 0$ . Then there exists  $0 \leq \gamma < 1$  satisfying

$$\inf_{n \in \mathbb{N}} \mu_1(h_n - \gamma) > -\left(\frac{N-2}{2}\right)^2 + 1$$

and such that if  $u \in H^1(\mathbb{R}^N)$  solves

$$(56) \quad -\Delta u(x) - V_\gamma(x)u(x) + \alpha u(x) = f(x), \quad \text{where } V_\gamma(x) := \sum_{n=1}^{\infty} \frac{h_n \left( \frac{x-a_n}{|x-a_n|} \right) - \gamma}{|x-a_n|^2} \phi_n(x),$$

then

$$(57) \quad \|(V_\gamma - \bar{V})u\|_{L^2(\mathbb{R}^N)} < \varepsilon.$$

If  $\inf_{n \in \mathbb{N}} \mu_1(h_n) > -\left(\frac{N-2}{2}\right)^2 + 1$  it is enough to choose  $\gamma = 0$ . If  $\inf_{n \in \mathbb{N}} \mu_1(h_n) = -\left(\frac{N-2}{2}\right)^2 + 1$ , from [12, Theorem 1.2], there exists a positive constant  $\tilde{C}$  independent on  $\gamma \in (0, 1)$ , such that all solutions of (56) can be estimated as

$$|u(x)| \leq \tilde{C} |x - a_n|^{-\frac{N-2}{2} + \sqrt{1+\gamma}} \|u\|_{H^1(\mathcal{E}_{0, \tilde{h}_n}^{\sigma, \delta'}(a_n))} \quad \text{in } \mathcal{E}_{0, \tilde{h}_n}^{\sigma, 2\delta}(a_n),$$

for some  $2\delta < \delta' < 1/2$  and for all  $n \in \mathbb{N}$ . We emphasize that, thanks to assumption (22), the constant  $\tilde{C}$  in the above estimate can be taken to be independent of  $n$ , as one can easily obtain by scanning through the proof of [12, Theorem 1.2] and checking the dependence of the estimate constant of the angular coefficient of the dipole. Consequently

$$\left\| \frac{\gamma \phi_n u}{|x - a_n|^2} \right\|_{L^2(\mathbb{R}^N)}^2 \leq \text{const}(\tilde{C}, N, \varepsilon, \sigma) \|u\|_{H^1(\mathcal{E}_{0, \tilde{h}_n}^{\sigma, \delta'}(a_n))}^2 \frac{\gamma^2}{\sqrt{1+\gamma}-1},$$

and hence

$$\begin{aligned} \|(V_\gamma - \bar{V})u\|_{L^2(\mathbb{R}^N)} &\leq \text{const}(\tilde{C}, N, \varepsilon, \sigma) \frac{\gamma}{\sqrt{\sqrt{1+\gamma}-1}} \|u\|_{H^1(\mathbb{R}^N)} \\ &\leq \text{const} \|f\|_{L^2(\mathbb{R}^N)} \frac{\gamma}{\sqrt{\sqrt{1+\gamma}-1}} \rightarrow 0 \quad \text{as } \gamma \rightarrow 0. \end{aligned}$$

Therefore it is possible to choose  $\gamma$  small enough in order to ensure that all solutions of (56) satisfy (57). For such a  $\gamma$ , the Lax-Milgram Theorem provides a unique  $w \in H^1(\mathbb{R}^N)$  weakly solving

$$-\Delta w(x) - V_\gamma(x)w(x) + \alpha w(x) = f(x) \quad \text{in } \mathbb{R}^N.$$

Being  $V_\gamma \in C^\infty(\mathbb{R}^N \setminus \{a_n\}_{n \in \mathbb{N}})$ , from classical elliptic regularity theory,  $w \in C^\infty(\mathbb{R}^N \setminus \{a_n\}_{n \in \mathbb{N}})$ . From [12] and arguing as in the proof of Theorem 1.6, we deduce that

$$(58) \quad w(x) \sim |x - a_n|^{\sigma_n}, \quad \text{and}$$

$$(59) \quad \nabla w(x) = \begin{cases} O(|x - a_n|^{\sigma_n - 1}), & \text{if } \mu_1(h_n) + \gamma < N - 1, \\ O(|x - a_n|^{-\tau}), & \text{if } \mu_1(h_n) + \gamma \geq N - 1, \end{cases}$$

as  $x \rightarrow a_n$ , where  $\sigma_n = -\frac{N-2}{2} + \sqrt{(\frac{N-2}{2})^2 + \mu_1(h_n) + \gamma}$  and  $0 < \tau < \frac{N-2}{2}$ . Since

$$\tilde{\sigma} := \inf_{n \in \mathbb{N}} \sigma_n > 2 - \frac{N}{2},$$

for all  $j \in \mathbb{N}$ , we can choose  $N_j \in \mathbb{N}$  such that  $N_j \rightarrow +\infty$  as  $j \rightarrow \infty$ ,  $N_j j^{4-N-2\tilde{\sigma}} \rightarrow 0$ , and  $N_j j^{2\tau-N+2} \rightarrow 0$ , and let  $R_j > 0$  such that  $R_j \rightarrow +\infty$  as  $j \rightarrow \infty$  and  $B(a_n, 1/j) \subset B(0, R_j)$  for all  $n = 1, \dots, N_j$ . Let  $\eta_j$  be a cut-off function such that  $\eta_j \in C_c^\infty(\mathbb{R}^N \setminus \{a_n\}_{n \in \mathbb{N}})$ ,  $0 \leq \eta_j \leq 1$ , and

$$\begin{aligned} \eta_j(x) &\equiv 0 \text{ in } \bigcup_{n=1}^{N_j} B\left(a_n, \frac{1}{2j}\right) \cup (\mathbb{R}^N \setminus B(0, 2R_j)), \quad \eta_j(x) \equiv 1 \text{ in } B(0, R_j) \setminus \bigcup_{n=1}^{N_j} B\left(a_n, \frac{1}{j}\right), \\ |\nabla \eta_j(x)| &\leq Cj \text{ in } \bigcup_{n=1}^{N_j} \left(B\left(a_n, \frac{1}{j}\right) \setminus B\left(a_n, \frac{1}{2j}\right)\right), \quad |\nabla \eta_j(x)| \leq \frac{C}{R_j} \text{ in } B(0, 2R_j) \setminus B(0, R_j), \\ |\Delta \eta_j(x)| &\leq Cj^2 \text{ in } \bigcup_{n=1}^{N_j} \left(B\left(a_n, \frac{1}{j}\right) \setminus B\left(a_n, \frac{1}{2j}\right)\right), \quad |\Delta \eta_j(x)| \leq \frac{C}{R_j^2} \text{ in } B(0, 2R_j) \setminus B(0, R_j), \end{aligned}$$

for some positive constant  $C$  independent of  $j$  and  $n$ . Setting  $f_j := \eta_j f - 2\nabla \eta_j \cdot \nabla w - w \Delta \eta_j$ , we have that  $\eta_j f \rightarrow f$  in  $L^2(\mathbb{R}^N)$ , and, from (58),

$$\begin{aligned} &\int_{\mathbb{R}^N} |\nabla \eta_j(x)|^2 |\nabla w(x)|^2 dx \\ &\leq \text{const } j^2 \sum_{\substack{n=1 \\ \mu_1(h_n) + \gamma < N-1}}^{N_j} \int_{B(a_n, \frac{1}{j}) \setminus B(a_n, \frac{1}{2j})} |x - a_n|^{2(\sigma_n - 1)} dx \\ &\quad + \text{const } j^2 \sum_{\substack{n=1 \\ \mu_1(h_n) + \gamma \geq N-1}}^{N_j} \int_{B(a_n, \frac{1}{j}) \setminus B(a_n, \frac{1}{2j})} |x - a_n|^{-2\tau} dx + \frac{\text{const}}{R_j^2} \int_{B(0, 2R_j) \setminus B(0, R_j)} |\nabla w(x)|^2 dx \\ &\leq \text{const} [N_j j^{4-N-2\tilde{\sigma}} + N_j j^{2\tau-N+2} + R_j^{-2} \|w\|_{H^1(\mathbb{R}^N)}^2] \end{aligned}$$

and, in a similar way,

$$\int_{\mathbb{R}^N} |\Delta \eta_j(x)|^2 |w(x)|^2 dx \leq \text{const} [N_j j^{4-N-2\bar{\sigma}} + R_j^{-4}].$$

By the choice of  $N_j$ , we deduce that  $f_j \rightarrow f$  in  $L^2(\mathbb{R}^N)$ . Hence, for  $j$  large,  $\|f_j - f\|_{L^2(\mathbb{R}^N)} < \varepsilon$ . The functions  $w_j := \eta_j w \in C_c^\infty(\mathbb{R}^N \setminus \{a_n\}_{n \in \mathbb{N}})$  solve

$$-\Delta w_j(x) - V(x)w_j(x) + b w_j(x) = g_j(x),$$

where  $g_j(x) := f_j(x) + (V_\gamma(x) - \bar{V}(x))w_j(x)$ , i.e.  $g_j \in \text{Range}(-\Delta - V + b)$ . Moreover, from (57) and  $|\eta_j| \leq 1$ , we deduce that

$$\|g_j - f_j\|_{L^2(\mathbb{R}^N)} = \|(V_\gamma - \bar{V})w_j\|_{L^2(\mathbb{R}^N)} \leq \|(V_\gamma - \bar{V})w\|_{L^2(\mathbb{R}^N)} < \varepsilon,$$

hence  $\|g_j - f\|_{L^2(\mathbb{R}^N)} < 2\varepsilon$  for large  $n$ . The density of  $\text{Range}(-\Delta - V + b)$  in  $C_c^\infty(\mathbb{R}^N \setminus \{a_n\}_{n \in \mathbb{N}})$  and consequently in  $L^2(\mathbb{R}^N)$  is proved.

The proof of non essential self-adjointness in the case  $\mu_1(h_n) < -(\frac{N-2}{2})^2 + 1$  for some  $n \in \mathbb{N}$  can be obtained just by mimicking the arguments of the proof of Theorem 1.6 and Corollary 6.2.  $\square$

**Remark 6.4.** If  $\{h_n : n \in \mathbb{N}\}$  is finite, i.e. if only a finite number of possible angular coefficients is repeated in the reticle, then the assumption of  $C^\infty$ -smoothness of  $h_n$  required in Theorem 6.3 can be removed, as one can easily check arguing by approximation as in the proof of Theorem 1.6.

#### APPENDIX

**Lemma A.1.** *Let  $\{h_n\}_{n \in \mathbb{N}} \subset L^\infty(\mathbb{S}^{N-1})$  and  $h \in L^\infty(\mathbb{S}^{N-1})$  such that*

$$h_n \rightarrow h \text{ a.e. in } \mathbb{S}^{N-1} \quad \text{and} \quad \sup_n \|h_n\|_{L^\infty(\mathbb{S}^{N-1})} < +\infty.$$

*Then*

- i)  $\lim_{n \rightarrow +\infty} \mu_1(h_n) = \mu_1(h)$ ;
- ii)  $\psi_1^{h_n}$  converge to  $\psi_1^h$  uniformly in  $\mathbb{S}^{N-1}$  and in  $H^1(\mathbb{S}^{N-1})$ ;
- iii)  $\frac{h_n(\frac{x}{|x|})}{|x|^2}$  converge to  $\frac{h(\frac{x}{|x|})}{|x|^2}$  in the Marcinkiewicz space  $L^{N/2, \infty}$ .

**PROOF.** For all  $n \in \mathbb{N}$ , let  $\psi_n = \psi_1^{h_n}$  be the positive  $L^2$ -normalized eigenfunction associated to the first eigenvalue  $\mu_1(h_n)$ , i.e.

$$(1) \quad -\Delta \psi_n - h_n \psi_n = \mu_1(h_n) \psi_n, \quad \text{in } \mathbb{S}^{N-1}, \quad \text{and} \quad \int_{\mathbb{S}^{N-1}} \psi_n^2 = 1.$$

Since  $\{h_n\}_n$  is bounded in  $L^\infty(\mathbb{S}^{N-1})$ , it is easy to verify that  $\{\mu_1(h_n)\}_n$  is bounded in  $\mathbb{R}$ , hence it admits a subsequence, still denoted as  $\{\mu_1(h_n)\}_n$ , such that  $\mu_1(h_n) \rightarrow \bar{\mu}$  as  $n \rightarrow +\infty$  for some  $\bar{\mu} \in \mathbb{R}$ . By a standard bootstrap argument, it follows that  $\{\psi_n\}_n$  is relatively compact in  $H^1(\mathbb{S}^{N-1})$  and bounded in  $C^{0, \alpha}(\mathbb{S}^{N-1})$  for some positive  $\alpha$ . In particular the sequence  $\{\psi_n\}_n$  is equicontinuous and hence, by the Ascoli-Arzelà Theorem, there exists  $\bar{\psi} \in H^1(\mathbb{S}^{N-1}) \cap C^0(\mathbb{S}^{N-1})$  such that  $\psi_n \rightarrow \bar{\psi}$  in  $H^1(\mathbb{S}^{N-1})$  and uniformly in  $\mathbb{S}^{N-1}$ .

Passing to the limit in (1), strong  $H^1$ -convergence of  $\psi_n$  to  $\psi$  and Dominated Convergence's Theorem yield that  $\bar{\psi}$  satisfies

$$-\Delta\bar{\psi} - h\bar{\psi} = \bar{\mu}\bar{\psi}, \quad \text{in } \mathbb{S}^{N-1}, \quad \text{and} \quad \int_{\mathbb{S}^{N-1}} \psi^2 = 1,$$

and therefore

$$\bar{\mu} \geq \mu_1(h).$$

On the other hand, for all  $\varphi \in H^1(\mathbb{S}^{N-1}) \setminus \{0\}$

$$\mu_1(h_n) \leq \frac{\int_{\mathbb{S}^{N-1}} |\nabla_{\mathbb{S}^{N-1}} \varphi(\theta)|^2 dV(\theta) - \int_{\mathbb{S}^{N-1}} h_n(\theta) \varphi^2(\theta) dV(\theta)}{\int_{\mathbb{S}^{N-1}} \varphi^2(\theta) dV(\theta)},$$

hence, letting  $n \rightarrow +\infty$ ,

$$\bar{\mu} \leq \frac{\int_{\mathbb{S}^{N-1}} |\nabla_{\mathbb{S}^{N-1}} \varphi(\theta)|^2 dV(\theta) - \int_{\mathbb{S}^{N-1}} h(\theta) \varphi^2(\theta) dV(\theta)}{\int_{\mathbb{S}^{N-1}} \varphi^2(\theta) dV(\theta)},$$

which implies that  $\bar{\mu} \leq \mu_1(h)$ . Then  $\mu_1(h) = \bar{\mu}$  and  $\bar{\psi} = \psi_1^h$ . Statements i–ii) follow from above and the Uryson property.

A direct calculation shows that

$$\left\| \frac{(h_n - h)\left(\frac{x}{|x|}\right)}{|x|^2} \right\|_{L^{N/2, \infty}(\mathbb{R}^N)} = N^{-2/N} \|h_n - h\|_{L^{N/2}(\mathbb{S}^{N-1})}$$

and hence statement iii) follows from the assumption on  $\{h_n\}_n$  and the Dominated Convergence Theorem.  $\square$

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