ON ASYMPTOTIC BEHAVIOR OF SOLUTIONS TO SCHRÖDINGER EQUATIONS WITH SINGULAR DIPOLE–TYPE POTENTIALS

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Abstract. Asymptotics of solutions to Schrödinger equations with singular dipole-type potentials is investigated. We evaluate the exact behavior near the singularity of solutions to elliptic equations with potentials which are purely angular multiples of radial inverse-square functions. Both the linear and the semilinear (critical and subcritical) cases are considered.

Dedicated to Prof. Norman Dancer on the occasion of his 60th birthday.

1. Introduction and statement of the main results

In nonrelativistic molecular physics, the interaction between an electric charge and the dipole moment $\mathbf{D} \in \mathbb{R}^N$ of a molecule is described by an inverse square potential with an anisotropic coupling strength. In particular the Schrödinger equation for the wave function of an electron interacting with a polar molecule (supposed to be point-like) can be written as

$$\Big(-\frac{\hbar}{2m}\,\Delta + e\,\frac{x\cdot\mathbf{D}}{|x|^3} - E\Big)\Psi = 0,$$

where e and m denote respectively the charge and the mass of the electron and $\mathbf{D}$ is the dipole moment of the molecule, see [11].

We aim to describe the asymptotic behavior near the singularity of solutions to equations associated to dipole-type Schrödinger operators of the form

$$L_{\lambda,\mathbf{d}} := -\Delta - \frac{\lambda\,(x\cdot\mathbf{d})}{|x|^3}$$

in $\mathbb{R}^N$, where $N \geqslant 3$, $\lambda = \frac{2me|\mathbf{D}|}{\hbar}$, being $|\mathbf{D}|$ the magnitude of the dipole moment $\mathbf{D}$, and $\mathbf{d} = \mathbf{D}/|\mathbf{D}|$ denotes the orientation of $\mathbf{D}$. A precise estimate of such a behavior is indeed an important tool in establishing fundamental properties of Schrödinger operators, such as positivity, essential self-adjointness, and spectral properties, see e.g. [7].

We emphasize that, from the mathematical point of view, potentials of the form $\frac{\lambda\,(x\cdot\mathbf{d})}{|x|^3}$ have the same order of homogeneity as inverse square potentials and consequently share many features with them, such as invariance by scaling and Kelvin transform, as well as no inclusion in the

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Kato class. We mention that Schrödinger equations with Hardy-type singular potentials have been largely studied, see e.g. [1, 6, 9, 10, 16, 17] and references therein.

More precisely, in this paper we deal with a more general class of Schrödinger operators including those with dipole-type potentials, namely with operators whose potentials are purely angular multiples of radial inverse-square potentials:

$$\mathcal{L}_a := -\Delta - \frac{a(x/|x|)}{|x|^2}$$

in $\mathbb{R}^N$, where $N \geqslant 3$ and $a \in L^\infty(\mathbb{S}^{N-1})$.

The problem of establishing the asymptotic behavior of solutions to elliptic equations near an isolated singular point has been studied by several authors in a variety of contexts, see e.g. [13] for Fuchsian type elliptic operators and [5] for Fuchsian type weighted operators. The asymptotics we derive in this work is not contained in the aforementioned papers, which prove the existence of the limit at the singularity of any quotient of two positive solutions in some linear and semilinear cases which, however, do not not include the perturbed linear case and the critical nonlinear case treated here. Moreover, besides proving the existence of such a limit, we also obtain a Cauchy type representation formula for it, see (4) and (8).

We also quote [12], where asymptotics at infinity is established for perturbed inverse square potentials and in some particular nonradial case. Hölder continuity results for degenerate elliptic equations with singular weights and asymptotic analysis of the behavior of solutions near the pole are contained in [8].

As a natural setting to study the properties of operators $\mathcal{L}_a$, we introduce the functional space $\mathcal{D}^{1,2}(\mathbb{R}^N)$ defined as the completion of $C^\infty_{\rm c}(\mathbb{R}^N)$ with respect to the Dirichlet norm

$$\|u\|_{\mathcal{D}^{1,2}(\mathbb{R}^N)} := \bigg(\int_{\mathbb{R}^N} |\nabla u(x)|^2\, dx\bigg)^{1/2}.$$

In order to discuss the positivity properties of the Schrödinger operator $\mathcal{L}_a$ in $\mathcal{D}^{1,2}(\mathbb{R}^N)$, we consider the best constant in the associated Hardy-type inequality

$$\Lambda_N(a) := \sup_{u\in\mathcal{D}^{1,2}(\mathbb{R}^N)\setminus\{0\}} \frac{\displaystyle\int_{\mathbb{R}^N} |x|^{-2} a(x/|x|)\, u^2(x)\, dx}{\displaystyle\int_{\mathbb{R}^N} |\nabla u(x)|^2\, dx}. \tag{1}$$

By the classical Hardy inequality, $\Lambda_N(a) \leqslant \frac{4}{(N-2)^2} \operatorname{ess\,sup}_{\mathbb{S}^{N-1}} a$, where $\operatorname{ess\,sup}_{\mathbb{S}^{N-1}} a$ denotes the essential supremum of a in $\mathbb{S}^{N-1}$. We also notice that, in the dipole case, by rotation invariance, $\Lambda_N\big(\frac{\lambda\, x}{|x|}\cdot \mathbf{d}\big)$ does not depend on $\mathbf{d}$.

It is easy to verify that the quadratic form associated to $\mathcal{L}_a$ is positive definite in $\mathcal{D}^{1,2}(\mathbb{R}^N)$ if and only if $\Lambda_N(a) < 1$. The relation between the value $\Lambda_N(a)$ and the first eigenvalue of the angular component of the operator on the sphere $\mathbb{S}^{N-1}$ is discussed in section 2, see Lemmas 2.4 and 2.5. More precisely, Lemma 2.5 ensures that the quadratic form associated to $\mathcal{L}_a$ is positive definite if and only if

$$\mu_1 > -\bigg(\frac{N-2}{2}\bigg)^2,$$

where $\mu_1 = \mu_1(a, N)$ is the first eigenvalue of the operator $-\Delta_{\mathbb{S}^{N-1}} - a(\theta)$ on $\mathbb{S}^{N-1}$ (see Lemma 2.1). We denote by ψ_1 the associated positive L^2-normalized eigenfunction and set

$$\sigma = \sigma(a, N) := -\frac{N-2}{2} + \sqrt{\left(\frac{N-2}{2}\right)^2 + \mu_1}. \tag{2}$$

In the spirit of the well-known Riemann removable singularity theorem, we describe the behavior of solutions to linear Schrödinger equations with a dipole-type singularity localized in a neighborhood of 0.

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^N$ be a bounded open set such that $0 \in \Omega$ and let $q \in L^\infty_{\rm loc}(\Omega \setminus \{0\})$ be such that $q(x) = O(|x|^{-(2-\varepsilon)})$ as $|x| \to 0$ for some $\varepsilon > 0$. Assume that $a \in L^\infty(\mathbb{S}^{N-1})$ satisfies $\Lambda_N(a) < 1$ and $u \in H^1(\Omega)$, $u \geqslant 0$ a.e. in Ω, $u \not\equiv 0$, weakly solves*

$$-\Delta u(x) - \frac{a(x/|x|)}{|x|^2}\, u(x) = q(x)\, u(x) \quad \text{in } \Omega, \tag{3}$$

i.e.

$$\int_\Omega \nabla u(x) \cdot \nabla w(x)\, dx - \int_\Omega \frac{a(x/|x|)}{|x|^2}\, u(x) w(x)\, dx = \int_\Omega q(x) u(x) w(x)\, dx, \quad \text{for all } w \in H^1_0(\Omega).$$

Then the function

$$x \mapsto \frac{u(x)}{|x|^\sigma \psi_1(x/|x|)}$$

is continuous in Ω and

$$\lim_{|x|\to 0} \frac{u(x)}{|x|^\sigma \psi_1\big(\frac{x}{|x|}\big)} = \int_{\mathbb{S}^{N-1}} \bigg(R^{-\sigma} u(R\theta) + \int_0^R \frac{s^{1-\sigma}}{2\sigma + N - 2}\, q(s\,\theta) u(s\,\theta)\, ds - R^{-2\sigma-N+2} \int_0^R \frac{s^{N-1+\sigma}}{2\sigma+N-2}\, q(s\,\theta) u(s\,\theta)\, ds \bigg) \psi_1(\theta)\, dV(\theta), \tag{4}$$

for all $R > 0$ such that $\overline{B(0,R)} := \{x \in \mathbb{R}^N : |x| \leqslant R\} \subset \Omega$.

We notice that (4) is actually a *Cauchy's integral type formula* for u. Moreover the term at the right hand side is independent of R, see also Remark 4.2. In the case in which the perturbation q is radial then an analogous formula holds also for changing sign solutions to (3), see Remark 4.3.

If the perturbing potential q satisfies some proper summability condition, instead of the stronger control on the blow-up rate at the singularity required in Theorem 1.1, a Brezis-Kato type argument, see [2], allows us to derive an upper estimate on the behavior of solutions. For any $q \geqslant 1$, we denote as $L^q(\varphi^{2^*}, \Omega)$ the weighted L^q-space endowed with the norm

$$\|u\|_{L^q(\varphi^{2^*},\Omega)} := \bigg(\int_\Omega \varphi^{2^*}(x) |u(x)|^s\, dx \bigg)^{1/q},$$

where $2^* = \frac{2N}{N-2}$ is the critical Sobolev exponent and and φ denotes the weight function

$$\varphi(x) := |x|^\sigma \psi_1(x/|x|). \tag{5}$$

The following Brezis-Kato type result holds.

Theorem 1.2. *Let $\Omega \subset \mathbb{R}^N$ be a bounded open set such that $0 \in \Omega$, $a \in L^\infty(\mathbb{S}^{N-1})$ satisfying $\Lambda_N(a) < 1$, and $V \in L^s(\varphi^{2^*}, \Omega)$ for some $s > N/2$. Then, for any $\Omega' \Subset \Omega$, there exists a positive constant*

$$C = C\big(N, a, \|V\|_{L^s(\varphi^{2^*},\Omega)}, \operatorname{dist}(\Omega', \partial\Omega), \operatorname{diam}\Omega\big)$$

depending only on N, a, $\|V\|_{L^s(\varphi^{2^*},\Omega)}$, $\operatorname{dist}(\Omega', \partial\Omega)$, and $\operatorname{diam}\Omega$, such that for any weak $H^1(\Omega)$-solution u of

$$-\Delta u(x) - \frac{a(x/|x|)}{|x|^2}\, u(x) = \varphi^{2^*-2}(x) V(x) u(x), \quad \text{in } \Omega, \tag{6}$$

i.e. for any $u \in H^1(\Omega)$ satisfying

$$\int_\Omega \nabla u(x) \cdot \nabla w(x)\, dx - \int_\Omega \frac{a(x/|x|)}{|x|^2}\, u(x) w(x)\, dx = \int_\Omega \varphi^{2^*-2}(x) V(x) u(x) w(x)\, dx,$$

for all $w \in H^1_0(\Omega)$, there holds $\frac{u}{\varphi} \in L^\infty(\Omega')$ and

$$\left\| \frac{u}{\varphi} \right\|_{L^\infty(\Omega')} \leqslant C\, \|u\|_{L^{2^*}(\Omega)}.$$

The Brezis-Kato procedure can be applied also to semilinear problems with at most critical growth, thus providing an upper bound for solutions and then reducing the semilinear problem to a linear one with enough control on the potential at the singularity to apply Theorem 1.1 and to recover the exact asymptotic behavior.

Theorem 1.3. *Let Ω be a bounded open set containing 0, $a \in L^\infty(\mathbb{S}^{N-1})$ satisfying $\Lambda_N(a) < 1$, and $f : \Omega \times \mathbb{R} \to \mathbb{R}$ such that, for some positive constant C,*

$$\left| \frac{f(x,u)}{u} \right| \leqslant C\left(1 + |u|^{2^*-2}\right) \quad \text{for a.e. } (x,u) \in \Omega \times \mathbb{R}.$$

Assume that $u \in H^1(\Omega)$, $u \geqslant 0$ a.e. in Ω, $u \not\equiv 0$, weakly solves

$$-\Delta u(x) - \frac{a(x/|x|)}{|x|^2}\, u(x) = f(x, u(x)), \quad \text{in } \Omega, \tag{7}$$

i.e.

$$\int_\Omega \nabla u(x) \cdot \nabla w(x)\, dx - \int_\Omega \frac{a(x/|x|)}{|x|^2}\, u(x) w(x)\, dx = \int_\Omega f(x, u(x)) w(x)\, dx, \quad \text{for all } w \in H^1_0(\Omega).$$

Then the function

$$x \mapsto \frac{u(x)}{|x|^\sigma \psi_1(x/|x|)}$$

is continuous in Ω and

$$\lim_{|x|\to 0} \frac{u(x)}{|x|^\sigma \psi_1\big(\frac{x}{|x|}\big)} = \int_{\mathbb{S}^{N-1}} \bigg(r^{-\sigma} u(r\theta) + \int_0^r \frac{s^{1-\sigma}}{2\sigma + N - 2}\, f\big(s\,\theta, u(s\,\theta)\big)\, ds - r^{-2\sigma - N + 2} \int_0^r \frac{s^{N-1+\sigma}}{2\sigma + N - 2}\, f\big(s\,\theta, u(s\,\theta)\big)\, ds \bigg) \psi_1(\theta)\, dV(\theta), \tag{8}$$

for all $r > 0$ such that $\overline{B(0,r)} := \{x \in \mathbb{R}^N : |x| \leqslant r\} \subset \Omega$.

Notation. We list below some notation used throughout the paper.

- $B(a,r)$ denotes the ball $\{x \in \mathbb{R}^N : |x-a| < r\}$ in $\mathbb{R}^N$ with center at a and radius r.
- dV denotes the volume element on the sphere $\mathbb{S}^{N-1}$.
- ω_N denotes the volume of the unit sphere $\mathbb{S}^{N-1}$, i.e. $\omega_N = \int_{\mathbb{S}^{N-1}} dV(\theta)$.
- for any $a \in L^1(\mathbb{S}^{N-1})$, $⨍_{\mathbb{S}^{N-1}} a(\theta)\, dV(\theta)$ denotes the mean of a on $\mathbb{S}^{N-1}$, i.e.

$$⨍_{\mathbb{S}^{N-1}} a(\theta)\, dV(\theta) = \frac{1}{\omega_N} \int_{\mathbb{S}^{N-1}} a(\theta)\, dV(\theta).$$

- the symbol ess sup stands for essential supremum.

2. Spectrum of the angular component

Due to the structure of the dipole-type potential of equation (3), a natural approach to describe the solutions seems to be the separation of variables. To employ such a technique, we need, as a starting point, the description of the spectrum of the angular part of dipole Schrödinger operators.

Lemma 2.1. *Let $a \in L^\infty(\mathbb{S}^{N-1})$. Then the operator $-\Delta_{\mathbb{S}^{N-1}} - a(\theta)$ on $\mathbb{S}^{N-1}$ admits a diverging sequence of eigenvalues $\mu_1 < \mu_2 \leqslant \cdots \leqslant \mu_k < \cdots$ the first of which has the following properties:*

(i) *μ_1 is simple;*

(ii) *μ_1 can be characterized as*

$$\mu_1 = \min_{\psi \in H^1(\mathbb{S}^{N-1})\setminus\{0\}} \frac{\int_{\mathbb{S}^{N-1}} |\nabla_{\mathbb{S}^{N-1}}\psi(\theta)|^2\, dV(\theta) - \int_{\mathbb{S}^{N-1}} a(\theta)\psi^2(\theta)\, dV(\theta)}{\int_{\mathbb{S}^{N-1}} \psi^2(\theta)\, dV(\theta)}; \tag{9}$$

(iii) *μ_1 is attained by a smooth positive eigenfunction ψ_1 such that $\min_{\mathbb{S}^{N-1}} \psi_1 > 0$;*

(iv) *if, for some $\kappa \in \mathbb{R}$, $a(\theta) = \kappa$ for a.e. $\theta \in \mathbb{S}^{N-1}$, then $\mu_1 = -\kappa$.*

(v) *if a is not constant, then $-\operatorname{ess\,sup}_{\mathbb{S}^{N-1}} a < \mu_1 < -⨍_{\mathbb{S}^{N-1}} a(\theta)\, dV(\theta)$.*

Proof. We prove assertion (v), being (i), (ii), (iii), and (iv) quite standard.

Since the function $\psi \equiv 1$ satisfies

$$\int_{\mathbb{S}^{N-1}} |\nabla_{\mathbb{S}^{N-1}}\psi(\theta)|^2\, dV(\theta) - \int_{\mathbb{S}^{N-1}} a(\theta)\psi^2(\theta)\, dV(\theta) = -\int_{\mathbb{S}^{N-1}} a(\theta)\, dV(\theta),$$

we deduce that $\mu_1 \leqslant -⨍_{\mathbb{S}^{N-1}} a(\theta)\, dV(\theta)$. In order to prove the strict inequality, we argue by contradiction and assume that $\mu_1 = -⨍_{\mathbb{S}^{N-1}} a(\theta)\, dV(\theta)$. Then $\psi_1 \equiv 1$ attains the minimum value in (9) but, since a is not constant, it does not satisfy equation $-\Delta_{\mathbb{S}^{N-1}}\psi_1 - a(\theta)\psi_1 = 0$, a contradiction. We can thereby conclude that $\mu_1 < -⨍_{\mathbb{S}^{N-1}} a(\theta)\, dV(\theta)$.

From (9), [17, Lemma 1.1], and the optimality of the best constant in Hardy's inequality, it follows that

$$\mu_1 > \inf_{\psi \in H^1(\mathbb{S}^{N-1})\setminus\{0\}} \frac{\int_{\mathbb{S}^{N-1}} |\nabla_{\mathbb{S}^{N-1}}\psi(\theta)|^2\, dV(\theta) - \operatorname{ess\,sup}_{\mathbb{S}^{N-1}} a \int_{\mathbb{S}^{N-1}} \psi^2(\theta)\, dV(\theta)}{\int_{\mathbb{S}^{N-1}} \psi^2(\theta)\, dV(\theta)}$$

$$= \inf_{u \in \mathcal{D}^{1,2}(\mathbb{R}^N)\setminus\{0\}} \frac{\int_{\mathbb{R}^N} |\nabla u(x)|^2\, dx - \Big(\big(\frac{N-2}{2}\big)^2 + \operatorname{ess\,sup}_{\mathbb{S}^{N-1}} a\Big) \int_{\mathbb{R}^N} \frac{u^2(x)}{|x|^2}\, dx}{\int_{\mathbb{R}^N} \frac{u^2(x)}{|x|^2}\, dx} = -\operatorname*{ess\,sup}_{\mathbb{S}^{N-1}} a,$$

thus proving the left part of the inequality stated in (v). □

The asymptotic behavior of eigenvalues μ_k as $k \to +\infty$ is described by Weyl's law, which is recalled in the theorem below. We refer to [14, 15] for a proof.

Theorem 2.2 (Weyl's law). *For $a \in L^\infty(\mathbb{S}^{N-1})$, let $\{\mu_k\}_{k\geqslant 1}$ be the eigenvalues of the operator $-\Delta_{\mathbb{S}^{N-1}} - a(\theta)$ on $\mathbb{S}^{N-1}$. Then*

$$\mu_k = C(N,a) k^{2/(N-1)}\big(1+o(1)\big) \quad \text{as } k \to +\infty, \tag{10}$$

for some positive constant $C(N,a)$ depending only on N and a.

The following lemma provides an estimate of the L^∞-norm of eigenfunctions of the operator $-\Delta_{\mathbb{S}^{N-1}} - a(\theta)$ in terms of the corresponding eigenvalues.

Lemma 2.3. *For $a \in L^\infty(\mathbb{S}^{N-1})$ and $k \in \mathbb{N}\setminus\{0\}$, let ψ_k be a L^2-normalized eigenfunction of the Schrödinger operator $-\Delta_{\mathbb{S}^{N-1}} - a(\theta)$ on the sphere associated to the k-th eigenvalue μ_k, i.e.*

$$\begin{cases} -\Delta_{\mathbb{S}^{N-1}}\psi_k(\theta) - a(\theta)\,\psi_k(\theta) = \mu_k\,\psi_k(\theta), & \text{in } \mathbb{S}^{N-1}, \\ \int_{\mathbb{S}^{N-1}} |\psi_k(\theta)|^2\, dV(\theta) = 1. \end{cases} \tag{11}$$

Then, there exists a constant C_1 depending only on N and a such that

$$|\psi_k(\theta)| \leqslant C_1\,|\mu_k|^{\lfloor (N-1)/4\rfloor + 1},$$

where $\lfloor\cdot\rfloor$ denotes the floor function, i.e. $\lfloor x\rfloor := \min\{j \in \mathbb{Z} : \ j \leqslant x\}$.

Proof. Using classical elliptic regularity theory and bootstrap methods, we can easily prove that for any $j \in \mathbb{N}$ there exists a constant $C(N,j)$, depending only on j and N, such that

$$\|\psi_k\|_{W^{2,\frac{2(N-1)}{(N-1)-4(j-1)}}(\mathbb{S}^{N-1})} \leqslant C(N,j)\big(\mu_k + \|a\|_{L^\infty(\mathbb{S}^{N-1})}\big)^j.$$

Choosing $j = \big\lfloor \frac{N-1}{4}\big\rfloor + 1$, by Sobolev's inclusions we deduce that

$$W^{2,\frac{2(N-1)}{(N-1)-4(j-1)}}(\mathbb{S}^{N-1}) \hookrightarrow C^{0,\alpha}(\mathbb{S}^{N-1}) \hookrightarrow L^\infty(\mathbb{S}^{N-1}),$$

for any $0 < \alpha < 2\big(1 - \frac{N-1}{4} + \big\lfloor\frac{N-1}{4}\big\rfloor\big)$, thus implying the required estimate. □

Arguing as in the proof of [17, Lemma 1.1], we can deduce the following characterization of $\Lambda_N(a)$.

Lemma 2.4. *For $a \in L^\infty(\mathbb{S}^{N-1})$, let $\Lambda_N(a)$ be defined in (1). Then*

$$\Lambda_N(a) := \max_{\psi \in H^1(\mathbb{S}^{N-1})\setminus\{0\}} \frac{\int_{\mathbb{S}^{N-1}} a(\theta)\,\psi^2(\theta)\,dV(\theta)}{\int_{\mathbb{S}^{N-1}} |\nabla_{\mathbb{S}^{N-1}}\psi(\theta)|^2\,dV(\theta) + \big(\frac{N-2}{2}\big)^2\int_{\mathbb{S}^{N-1}}\psi^2(\theta)\,dV(\theta)}. \tag{12}$$

We notice that the supremum in (12) is achieved due to the compactness of the embedding $H^1(\mathbb{S}^{N-1}) \hookrightarrow L^2(\mathbb{S}^{N-1})$. As a direct consequence of the above lemma, it is possible to compare $\Lambda_N(a)$ with the best constant in Hardy's inequality

$$\frac{4}{(N-2)^2} = \sup_{u \in \mathcal{D}^{1,2}(\mathbb{R}^N)\setminus\{0\}} \frac{\int_{\mathbb{R}^N} u^2(x)|x|^{-2}\,dx}{\int_{\mathbb{R}^N} |\nabla u(x)|^2\,dx}.$$

Indeed, if a is not constant, there holds

$$\frac{4}{(N-2)^2} \fint_{\mathbb{S}^{N-1}} a(\theta)\,dV(\theta) < \Lambda_N(a) < \operatorname*{ess\,sup}_{\mathbb{S}^{N-1}} a\,\frac{4}{(N-2)^2},$$

whereas, if $a(\theta) = \kappa$ for a.e. $\theta \in \mathbb{S}^{N-1}$ and for some $\kappa \in \mathbb{R}$, then $\Lambda_N(a) = 4\kappa/(N-2)^2$.

Let us consider the quadratic form associated to the Schrödinger operator $\mathcal{L}_a$, i.e.

$$Q_a(u) := \int_{\mathbb{R}^N} |\nabla u(x)|^2 dx - \int_{\mathbb{R}^N} \frac{a(x/|x|)\, u^2(x)}{|x|^2}\, dx.$$

The problem of positivity of Q_a is solved in the following lemma.

Lemma 2.5. *Let $a \in L^\infty(\mathbb{S}^{N-1})$. The following conditions are equivalent:*

i) Q_a *is positive definite, i.e.* $\displaystyle\inf_{u \in \mathcal{D}^{1,2}(\mathbb{R}^N)\setminus\{0\}} \frac{Q_a(u)}{\int_{\mathbb{R}^N} |\nabla u(x)|^2\, dx} > 0$;

ii) $\Lambda_N(a) < 1$;

iii) $\mu_1 > -\left(\frac{N-2}{2}\right)^2$ *where μ_1 is defined in (9).*

Proof. The equivalence between i) and ii) follows from the definition of $\Lambda_N(a)$, see (1). On the other hand, [17, Proposition 1.3 and Lemma 1.1] ensure that i) is equivalent to iii). □

Remark 2.6. *We notice that in the case of dipole potentials, namely if $a(\theta) = \lambda\, \frac{x \cdot \mathbf{d}}{|x|}$, then, rewriting (12) in spherical coordinates and exploiting the symmetry with respect to the dipole axis, $\Lambda_N(a)$ can be characterized as*

$$\Lambda_N\Big(\lambda\, \frac{x \cdot \mathbf{d}}{|x|}\Big) = \lambda \sup_{w \in H_0^1(0,\pi)} \frac{\int_0^\pi \cos\theta\, w^2(\theta)\, d\theta}{\int_0^\pi \Big[|w'(\theta)|^2 + \frac{(N-2)(N-4)}{4} (\sin\theta)^{-2} w^2(\theta)\Big]\, d\theta}.$$

In dimension $N = 3$, a Taylor's expansion of $\Lambda_N\big(\lambda\, \frac{x \cdot \mathbf{d}}{|x|}\big)$ near $\lambda = 0$ can be found in [11].

An approximation of $\Lambda_N\big(\lambda\, \frac{x \cdot \mathbf{d}}{|x|}\big)$ can be performed numerically with a finite difference method, see table 1.

3. A Brezis-Kato type lemma

In this section, we follow the procedure developed by Brezis and Kato in [2] to control from above the behavior of solutions to Schrödinger equations with dipole type potentials, in order to prove Theorem 1.2.

Let us consider the weight φ introduced in (5) and define the weighted H^1-space $H^1_\varphi(\Omega)$ as the closure of $C^\infty(\bar\Omega)$ with respect to

$$\|u\|^2_{H^1_\varphi(\Omega)} := \int_\Omega \varphi^2(x) \left(|\nabla u|^2 + |u|^2\right)\, dx,$$

and the space $\mathcal{D}^{1,2}_\varphi(\Omega)$ as the closure of $C^\infty_{\rm c}(\Omega)$ with respect to

$$\|u\|^2_{\mathcal{D}^{1,2}_\varphi(\Omega)} := \int_\Omega \varphi^2(x) |\nabla u|^2\, dx.$$

N	$\big(\Lambda_N(1)\big)^{-1}=\frac{(N-2)^2}{4}$	$\Big[\Lambda_N\big(\frac{x\cdot\mathbf{d}}{\lvert x\rvert}\big)\Big]^{-1}$
3	0.25	1.6398
4	1	3.7891
5	2.25	7.5831
6	4	12.6713
7	6.25	19.0569
8	9	26.7407
9	12.25	35.7231
10	16	46.0044
11	20.25	57.5845

N	$\big(\Lambda_N(1)\big)^{-1}=\frac{(N-2)^2}{4}$	$\Big[\Lambda_N\big(\frac{x\cdot\mathbf{d}}{\lvert x\rvert}\big)\Big]^{-1}$
12	25	70.4636
13	30.25	84.6417
14	36	100.1187
15	42.25	116.8948
16	49	134.9698
17	56.25	154.3439
18	64	175.017
19	72.25	196.9891
20	81	220.2603

TABLE 1. Some numerical approximations of $\Lambda_N\big(\frac{x\cdot\mathbf{d}}{|x|}\big)$ obtained by finite difference with 10000 steps.

By the Caffarelli-Kohn-Nirenberg inequality (see [3] and [4]) and the definition of φ, it follows that, for any $w\in\mathcal{D}^{1,2}_{\varphi}(\Omega)$,

$$\left(\int_\Omega \varphi^{2^*}(x)|w(x)|^{2^*}\,dx\right)^{2/2^*}\leqslant \mathcal{C}_{N,a}\int_\Omega \varphi^2(x)|\nabla w|^2\,dx, \tag{13}$$

for some positive constant $\mathcal{C}_{N,a}$ depending only on N and a.

Lemma 3.1. *Let Ω be a bounded open set containing 0 and $v\in H^1_\varphi(\Omega)\cap L^q(\varphi^{2^*},\Omega)$, $q>1$, be a weak solution to*

$$-\operatorname{div}(\varphi^2(x)\nabla v(x))=\varphi^{2^*}(x)V(x)v(x),\quad \text{in } \Omega, \tag{14}$$

where $V\in L^s(\varphi^{2^*},\Omega)$ for some $s>\frac{N}{2}$. Then, there exists a positive constant $\widetilde{C}=\widetilde{C}(a,N)$ depending only on a and N, such that for any $\Omega'\Subset\Omega$, $v\in L^{\frac{2^*q}{2}}(\varphi^{2^*},\Omega')$ and

$$\|v\|_{L^{\frac{2^*q}{2}}(\varphi^{2^*},\Omega')}\leqslant \widetilde{C}^{\frac1q}\big(\operatorname{diam}\Omega\big)^{\frac{\sigma(2-2^*)}{q}}\times \tag{15}$$
$$\times\left(\frac{8}{C(q)}\frac{4}{(\operatorname{dist}(\Omega',\partial\Omega))^2}+\frac{4(q+2)}{(\operatorname{dist}(\Omega',\partial\Omega))^2}+\frac{2\ell_q}{C(q)}\right)^{\frac1q}\|v\|_{L^q(\varphi^{2^*},\Omega)},$$

where $C(q):=\min\big\{\frac14,\frac{4}{q+4}\big\}$ and

$$\ell_q=\left[\max\left\{8\,\mathcal{C}_{N,a}\|V\|^{2s/N}_{L^s(\varphi^{2^*},\Omega)},\frac{q+4}{2}\mathcal{C}_{N,a}\|V\|^{2s/N}_{L^s(\varphi^{2^*},\Omega)}\right\}\right]^{\frac{N}{2s-N}}.$$

Proof. Hölder's inequality and (13) yield for any $w \in \mathcal{D}^{1,2}_{\varphi}(\Omega)$

$$\begin{aligned}(16)\qquad \int_\Omega \varphi^{2^*}(x)|V(x)|w^2(x)\,dx &\leqslant \ell_q \int\limits_{|V(x)|\leqslant \ell_q} \varphi^{2^*}(x)w^2(x)\,dx + \int\limits_{|V(x)|\geqslant \ell_q} \varphi^{\frac{4}{N-2}}(x)|V(x)|\varphi^2(x)w^2(x)\,dx\\
&\leqslant \ell_q \int_\Omega \varphi^{2^*}(x)w^2(x)\,dx + \bigg(\int_\Omega \varphi^{2^*}(x)w^{2^*}(x)\,dx\bigg)^{\frac{2}{2^*}} \bigg(\int\limits_{|V(x)|\geqslant \ell_q} \varphi^{2^*}(x)|V(x)|^{\frac{N}{2}}\,dx\bigg)^{\frac{2}{N}}\\
&\leqslant \ell_q \int_\Omega \varphi^{2^*}(x)w^2(x)\,dx + C_{N,a}\bigg(\int_\Omega \varphi^2(x)|\nabla w|^2\,dx\bigg)\bigg(\int\limits_{|V(x)|\geqslant \ell_q} \varphi^{2^*}(x)|V(x)|^{\frac{N}{2}}\,dx\bigg)^{\frac{2}{N}}.\end{aligned}$$

By Hölder's inequality and by the choice of ℓ_q it follows that

$$\begin{aligned}(17)\qquad \int\limits_{|V(x)|\geqslant \ell_q} \varphi^{2^*}(x)|V(x)|^{\frac{N}{2}}\,dx &\leqslant \bigg(\int_\Omega \varphi^{2^*}(x)|V(x)|^s\,dx\bigg)^{\frac{N}{2s}} \bigg(\int\limits_{|V(x)|\geqslant \ell_q} \varphi^{2^*}(x)\,dx\bigg)^{\frac{2s-N}{2s}}\\
&\leqslant \bigg(\int_\Omega \varphi^{2^*}(x)|V(x)|^s\,dx\bigg)^{\frac{N}{2s}} \bigg(\int\limits_{|V(x)|\geqslant \ell_q} \bigg(\frac{|V(x)|}{\ell_q}\bigg)^s \varphi^{2^*}(x)\,dx\bigg)^{\frac{2s-N}{2s}}\\
&\leqslant \|V\|^s_{L^s(\varphi^{2^*},\Omega)}\,\ell_q^{-s+\frac{N}{2}} \leqslant \min\left\{\frac{C_{N,a}^{-1}}{8}, \frac{2C_{N,a}^{-1}}{q+4}\right\}^{\frac{N}{2}},\end{aligned}$$

and hence from (16) we obtain that for any $w \in \mathcal{D}^{1,2}_{\varphi}(\Omega)$

$$(18)\qquad \int_\Omega \varphi^{2^*}(x)|V(x)|w^2(x)\,dx \leqslant \ell_q \int_\Omega \varphi^{2^*}(x)w^2(x)\,dx + \min\left\{\frac{1}{8}, \frac{2}{q+4}\right\}\bigg(\int_\Omega \varphi^2(x)|\nabla w|^2\,dx\bigg).$$

Let η be a nonnegative cut-off function such that

$$\mathrm{supp}(\eta) \Subset \Omega, \quad \eta \equiv 1 \text{ on } \Omega', \text{ and } |\nabla\eta(x)| \leqslant \frac{2}{\mathrm{dist}(\Omega', \partial\Omega)}.$$

Set $v^n := \min(n, |v|) \in \mathcal{D}^{1,2}_{\varphi}(\Omega)$ and test (14) with $v(v^n)^{q-2}\eta^2 \in \mathcal{D}^{1,2}_{\varphi}(\Omega)$. This leads to

$$\begin{aligned}(q-2)\int_\Omega \varphi^2(x)\eta^2(x)|\nabla v^n(x)|^2(v^n(x))^{q-2}\,dx + \int_\Omega \varphi^2(x)\eta^2(x)(v^n(x))^{q-2}|\nabla v(x)|^2\,dx\\
= \int_\Omega \varphi^{2^*}(x)V(x)\eta^2(x)v^2(x)(v^n(x))^{q-2}\,dx - 2\int_\Omega \varphi^2(x)\eta(x)v(x)(v^n(x))^{q-2}\nabla v(x)\cdot\nabla\eta(x)\,dx.\end{aligned}$$

We use the elementary inequality $2ab \leqslant 1/2a^2 + 4b^2$ and obtain

$$\begin{aligned}(19)\qquad (q-2)\int_\Omega \varphi^2(x)\eta^2(x)|\nabla v^n(x)|^2(v^n(x))^{q-2}\,dx + \frac{1}{2}\int_\Omega \varphi^2(x)\eta^2(x)(v^n(x))^{q-2}|\nabla v(x)|^2\,dx\\
\leqslant \int_\Omega \varphi^{2^*}(x)V(x)\eta^2(x)v^2(x)(v^n(x))^{q-2}\,dx + 4\int_\Omega \varphi^2(x)|\nabla\eta(x)|^2v^2(x)(v_n(x))^{q-2}\,dx.\end{aligned}$$

Furthermore, an explicit calculation gives

$$\begin{aligned}\big|\nabla\big((v^n)^{\frac q2-1}v\eta\big)\big|^2 \leqslant \frac{(q+4)(q-2)}{4}(v^n)^{q-2}\eta^2|\nabla v^n|^2+2(v^n)^{q-2}|\nabla v|^2\eta^2\\ +2(v^n)^{q-2}v^2|\nabla\eta|^2+\frac{q-2}{2}(v^n)^q|\nabla\eta|^2.\end{aligned}\tag{20}$$

Letting $C(q):=\min\left\{\frac14,\frac{4}{q+4}\right\}$, from (19) and (20) we get

$$\begin{aligned}C(q)\int_\Omega\varphi^2(x)\big|\nabla\big((v^n)^{\frac q2-1}v\eta\big)(x)\big|^2\,dx\leqslant 2(2+C(q))\int_\Omega\varphi^2(x)(v^n(x))^{q-2}v(x)^2|\nabla\eta(x)|^2\,dx\\ +C(q)\frac{q-2}{2}\int_\Omega\varphi^2(x)(v^n(x))^q|\nabla\eta(x)|^2\,dx+\int_\Omega\varphi^{2^*}(x)V(x)\eta^2(x)v^2(x)(v^n(x))^{q-2}\,dx.\end{aligned}\tag{21}$$

Estimate (18) applied to $\eta(v^n)^{\frac q2-1}v$ gives

$$\begin{aligned}\int_\Omega\varphi^{2^*}(x)|V(x)|\big[\eta(x)(v^n(x))^{\frac q2-1}v(x)\big]^2\,dx\leqslant\frac{C(q)}{2}\int_\Omega\varphi^2(x)\big|\nabla(\eta(v^n)^{\frac q2-1}v)(x)\big|^2\,dx\\ +\ell_q\int_\Omega\varphi^{2^*}(x)(v^n(x))^{q-2}v^2(x)\eta^2(x)\,dx.\end{aligned}\tag{22}$$

Using (22) to estimate the term with V in (21), (13) yields

$$\begin{aligned}\left(\int_\Omega\varphi^{2^*}(x)|v^n(x)|^{(\frac q2-1)2^*}|v(x)|^{2^*}\eta^{2^*}(x)\,dx\right)^{\frac{2}{2^*}}\leqslant\frac{2\ell_qC_{N,a}}{C(q)}\int_\Omega\varphi^{2^*}(x)\eta^2(x)|v^n(x)|^{q-2}v^2(x)\,dx\\ +\frac{4C_{N,a}(2+C(q))}{C(q)}\int_\Omega\varphi^2(x)|v^n(x)|^{q-2}v^2(x)|\nabla\eta(x)|^2\,dx\\ +C_{N,a}(q-2)\int_\Omega\varphi^2(x)|v^n(x)|^q|\nabla\eta(x)|^2\,dx.\end{aligned}$$

Letting $n\to\infty$ in the above inequality, (15) follows. □

Proof of Theorem 1.2. Let u be a weak $H^1(\Omega)$-solution to (6). It is easy to verify that $\varphi(x):=|x|^\sigma\psi_1(x/|x|)\in H^1(\Omega)$ satisfies (in a weak $H^1(\Omega)$-sense and in a classical sense in $\Omega\setminus\{0\}$)

$$-\Delta\varphi(x)-\frac{a(x/|x|)}{|x|^2}\,\varphi(x)=0.$$

Then $v:=\frac{u}{\varphi}\in H^1_\varphi(\Omega)$ turns out to be a weak solution to (14). Let $R>0$ be such that

$$\Omega'\Subset\Omega'+B(0,2R)\Subset\Omega.$$

Using Lemma 3.1 in $\Omega_1:=\Omega'+B(0,R(2-r_1))\Subset\Omega'+B(0,2R)$, $r_1=1$, with $q=q_1=2^*$, we infer that $v\in L^{\frac{(2^*)^2}{2}}(\varphi^{2^*},\Omega_1)$ and the following estimate holds

$$\|v\|_{L^{\frac{(2^*)^2}{2}}(\varphi^{2^*},\Omega_1)}\leqslant\widetilde{C}^{\frac{1}{q_1}}\big(\operatorname{diam}\Omega\big)^{\frac{\sigma(2-2^*)}{q_1}}\left(\frac{8}{C(q_1)}\frac{4}{(Rr_1)^2}+\frac{4(q_1+2)}{(Rr_1)^2}+\frac{2\ell_{q_1}}{C(q_1)}\right)^{\frac{1}{q_1}}\|v\|_{L^{2^*}(\varphi^{2^*},\Omega)}.$$

Using again Lemma 3.1 in $\Omega_2 := \Omega' + B(0, R(2 - r_1 - r_2)) \Subset \Omega_1$, $r_2 = \frac{1}{4}$, with $q = q_2 = (2^*)^2/2$, we infer that $v \in L^{\frac{(2^*)^3}{4}}(\varphi^{2^*}, \Omega_2)$ and

$$\begin{aligned}\|v\|_{L^{\frac{(2^*)^3}{4}}(\varphi^{2^*},\Omega_2)} &\leqslant \widetilde{C}^{\frac{1}{q_2}} \big(\operatorname{diam}\Omega\big)^{\frac{\sigma(2-2^*)}{q_2}} \left(\frac{8}{C(q_2)} \frac{4}{(Rr_2)^2} + \frac{4(q_2+2)}{(Rr_2)^2} + \frac{2\ell_{q_2}}{C(q_2)} \right)^{\frac{1}{q_2}} \|v\|_{L^{q_2}(\varphi^{2^*},\Omega_1)} \\ &\leqslant \Big[\widetilde{C}\big(\operatorname{diam}\Omega\big)^{\sigma(2-2^*)}\Big]^{\frac{1}{q_1}+\frac{1}{q_2}} \left(\frac{8}{C(q_1)} \frac{4}{(Rr_1)^2} + \frac{4(q_1+2)}{(Rr_1)^2} + \frac{2\ell_{q_1}}{C(q_1)} \right)^{\frac{1}{q_1}} \times \\ &\qquad \times \left(\frac{8}{C(q_2)} \frac{4}{(Rr_2)^2} + \frac{4(q_2+2)}{(Rr_2)^2} + \frac{2\ell_{q_2}}{C(q_2)} \right)^{\frac{1}{q_2}} \|v\|_{L^{2^*}(\varphi^{2^*},\Omega)}.\end{aligned}$$

Setting, for any $n \in \mathbb{N}$, $n \geqslant 1$,

$$q_n = \frac{1}{2}\left(\frac{2^*}{2}\right)^n, \quad \Omega_n := \Omega' + B\bigg(0, R\bigg(2 - \sum_{k=1}^n r_k\bigg)\bigg), \quad \text{and} \quad r_n = \frac{1}{n^2},$$

and using iteratively Lemma 3.1, we obtain that, for any $n \in \mathbb{N}$, $n \geqslant 1$,

$$\begin{aligned}(23)\quad \|v\|_{L^{q_{n+1}}(\varphi^{2^*},\Omega')} &\leqslant \|v\|_{L^{q_{n+1}}(\varphi^{2^*},\Omega_n)} \\ &\leqslant \|v\|_{L^{2^*}(\varphi^{2^*},\Omega)} \Big[\widetilde{C}\big(\operatorname{diam}\Omega\big)^{\sigma(2-2^*)}\Big]^{\sum_{k=1}^n \frac{1}{q_k}} \prod_{k=1}^n \left(\frac{8}{C(q_k)} \frac{4}{(Rr_k)^2} + \frac{4(q_k+2)}{(Rr_k)^2} + \frac{2\ell_{q_k}}{C(q_k)} \right)^{\frac{1}{q_k}}.\end{aligned}$$

We notice that

$$\prod_{k=1}^n \left(\frac{8}{C(q_k)} \frac{4}{(Rr_k)^2} + \frac{4(q_k+2)}{(Rr_k)^2} + \frac{2\ell_{q_k}}{C(q_k)} \right)^{\frac{1}{q_k}} = \exp\left[\sum_{k=1}^n b_k\right] \tag{24}$$

where

$$b_k = \frac{1}{q_k} \log\left(\frac{16k^4}{R^2 C(q_k)} + \frac{4k^4(q_k+2)}{R^2} + \frac{2\ell_{q_k}}{C(q_k)} \right),$$

and, for some constant $C = C\big(N, a, \|V\|_{L^s(\varphi^{2^*},\Omega)}, \operatorname{dist}(\Omega', \partial\Omega)\big) > 0$,

$$b_k \sim 2\left(\frac{2}{2^*}\right)^k \log\left[C\left(\frac{1}{2}\left(\frac{2^*}{2}\right)^k\right)^{\frac{2s}{2s-N}} \right] \quad \text{as } k \to +\infty.$$

Hence $\sum_{n=1}^\infty b_n$ converges to some positive sum depending only on $\|V\|_{L^s(\varphi^{2^*},\Omega)}$, $\operatorname{dist}(\Omega', \partial\Omega)$, N, and a, hence

$$\lim_{n\to+\infty} \Big[\widetilde{C}\big(\operatorname{diam}\Omega\big)^{\sigma(2-2^*)}\Big]^{\sum_{k=1}^n \frac{1}{q_k}} \prod_{k=1}^n \left(\frac{8}{C(q_k)} \frac{4}{(Rr_k)^2} + \frac{4(q_k+2)}{(Rr_k)^2} + \frac{2\ell_{q_k}}{C(q_k)} \right)^{\frac{1}{q_k}}$$

is finite and depends only on N, a, $\|V\|_{L^s(\varphi^{2^*},\Omega)}$, and $\operatorname{dist}(\Omega', \partial\Omega)$. Hence, from (23), we deduce that there exists a positive constant C depending only on N, a, $\|V\|_{L^s(\varphi^{2^*},\Omega)}$, $\operatorname{dist}(\Omega', \partial\Omega)$, and $\operatorname{diam}\Omega$, such that

$$\|v\|_{L^{q_{n+1}}(\varphi^{2^*},\Omega')} \leqslant C\, \|v\|_{L^{2^*}(\varphi^{2^*},\Omega)} \quad \text{for all } n \in \mathbb{N}.$$

Letting $n \to +\infty$ we deduce that v is essentially bounded in Ω' with respect to the measure $\varphi^{2^*}dx$ and

$$\|v\|_{L^\infty(\varphi^{2^*},\Omega')} \leqslant C\, \|v\|_{L^{2^*}(\varphi^{2^*},\Omega)} = C\, \|u\|_{L^{2^*}(\Omega)},$$

where $\|v\|_{L^\infty(\varphi^{2^*},\Omega')}$ denotes the essential supremum of v with respect to the measure $\varphi^{2^*}dx$. Since $\varphi^{2^*}dx$ is absolutely continuous with respect to the Lebesgue measure and viceversa, there holds $\|v\|_{L^\infty(\varphi^{2^*},\Omega')} = \|v\|_{L^\infty(\Omega')}$, hence $v \in L^\infty(\Omega')$ and

$$\|v\|_{L^\infty(\Omega')} \leqslant C\,\|u\|_{L^{2^*}(\Omega)},$$

thus completing the proof. □

If the potential V in equation (6) belongs to $L^{N/2}(\varphi^{2^*},\Omega)$ (but to $L^s(\varphi^{2^*},\Omega)$ for no $s > N/2$), although we can no more derive an L^∞- bound for u/φ, we can obtain for u/φ as high summability as we like.

Theorem 3.2. *Let Ω be a bounded domain containing 0, $a \in L^\infty(\mathbb{S}^{N-1})$ satisfying $\Lambda_N(a) < 1$, and $V \in L^{N/2}(\varphi^{2^*},\Omega)$. Then, for any $\Omega' \Subset \Omega$ and for any weak $H^1(\Omega)$-solution u to (6), there holds $\frac{u}{\varphi} \in L^q(\varphi^{2^*},\Omega')$ for all $1 \leqslant q < +\infty$.*

Proof. The proof follows closely the proofs of Theorem 1.2 and Lemma 3.1. However, since we only require $V \in L^{N/2}(\varphi^{2^*},\Omega)$, we have that for any q there exists ℓ_q such that

$$\int_{|V(x)|\geqslant \ell_q} \varphi^{2^*}(x)|V(x)|^{\frac{N}{2}}\,dx \leqslant \min\left\{\frac{C_{N,a}^{-1}}{8}, \frac{2C_{N,a}^{-1}}{q+4}\right\}^{\frac{N}{2}},$$

but we can no more estimate ℓ_q in terms of q, as we did in (17) thanks to the summability assumption $V \in L^s(\varphi^{2^*},\Omega)$ for some $s > N/2$. Hence we still arrive at an estimate of type (23) but we have no control on the product in (24) as $n \to +\infty$. □

4. Behavior of solutions at singularities

The procedure followed in this section to prove Theorem 1.1 relies in comparison methods and separation of variables. Indeed we will evaluate the asymptotics of solutions to problem (3) by trapping them between functions which solve analogous problems with radial perturbing potentials. To this aim, the first step consists in deriving the asymptotic behavior of solutions to Schrödinger equations with a potential which is given by a radial perturbation of the dipole-type singular term. In this case, it is possible to expand the solution in Fourier series, thus separating the radial and angular variables, and to estimate the behavior of the Fourier coefficients in order to establish which of them is dominant near the singularity.

Proposition 4.1. *Let $a \in L^\infty(\mathbb{S}^{N-1})$ be such that $\Lambda_N(a) < 1$, $R > 0$, and $u \in H^1(B(0,R))$, $u \geqslant 0$ a.e. in $B(0,R)$, $u \not\equiv 0$, be a weak H^1-solution to*

$$-\Delta u(x) = \left[\frac{a(x/|x|)}{|x|^2} + h(|x|)\right]u(x) \quad \text{in } B(0,R), \tag{25}$$

where $h \in L^\infty_{\rm loc}(0,R) \cap L^p(0,R)$ for some $p > N/2$. Then, for any $r \in (0,R)$, there exists a positive constant C (depending on h, R, r, a, ε, and u) such that

$$\frac{1}{C}|x|^\sigma \leqslant u(x) \leqslant C|x|^\sigma \quad \text{for all } x \in B(0,r)\setminus\{0\},$$

where σ is defined in (2). Moreover, for any $\theta \in \mathbb{S}^{N-1}$ and $r \in (0,R)$

$$\lim_{\rho\to 0^+} u(\rho\,\theta)\rho^{-\sigma} = f(r,u,h,N,a)\,\psi_1(\theta), \tag{26}$$

where

$$f(r,v,h,N,a)=\int_{\mathbb{S}^{N-1}}\bigg(r^{-\sigma}u(r\eta)+\int_0^r\frac{s^{1-\sigma}}{2\sigma+N-2}\,h(s)u(s\,\eta)\,ds \\ -r^{-2\sigma-N+2}\int_0^r\frac{s^{N-1+\sigma}}{2\sigma+N-2}\,h(s)u(s\,\eta)\,ds\bigg)\psi_1(\eta)\,dV(\eta). \tag{27}$$

and there exists a positive constant $\bar{C}$ *(depending on* h*,* R*,* r*,* a*,* ε*, but not on* u*) such that*

$$u(\rho\,\theta)\leqslant \bar{C}\,\|u\|_{H^1(B(0,R))}\rho^{\sigma},\quad \textit{for all } 0<\rho<r. \tag{28}$$

Proof. Let $r\in(0,R)$. We can assume, without loss of generality, that $R>1$ and $r=1$. Indeed, setting $w(x):=u(rx)$, we notice that $w\in H^1(B(0,R/r))$ and weakly solves

$$-\Delta w(x)=\left[\frac{a(x/|x|)}{|x|^2}+\tilde{h}(|x|)\right]w(x)\quad \text{in } B(0,R/r),$$

where $\tilde{h}(\rho):=r^2h(r\rho)$ satisfies $\tilde{h}\in L^\infty_{\rm loc}(0,R/r)\cap L^p(0,R/r)$. Hence, it is enough to prove the statement for $R>1$ and $r=1$, being the general case easily obtainable from scaling.

Let $R>1$, $r=1$ and $u\in H^1(B(0,R))$, $u\geqslant 0$ a.e. in $B(0,R)$, $u\not\equiv 0$, be a weak solution of (25). By standard regularity theory, $u\in C^0\big(\overline{B(0,1)}\setminus B(0,s)\big)$ for any $s\in(0,1)$. For any $k\in\mathbb{N}\setminus\{0\}$, let ψ_k be a L^2-normalized eigenfunction of the operator $-\Delta_{\mathbb{S}^{N-1}}-a(\theta)$ on the sphere associated to the k-th eigenvalue μ_k, i.e. satisfying (11). We can choose the functions ψ_k in such a way that they form an orthonormal basis of $L^2(\mathbb{S}^{N-1})$, hence u can be expanded as

$$u(x)=u(\rho\,\theta)=\sum_{k=1}^{\infty}\varphi_k(\rho)\psi_k(\theta), \tag{29}$$

where $\rho=|x|\in(0,1]$, $\theta=x/|x|\in\mathbb{S}^{N-1}$, and

$$\varphi_k(\rho)=\int_{\mathbb{S}^{N-1}}u(\rho\,\theta)\psi_k(\theta)\,dV(\theta). \tag{30}$$

The Parseval identity yields

$$\int_{\mathbb{S}^{N-1}}|u(\rho\,\theta)|^2\,dV(\theta)=\sum_{k=1}^{\infty}|\varphi_k(\rho)|^2,\quad \text{for all } 0<\rho\leqslant 1,$$

and hence

$$\|u\|^2_{L^2(B(0,1))}=\int_0^1\rho^{N-1}\bigg(\sum_{k=1}^{\infty}|\varphi_k(\rho)|^2\bigg)\,d\rho=\sum_{k=1}^{\infty}\int_0^1\rho^{N-1}|\varphi_k(\rho)|^2\,d\rho. \tag{31}$$

Equations (25) and (11) imply that, for every k,

$$\varphi_k''(\rho)+\frac{N-1}{\rho}\varphi_k'(\rho)-\frac{\mu_k}{\rho^2}\varphi_k(\rho)=h(\rho)\varphi_k(\rho),\quad \text{in } (0,1).$$

A direct calculation shows that, for some $c_1^k,c_2^k\in\mathbb{R}$,

$$\varphi_k(\rho)=\rho^{\sigma_k^+}\bigg(c_1^k+\int_\rho^1\frac{s^{-\sigma_k^+ +1}}{\sigma_k^+-\sigma_k^-}h(s)\varphi_k(s)\,ds\bigg)+\rho^{\sigma_k^-}\bigg(c_2^k+\int_\rho^1\frac{s^{-\sigma_k^- +1}}{\sigma_k^- -\sigma_k^+}h(s)\varphi_k(s)\,ds\bigg), \tag{32}$$

where

$$\sigma_k^+ = -\frac{N-2}{2} + \sqrt{\left(\frac{N-2}{2}\right)^2 + \mu_k} \quad \text{and} \quad \sigma_k^- = -\frac{N-2}{2} - \sqrt{\left(\frac{N-2}{2}\right)^2 + \mu_k}. \tag{33}$$

For the sake of notation, we set

$$A_k(\rho) = \rho^{\sigma_k^+} \int_\rho^1 \frac{s^{-\sigma_k^+ + 1}}{\sigma_k^+ - \sigma_k^-} h(s)\varphi_k(s)\, ds \tag{34}$$

and

$$B_k(\rho) = \rho^{\sigma_k^-} \left(c_2^k + \int_\rho^1 \frac{s^{-\sigma_k^- + 1}}{\sigma_k^- - \sigma_k^+} h(s)\varphi_k(s)\, ds \right),$$

so that

$$\varphi_k(\rho) = c_1^k \rho^{\sigma_k^+} + A_k(\rho) + B_k(\rho). \tag{35}$$

Without loss of generality, we can assume that

$$\frac{N}{2} < p < \frac{N}{2 - \frac{1}{3}\sqrt{\left(\frac{N-2}{2}\right)^2 + \mu_1}},$$

so, setting $\varepsilon = 2 - \frac{N}{p}$, $0 < \varepsilon < \frac{1}{3}(\sigma_k^+ + \frac{N-2}{2})$, for every k. From Hölder's inequality and Lemma 2.3, it follows that

$$\begin{aligned}
|A_k(\rho)| &= \rho^{\sigma_k^+} \left| \int_\rho^1 \frac{s^{-\sigma_k^+ + 1}}{\sigma_k^+ - \sigma_k^-} h(s) \left(\int_{\mathbb{S}^{N-1}} u(s\,\theta)\psi_k(\theta)\, dV(\theta) \right) ds \right| \qquad (36)\\
&\leqslant \frac{C_1 \rho^{\sigma_k^+} |\mu_k|^{\left\lfloor \frac{N-1}{4} \right\rfloor + 1}}{\sigma_k^+ - \sigma_k^-} \int_\rho^1 s^{-\sigma_k^+ + 1 - \frac{N-1}{p} - \frac{N-1}{2^*}} \left(\int_{\mathbb{S}^{N-1}} s^{\frac{N-1}{p} + \frac{N-1}{2^*}} |h(s)||u(s\,\theta)|\, dV(\theta) \right) ds \\
&\leqslant \frac{\omega_N^{\frac{N+2}{2N}} C_1 \rho^{\sigma_k^+} |\mu_k|^{\left\lfloor \frac{N-1}{4} \right\rfloor + 1}}{\sigma_k^+ - \sigma_k^-} \frac{\|h\|_{L^p(B(0,1))}}{\omega_N} \|u\|_{L^{2^*}(B(0,1))} \left(\int_\rho^1 s^{(-\sigma_k^+ + 1 - \frac{N-1}{p} - \frac{N-1}{2^*})\frac{2^* p}{2^* p - 2^* - p}}\, ds \right)^{1 - \frac{1}{p} - \frac{1}{2^*}} \\
&= \frac{\omega_N^{\frac{N+2}{2N}} C_1 \rho^{\sigma_k^+} |\mu_k|^{\left\lfloor \frac{N-1}{4} \right\rfloor + 1}}{\sigma_k^+ - \sigma_k^-} \frac{\|h\|_{L^p(B(0,1))}}{\omega_N} \|u\|_{L^{2^*}(B(0,1))} \left[\frac{\rho^{\frac{2^* p}{2^* p - 2^* - p}(-\sigma_k^+ - \frac{N-2}{2} + 2 - \frac{N}{p})}}{\frac{2^* p}{2^* p - 2^* - p}\left(\frac{N}{p} - 2 + \sigma_k^+ + \frac{N-2}{2}\right)} \right]^{1 - \frac{1}{p} - \frac{1}{2^*}} \\
&\leqslant \frac{\omega_N^{\frac{N+2}{2N}} C_1 |\mu_k|^{\left\lfloor \frac{N-1}{4} \right\rfloor + 1}}{\sigma_k^+ - \sigma_k^-} \frac{\|h\|_{L^p(B(0,1))}}{\omega_N} \|u\|_{L^{2^*}(B(0,1))} \frac{\rho^{\varepsilon - \frac{N-2}{2}}}{\left[\frac{2^* p}{2^* p - 2^* - p}\left(\sigma_k^+ + \frac{N-2}{2} - \varepsilon\right)\right]^{1 - \frac{1}{p} - \frac{1}{2^*}}},
\end{aligned}$$

where $\omega_N = \int_{\mathbb{S}^{N-1}} dV(\theta)$. In particular

$$A_k(\rho) = o(\rho^{\sigma_k^-}) \quad \text{as } \rho \to 0^+. \tag{37}$$

Moreover

(38)

$$
\begin{aligned}
&\int_0^1 \left| s^{-\sigma_k^-+1} h(s)\varphi_k(s)\right| ds \\
&\leqslant C_1|\mu_k|^{\lfloor \frac{N-1}{4}\rfloor+1} \int_0^1 s^{-\sigma_k^-+1-\frac{N-1}{p}-\frac{N-1}{2^*}} \left(\int_{\mathbb{S}^{N-1}} s^{\frac{N-1}{p}+\frac{N-1}{2^*}} |h(s)||u(s\,\theta)|\, dV(\theta)\right) ds \\
&\leqslant \omega_N^{\frac{N+2}{2N}} C_1|\mu_k|^{\lfloor \frac{N-1}{4}\rfloor+1} \frac{\|h\|_{L^p(B(0,1))}}{\omega_N} \|u\|_{L^{2^*}(B(0,1))} \left[\int_0^1 s^{(-\sigma_k^-+1-\frac{N-1}{p}-\frac{N-1}{2^*})\frac{2^*p}{2^*p-2^*-p}}\, ds\right]^{1-\frac{1}{p}-\frac{1}{2^*}} \\
&= \omega_N^{\frac{N+2}{2N}} C_1|\mu_k|^{\lfloor \frac{N-1}{4}\rfloor+1} \frac{\|h\|_{L^p(B(0,1))}}{\omega_N} \|u\|_{L^{2^*}(B(0,1))} \frac{1}{\left[\frac{2^*p}{2^*p-2^*-p}\left(\varepsilon-\sigma_k^- - \frac{N-2}{2}\right)\right]^{1-\frac{1}{p}-\frac{1}{2^*}}} < \infty
\end{aligned}
$$

due to inequality $\sigma_k^- < -\frac{N-2}{2}$. As a consequence,

$$
\lim_{\rho\to 0^+} \int_\rho^1 \frac{s^{-\sigma_k^-+1}}{\sigma_k^- - \sigma_k^+}\, h(s)\varphi_k(s)\, ds \tag{39}
$$

is finite. Since $u \in L^{2^*}(B(0,1))$, from (37), (39), and the fact that $\rho^{\sigma_k^-}\psi_k(\theta) \notin L^{2^*}(B(0,1))$, we conclude that there must be

$$
c_2^k = -\int_0^1 \frac{s^{-\sigma_k^-+1}}{\sigma_k^- - \sigma_k^+}\, h(s)\varphi_k(s)\, ds, \tag{40}
$$

hence

$$
B_k(\rho) = \rho^{\sigma_k^-} \int_0^\rho \frac{s^{-\sigma_k^-+1}}{\sigma_k^+ - \sigma_k^-} h(s)\varphi_k(s)\, ds. \tag{41}
$$

Since $u \in C^0\big(\overline{B(0,1)} \setminus B(0,s)\big)$ for any $s \in (0,1)$, it makes sense to evaluate φ_k at $\rho = 1$ and, from (32) and (40), we have that

$$
c_1^k = \varphi_k(1) + \int_0^1 \frac{s^{-\sigma_k^-+1}}{\sigma_k^- - \sigma_k^+}\, h(s)\varphi_k(s)\, ds. \tag{42}
$$

From (32), (41), and (42), we deduce that

$$
\begin{aligned}
\varphi_k(\rho) = \rho^{\sigma_k^+}&\left(\varphi_k(1) + \int_0^1 \frac{s^{-\sigma_k^-+1}}{\sigma_k^- - \sigma_k^+}\, h(s)\varphi_k(s)\, ds + \int_\rho^1 \frac{s^{-\sigma_k^++1}}{\sigma_k^+ - \sigma_k^-} h(s)\varphi_k(s)\, ds\right) \\
&+ \rho^{\sigma_k^-} \int_0^\rho \frac{s^{-\sigma_k^-+1}}{\sigma_k^+ - \sigma_k^-} h(s)\varphi_k(s)\, ds.
\end{aligned} \tag{43}
$$

From above, (38), and standard elliptic estimates (which allow to estimate u outside the singularity in terms of its H^1-norm) we obtain that, for some positive constant $\tilde c$ depending only on N, R, a, and h,

$$
|c_1^k| \leqslant \tilde c\, |\mu_k|^{\lfloor \frac{N-1}{4}\rfloor+1} \|u\|_{H^1(B(0,R))} \Big[1 + \Big[\tfrac{2^*p}{2^*p-2^*-p}(\varepsilon - \sigma_k^- - \tfrac{N-2}{2})\Big]^{-1+\frac{1}{p}+\frac{1}{2^*}}\Big]. \tag{44}
$$

Arguing as in (38), we find that
(45)
$$|B_k(\rho)| \leqslant \frac{\omega_N^{\frac{N+2}{2N}} C_1 |\mu_k|^{\lfloor \frac{N-1}{4} \rfloor + 1}}{\sigma_k^+ - \sigma_k^-} \frac{\|h\|_{L^p(B(0,1))}}{\omega_N} \|u\|_{L^{2^*}(B(0,1))} \frac{\rho^{\varepsilon - \frac{N-2}{2}}}{\big[\frac{2^* p}{2^* p - 2^* - p}(\varepsilon - \sigma_k^- - \frac{N-2}{2})\big]^{1 - \frac{1}{p} - \frac{1}{2^*}}}.$$

From (35), (36), and (45), we can estimate φ_k as

$$|\varphi_k(\rho)| \leqslant |c_1^k| \rho^{\sigma_k^+} + \frac{\alpha_k |\mu_k|^{\lfloor \frac{N-1}{4} \rfloor + 1}}{\sigma_k^+ - \sigma_k^-} \frac{\|h\|_{L^p(B(0,1))}}{\omega_N} \|u\|_{L^{2^*}(B(0,1))} \rho^{\varepsilon - \frac{N-2}{2}} \tag{46}$$

where

$$\alpha_k := \frac{\omega_N^{\frac{N+2}{2N}} C_1}{\big[\frac{2^* p}{2^* p - 2^* - p}(\sigma_k^+ + \frac{N-2}{2} - \varepsilon)\big]^{1 - \frac{1}{p} - \frac{1}{2^*}}} + \frac{\omega_N^{\frac{N+2}{2N}} C_1}{\big[\frac{2^* p}{2^* p - 2^* - p}(\varepsilon - \sigma_k^- - \frac{N-2}{2})\big]^{1 - \frac{1}{p} - \frac{1}{2^*}}}.$$

Claim 1: there holds

(47)
$$\begin{aligned} |A_k(\rho) + B_k(\rho)| &\leqslant |c_1^k| \rho^{\sigma_k^+} \sum_{i=1}^{j_k - 1} \bigg(\frac{\|h\|_{L^p(B(0,1))}}{\omega_N (\sigma_k^+ - \sigma_k^-)} \frac{2}{\varepsilon^{\frac{p-1}{p}}} \bigg)^i \\ &+ \alpha_k |\mu_k|^{\lfloor \frac{N-1}{4} \rfloor + 1} \|u\|_{L^{2^*}(B(0,1))} \bigg(\frac{\|h\|_{L^p(B(0,1))}}{\omega_N (\sigma_k^+ - \sigma_k^-)} \bigg)^{j_k} \rho^{j_k \varepsilon - \frac{N-2}{2}} \prod_{i=2}^{j_k} \frac{2}{\Big[\sqrt{\big(\frac{N-2}{2}\big)^2 + \mu_k} - i\varepsilon \Big]^{\frac{p-1}{p}}}, \end{aligned}$$

where

$$j_k := \bigg\lfloor \frac{1}{\varepsilon} \sqrt{\Big(\frac{N-2}{2}\Big)^2 + \mu_k} \bigg\rfloor - 1,$$

i.e. j_k is the unique integer number such that

$$\sqrt{\Big(\frac{N-2}{2}\Big)^2 + \mu_k} - 2\varepsilon < j_k \varepsilon \leqslant \sqrt{\Big(\frac{N-2}{2}\Big)^2 + \mu_k} - \varepsilon.$$

Notice that $\varepsilon \leqslant \frac{1}{3}\big(\sigma_k^+ + \frac{N-2}{2}\big)$ implies $j_k \geqslant 2$.

To prove the claim, we observe that, from (34) and (46) it follows that

$$\begin{aligned} |A_k(\rho)| &\leqslant |c_1^k| \rho^{\sigma_k^+} \frac{\|h\|_{L^p(B(0,1))}}{\omega_N (\sigma_k^+ - \sigma_k^-)} \Big(\int_\rho^1 s^{(1 - \frac{N-1}{p}) \frac{p}{p-1}} \, ds \Big)^{\frac{p-1}{p}} \\ &+ \alpha_k |\mu_k|^{\lfloor \frac{N-1}{4} \rfloor + 1} \|u\|_{L^{2^*}(B(0,1))} \bigg(\frac{\|h\|_{L^p(B(0,1))}}{\omega_N (\sigma_k^+ - \sigma_k^-)} \bigg)^2 \rho^{\sigma_k^+} \Big(\int_\rho^1 s^{(-\sigma_k^+ + 1 + \varepsilon - \frac{N-2}{2} - \frac{N-1}{p}) \frac{p}{p-1}} \, ds \Big)^{\frac{p-1}{p}} \\ &\leqslant |c_1^k| \rho^{\sigma_k^+} \frac{\|h\|_{L^p(B(0,1))}}{\omega_N (\sigma_k^+ - \sigma_k^-)} \frac{1}{(\varepsilon \frac{p}{p-1})^{\frac{p-1}{p}}} \\ &+ \alpha_k |\mu_k|^{\lfloor \frac{N-1}{4} \rfloor + 1} \|u\|_{L^{2^*}(B(0,1))} \bigg(\frac{\|h\|_{L^p(B(0,1))}}{\omega_N (\sigma_k^+ - \sigma_k^-)} \bigg)^2 \frac{\rho^{2\varepsilon - \frac{N-2}{2}}}{\Big[\big(\sigma_k^+ - 2\varepsilon + \frac{N-2}{2}\big) \frac{p}{p-1} \Big]^{\frac{p-1}{p}}}. \end{aligned} \tag{48}$$

In a similar way, from (41) and (46) we deduce that

$$|B_k(\rho)| \leqslant |c_1^k| \rho^{\sigma_k^+} \frac{\|h\|_{L^p(B(0,1))}}{\omega_N(\sigma_k^+ - \sigma_k^-)} \frac{1}{\left[(\varepsilon + \sigma_k^+ - \sigma_k^-)\frac{p}{p-1}\right]^{\frac{p-1}{p}}} + \alpha_k |\mu_k|^{\left\lfloor \frac{N-1}{4} \right\rfloor + 1} \|u\|_{L^{2^*}(B(0,1))} \left(\frac{\|h\|_{L^p(B(0,1))}}{\omega_N(\sigma_k^+ - \sigma_k^-)} \right)^2 \frac{\rho^{2\varepsilon - \frac{N-2}{2}}}{\left[\left(2\varepsilon - \sigma_k^- - \frac{N-2}{2}\right)\frac{p}{p-1}\right]^{\frac{p-1}{p}}}. \tag{49}$$

Summing up (48) and (49), we obtain

$$|A_k(\rho) + B_k(\rho)| \leqslant |c_1^k| \rho^{\sigma_k^+} \frac{\|h\|_{L^p(B(0,1))}}{\omega_N(\sigma_k^+ - \sigma_k^-)} \frac{2}{\varepsilon^{\frac{p-1}{p}}} + \alpha_k |\mu_k|^{\left\lfloor \frac{N-1}{4} \right\rfloor + 1} \|u\|_{L^{2^*}(B(0,1))} \left(\frac{\|h\|_{L^p(B(0,1))}}{\omega_N(\sigma_k^+ - \sigma_k^-)} \right)^2 \frac{2\rho^{2\varepsilon - \frac{N-2}{2}}}{\left(\sigma_k^+ + \frac{N-2}{2} - 2\varepsilon\right)^{\frac{p-1}{p}}},$$

and hence, from (35),

$$|\varphi_k(\rho)| \leqslant |c_1^k| \rho^{\sigma_k^+} \left(1 + \frac{\|h\|_{L^p(B(0,1))}}{\omega_N(\sigma_k^+ - \sigma_k^-)} \frac{2}{\varepsilon^{\frac{p-1}{p}}} \right) + \alpha_k |\mu_k|^{\left\lfloor \frac{N-1}{4} \right\rfloor + 1} \|u\|_{L^{2^*}(B(0,1))} \left(\frac{\|h\|_{L^p(B(0,1))}}{\omega_N(\sigma_k^+ - \sigma_k^-)} \right)^2 \frac{2\rho^{2\varepsilon - \frac{N-2}{2}}}{\left(\sqrt{\left(\frac{N-2}{2}\right)^2 + \mu_k} - 2\varepsilon\right)^{\frac{p-1}{p}}}. \tag{50}$$

Using (50), we can improve our estimates of $A_k(\rho)$ and $B_k(\rho)$ thus obtaining

$$|A_k(\rho)| \leqslant |c_1^k| \rho^{\sigma_k^+} \left[\frac{\|h\|_{L^p(B(0,1))}}{\omega_N(\sigma_k^+ - \sigma_k^-)} \frac{1}{\varepsilon^{\frac{p-1}{p}}} + \left(\frac{\|h\|_{L^p(B(0,1))}}{\omega_N(\sigma_k^+ - \sigma_k^-)} \right)^2 \frac{2}{\varepsilon^{2\frac{p-1}{p}}} \right] + \alpha_k |\mu_k|^{\left\lfloor \frac{N-1}{4} \right\rfloor + 1} \|u\|_{L^{2^*}(B(0,1))} \left(\frac{\|h\|_{L^p(B(0,1))}}{\omega_N(\sigma_k^+ - \sigma_k^-)} \right)^3 \times \\ \times \frac{\rho^{3\varepsilon - \frac{N-2}{2}}}{\left[\sqrt{\left(\frac{N-2}{2}\right)^2 + \mu_k} - 3\varepsilon\right]^{\frac{p-1}{p}}} \frac{2}{\left[\sqrt{\left(\frac{N-2}{2}\right)^2 + \mu_k} - 2\varepsilon\right]^{\frac{p-1}{p}}}$$

and

$$|B_k(\rho)| \leqslant |c_1^k| \rho^{\sigma_k^+} \left[\frac{\|h\|_{L^p(B(0,1))}}{\omega_N(\sigma_k^+ - \sigma_k^-)} \frac{1}{\varepsilon^{\frac{p-1}{p}}} + \left(\frac{\|h\|_{L^p(B(0,1))}}{\omega_N(\sigma_k^+ - \sigma_k^-)} \right)^2 \frac{2}{\varepsilon^{2\frac{p-1}{p}}} \right] + \alpha_k |\mu_k|^{\left\lfloor \frac{N-1}{4} \right\rfloor + 1} \|u\|_{L^{2^*}(B(0,1))} \left(\frac{\|h\|_{L^p(B(0,1))}}{\omega_N(\sigma_k^+ - \sigma_k^-)} \right)^3 \times \\ \times \frac{\rho^{3\varepsilon - \frac{N-2}{2}}}{\left[\sqrt{\left(\frac{N-2}{2}\right)^2 + \mu_k} + 3\varepsilon\right]^{\frac{p-1}{p}}} \frac{2}{\left[\sqrt{\left(\frac{N-2}{2}\right)^2 + \mu_k} - 2\varepsilon\right]^{\frac{p-1}{p}}}.$$

Summing up we find that

$$
\begin{aligned}
|A_k(\rho)+B_k(\rho)| &\leqslant |c_1^k|\rho^{\sigma_k^+}\left[\frac{\|h\|_{L^p(B(0,1))}}{\omega_N(\sigma_k^+-\sigma_k^-)}\frac{2}{\varepsilon^{\frac{p-1}{p}}}+\left(\frac{\|h\|_{L^p(B(0,1))}}{\omega_N(\sigma_k^+-\sigma_k^-)}\right)^2\frac{4}{\varepsilon^{2\frac{p-1}{p}}}\right]\\
&+\alpha_k|\mu_k|^{\lfloor\frac{N-1}{4}\rfloor+1}\|u\|_{L^{2^*}(B(0,1))}\left(\frac{\|h\|_{L^p(B(0,1))}}{\omega_N(\sigma_k^+-\sigma_k^-)}\right)^3\times\\
&\times\frac{4\rho^{3\varepsilon-\frac{N-2}{2}}}{\left[\sqrt{\left(\frac{N-2}{2}\right)^2+\mu_k}-3\varepsilon\right]^{\frac{p-1}{p}}\left[\sqrt{\left(\frac{N-2}{2}\right)^2+\mu_k}-2\varepsilon\right]^{\frac{p-1}{p}}}.
\end{aligned}
$$

An iteration of the above argument (j_k-1) times easily leads to estimate (47). Claim 1 is thereby proved.

Claim 2: the function $s\mapsto s^{-\sigma_k^+ +1}h(s)\varphi_k(s)$ belongs to $L^1(0,1)$ and

$$\lim_{\rho\to 0^+}\rho^{-\sigma_k^+}\varphi_k(\rho)=c_1^k+\int_0^1\frac{s^{-\sigma_k^+ +1}}{\sigma_k^+-\sigma_k^-}h(s)\varphi_k(s)\,ds. \tag{51}$$

Indeed, from (47), (35), (44), and the choice of j_k, it follows that

$$|\varphi_k(\rho)|\leqslant d_k\|u\|_{H^1(B(0,1))}\rho^{j_k\varepsilon-\frac{N-2}{2}}, \tag{52}$$

for some positive constant d_k depending on k (and on a, R, h N). We distinguish now two cases. If $j_k\varepsilon<\sqrt{\left(\frac{N-2}{2}\right)^2+\mu_k}-\varepsilon$, then from (34), (41), and (52) we derive that

$$|A_k(\rho)+B_k(\rho)|\leqslant d_k'\|u\|_{H^1(B(0,1))}\rho^{(j_k+1)\varepsilon-\frac{N-2}{2}},$$

for some other positive constant d_k' depending on k (and on a, R, h, N), and hence, by (44) and the choice of j_k,

$$|\varphi_k(\rho)|\leqslant d_k''\|u\|_{H^1(B(0,1))}\rho^{(j_k+1)\varepsilon-\frac{N-2}{2}}. \tag{53}$$

Estimate (53) and the choice of j_k imply that the function $s\mapsto s^{-\sigma_k^+ +1}h(s)\varphi_k(s)$ belongs to $L^1(0,1)$. Moreover, from (41) and (53) it follows that

$$|B_k(\rho)|\leqslant d_k'''\|u\|_{H^1(B(0,1))}\rho^{(j_k+2)\varepsilon-\frac{N-2}{2}}=o(\rho^{\sigma_k^+})\quad\text{as }\rho\to 0^+, \tag{54}$$

and the claim is proved. If $j_k\varepsilon=\sqrt{\left(\frac{N-2}{2}\right)^2+\mu_k}-\varepsilon$, then from (34), (41), and (52) we derive that

$$|A_k(\rho)+B_k(\rho)|\leqslant\gamma_k\|u\|_{H^1(B(0,1))}\rho^{\sigma_k^+}|\log\rho|^{\frac{p-1}{p}},$$

for some other positive constant γ_k depending on k (and on a, R, h, N), and hence,

$$|\varphi_k(\rho)|\leqslant\gamma_k'\|u\|_{H^1(B(0,1))}\rho^{\sigma_k^+}|\log\rho|^{\frac{p-1}{p}}. \tag{55}$$

Estimate (55) implies that the function $s\mapsto s^{-\sigma_k^+ +1}h(s)\varphi_k(s)$ belongs to $L^1(0,1)$. Moreover, from (41) and (55) it follows that

$$|B_k(\rho)|\leqslant\gamma_k''\|u\|_{H^1(B(0,1))}\rho^{\sigma_k^-}\left(\int_0^\rho s^{(\sigma_k^+-\sigma_k^-+\varepsilon)\frac{p}{p-1}-1}|\log s|\,ds\right)^{\frac{p-1}{p}}=o(\rho^{\sigma_k^+})\quad\text{as }\rho\to 0^+, \tag{56}$$

and claim 2 is proved also in this case.

Let us fix $\bar{k}$ such that

$$\frac{\|h\|_{L^p(B(0,1))}}{\omega_N(\sigma_k^+-\sigma_k^-)}\frac{2}{\varepsilon^{\frac{p-1}{p}}}=\frac{\|h\|_{L^p(B(0,1))}}{\omega_N\varepsilon^{\frac{p-1}{p}}\sqrt{\left(\frac{N-2}{2}\right)^2+\mu_k}}<\frac{1}{3},\quad \sigma_k^+>4\varepsilon\quad\text{and}\quad\frac{\sigma_k^+}{2}>\sigma_1^+,\quad\forall\,k\geqslant\bar{k}. \tag{57}$$

From (47) and (57), it follows that, for all $k\geqslant\bar{k}$ and for some positive constant C_2 (depending only on N, h, and a),

$$|A_k(\rho)+B_k(\rho)|\leqslant|c_1^k|\rho^{\sigma_k^+}\sum_{i=1}^{\infty}\left(\frac{1}{3}\right)^i$$
$$+C_2\|u\|_{H^1(B(0,1))}\left(\frac{\|h\|_{L^p(B(0,1))}}{\omega_N(\sigma_k^+-\sigma_k^-)}\right)^{j_k}|\mu_k|^{\left\lfloor\frac{N-1}{4}\right\rfloor+1}\frac{\rho^{j_k\varepsilon-\frac{N-2}{2}}\left(\frac{2}{\varepsilon}\right)^{j_k-1}}{\prod_{i=2}^{j_k}\left(\left\lfloor\varepsilon^{-1}\sqrt{\left(\frac{N-2}{2}\right)^2+\mu_k}\right\rfloor-i\right)},$$

which yields, for all $k\geqslant\bar{k}$,

$$|A_k(\rho)+B_k(\rho)|\leqslant\frac{1}{2}|c_1^k|\rho^{\sigma_k^+}+b_k\rho^{\sigma_k^+/2}, \tag{58}$$

where

$$b_k=C_2\|u\|_{H^1(B(0,1))}\left(\frac{\|h\|_{L^p(B(0,1))}}{\omega_N(\sigma_k^+-\sigma_k^-)}\right)^{j_k}|\mu_k|^{\left\lfloor\frac{N-1}{4}\right\rfloor+1}\frac{\left(\frac{2}{\varepsilon}\right)^{j_k-1}}{(j_k-1)!}.$$

From (10), $\mu_k\sim k^{2/(N-1)}$ and $j_k\sim k^{1/(N-1)}$ as $k\to+\infty$. Hence we have that

$$|b_k|\leqslant C_3\|u\|_{H^1(B(0,1))}\exp\left(-C_4\,k^{1/(N-1)}\right), \tag{59}$$

for some positive constants C_3 and C_4 depending only on N, h, and a. In view of (35) and (58), we deduce that

$$\frac{1}{2}|c_1^k|\rho^{\sigma_k^+}\leqslant|\varphi_k(\rho)|+|b_k|\rho^{\sigma_k^+/2},\quad\text{for all }k\geqslant\bar{k},$$

and consequently

$$\frac{1}{4}\sum_{k=\bar{k}}^{\infty}|c_1^k|^2\int_0^1\rho^{2\sigma_k^++N-1}\,d\rho\leqslant2\sum_{k=\bar{k}}^{\infty}\int_0^1|\varphi_k(\rho)|^2\rho^{N-1}\,d\rho+2\sum_{k=\bar{k}}^{\infty}|b_k|^2\int_0^1\rho^{\sigma_k^++N-1}\,d\rho.$$

Hence, from (31) and (59), we obtain that

$$\sum_{k=\bar{k}}^{\infty}\frac{|c_1^k|^2}{N+2\sigma_k^+}\leqslant8\|u\|^2_{L^2(B(0,1))}+8\sum_{k=\bar{k}}^{\infty}\frac{|b_k|^2}{N+\sigma_k^+}<+\infty. \tag{60}$$

From (29),

$$u(\rho\,\theta)\rho^{-\sigma_1^+}=\rho^{-\sigma_1^+}\varphi_1(\rho)\psi_1(\theta)+\sum_{k=2}^{\bar{k}-1}\rho^{\sigma_k^+-\sigma_1^+}\rho^{-\sigma_k^+}\varphi_k(\rho)\psi_k(\theta)+\sum_{k=\bar{k}}^{\infty}\rho^{-\sigma_1^+}\varphi_k(\rho)\psi_k(\theta). \tag{61}$$

From (51) we deduce that

$$\lim_{\rho\to0^+}\rho^{-\sigma_1^+}\varphi_1(\rho)\psi_1(\theta)=\left[c_1^1+\int_0^1\frac{s^{-\sigma_1^++1}}{\sigma_1^+-\sigma_1^-}h(s)\varphi_1(s)\,ds\right]\psi_1(\theta) \tag{62}$$

and

$$\lim_{\rho\to 0^+}\sum_{k=2}^{\bar k-1}\rho^{\sigma_k^+-\sigma_1^+}\rho^{-\sigma_k^+}\varphi_k(\rho)\psi_k(\theta)=0. \tag{63}$$

From (35), (58), (60), (59), and (57), we deduce that there exists some positive constant C_5 depending only on N, h, and a, such that, for all $\rho\in(0,1/2)$,

$$\begin{aligned}\sum_{k=\bar k}^{\infty}\rho^{-\sigma_1^+}|\varphi_k(\rho)||\psi_k(\theta)| &\leqslant C_1\sum_{k=\bar k}^{\infty}\rho^{-\sigma_1^+}\left(\frac{3}{2}|c_1^k|\rho^{\sigma_k^+}+|b_k|\rho^{\sigma_k^+/2}\right)|\mu_k|^{\lfloor (N-1)/4\rfloor+1}\\ &\leqslant \frac{3}{2}\,C_1\rho^{\sigma_2^+-\sigma_1^+}\left(\sum_{k=\bar k}^{\infty}\frac{|c_1^k|^2}{N+2\sigma_k^+}\right)^{\frac12}\left(\sum_{k=\bar k}^{\infty}(N+2\sigma_k^+)\rho^{2(\sigma_k^+-\sigma_2^+)}|\mu_k|^{2\lfloor (N-1)/4\rfloor+2}\right)^{\frac12}\\ &\quad + C_1\rho^{(\sigma_{\bar k}^+/2)-\sigma_1^+}\sum_{k=\bar k}^{\infty}|b_k||\mu_k|^{\lfloor (N-1)/4\rfloor+1}\\ &\leqslant C_5\|u\|_{H^1(B(0,1))}\big(\rho^{2(\sigma_k^+-\sigma_2^+)}+\rho^{(\sigma_{\bar k}^+/2)-\sigma_1^+}\big),\end{aligned} \tag{64}$$

which implies

$$\lim_{\rho\to 0^+}\sum_{k=\bar k}^{\infty}\rho^{-\sigma_1^+}\varphi_k(\rho)\psi_k(\theta)=0. \tag{65}$$

Collecting (61), (62), (63), and (65), we finally obtain that

$$\lim_{\rho\to 0^+}u(\rho\,\theta)\rho^{-\sigma_1^+}=\left[c_1^1+\int_0^1\frac{s^{-\sigma_1^++1}}{\sigma_1^+-\sigma_1^-}h(s)\varphi_1(s)\,ds\right]\psi_1(\theta). \tag{66}$$

We notice that, in view of (32), (30), and (40),

$$\begin{aligned}&c_1^1+\int_0^1\frac{s^{-\sigma_1^++1}}{\sigma_1^+-\sigma_1^-}h(s)\varphi_1(s)\,ds\\ &=\varphi_1(1)-c_2^1+\int_0^1\frac{s^{-\sigma_1^++1}}{\sigma_1^+-\sigma_1^-}h(s)\varphi_1(s)\,ds\\ &=\int_{\mathbb{S}^{N-1}}u(\eta)\psi_1(\eta)\,dV(\eta)+\int_{\mathbb{S}^{N-1}}\left[\int_0^1\frac{s\left(s^{-\sigma_1^+}-s^{-\sigma_1^-}\right)}{\sigma_1^+-\sigma_1^-}\,h(s)u(s\,\eta)\,ds\right]\psi_1(\eta)\,dV(\eta).\end{aligned} \tag{67}$$

The limit in (26) follows now from (66) and (67) in the case $r=1$ and by a change of variable inside the integral for $r\neq 1$. Moreover estimates (44), (53), (55), (54), (56), the definition of j_k, and (64), imply that, for some $C_6>0$ depending only on N, R, h, and a,

$$\begin{aligned}u(\rho\,\theta)\rho^{-\sigma_1^+}&=\left(c_1^1+\int_\rho^1\frac{s^{-\sigma_k^++1}}{\sigma_k^+-\sigma_k^-}h(s)\varphi_1(s)\right)\psi_1(\theta)+\rho^{-\sigma_1^+}\left[B_1(\rho)\psi_1(\theta)+\sum_{k>1}^{\infty}\varphi_k(\rho)\psi_k(\theta)\right]\\ &\leqslant C_6\|u\|_{H^1(B(0,R))},\end{aligned} \tag{68}$$

for all $0 < \rho < 1/2$. On the other hand, standard elliptic estimates in $B(0,1) \setminus B(0,1/2)$ yield, for some $C_7 > 0$ depending only on N, R, h, and a,

$$u(\rho\,\theta)\rho^{-\sigma_1^+} \leqslant C_7 \|u\|_{H^1(B(0,R))}, \quad \text{for all } \frac{1}{2} \leqslant \rho < 1. \tag{69}$$

Estimate (28) follows from (68) and (69).

From (66) and the positivity of u, it follows easily that $\Big[c_1^1 + \int_0^1 \frac{s^{-\sigma_1^+ +1}}{\sigma_1^+ - \sigma_1^-} h(s)\varphi_1(s)\,ds\Big] \geqslant 0$. Since

$$0 < \min_{\mathbb{S}^{N-1}} \psi_1 \leqslant \psi_1(\theta) \leqslant \max_{\mathbb{S}^{N-1}} \psi_1, \quad \text{for all } \theta \in \mathbb{S}^{N-1},$$

and, by standard regularity theory, $u \in C^0\big(\overline{B(0,1)} \setminus B(0,s)\big)$ for any $s \in (0,1)$, the proof of Proposition 4.1 will be complete if we show that

$$c_1^1 + \int_0^1 \frac{s^{-\sigma_1^+ +1}}{\sigma_1^+ - \sigma_1^-} h(s)\varphi_1(s)\,ds > 0. \tag{70}$$

In order to obtain (70), we need to prove the following

Claim 3: if $k \in \mathbb{N} \setminus \{0\}$ and

$$c_1^k + \int_0^1 \frac{s^{-\sigma_k^+ +1}}{\sigma_k^+ - \sigma_k^-} h(s)\varphi_k(s)\,ds = 0.$$

then $\varphi_k(\rho) = 0$ for all $\rho \in (0,1)$. Indeed, if $c_1^k + \int_0^1 \frac{s^{-\sigma_k^+ +1}}{\sigma_k^+ - \sigma_k^-} h(s)\varphi_k(s)\,ds = 0$, then

$$\varphi_k(\rho) = -\rho^{\sigma_k^+} \int_0^\rho \frac{s^{-\sigma_k^+ +1}}{\sigma_k^+ - \sigma_k^-} h(s)\varphi_k(s)\,ds + \rho^{\sigma_k^-} \int_0^\rho \frac{s^{-\sigma_k^- +1}}{\sigma_k^+ - \sigma_k^-} h(s)\varphi_k(s)\,ds. \tag{71}$$

From claim 2 and (51), we know that there exists a constant ℓ_k depending on k (and on a, R, h, u, N) such that

$$|\varphi_k(\rho)| \leqslant \ell_k \rho^{\sigma_k^+}.$$

Using the above estimate in (71), we can improve such an estimate as

$$|\varphi_k(\rho)| \leqslant \ell_k \rho^{\sigma_k^+} \frac{2\|h\|_{L^p(B(0,1))}}{\omega_N(\sigma_k^+ - \sigma_k^-)} \frac{\rho^\varepsilon}{\varepsilon}.$$

Using the above estimate in (71), we can obtain the following further improvement

$$|\varphi_k(\rho)| \leqslant \ell_k \rho^{\sigma_k^+} \bigg(\frac{2\|h\|_{L^p(B(0,1))}}{\omega_N(\sigma_k^+ - \sigma_k^-)\varepsilon} \bigg)^2 \frac{(\rho^\varepsilon)^2}{(2\cdot 1)^{\frac{p-1}{p}}}.$$

Arguing by induction, we can easily prove that, for all $j \in \mathbb{N}$,

$$|\varphi_k(\rho)| \leqslant \ell_k \rho^{\sigma_k^+} \bigg(\frac{2\|h\|_{L^p(B(0,1))}}{\omega_N(\sigma_k^+ - \sigma_k^-)\varepsilon} \bigg)^j \frac{(\rho^\varepsilon)^j}{(j!)^{\frac{p-1}{p}}},$$

and letting $j \to +\infty$, we deduce that $\varphi_k(\rho) = 0$ for all $\rho \in (0,1)$. Claim 3 is thereby proved.

We are now in position to prove (70). Arguing by contradiction, let us assume that

$$c_1^1 + \int_0^1 \frac{s^{-\sigma_1^+ +1}}{\sigma_1^+ - \sigma_1^-} h(s)\varphi_1(s)\,ds = 0 \tag{72}$$

and let $k_0 > 1$ be the smallest index for which

$$c_1^{k_0} + \int_0^1 \frac{s^{-\sigma_{k_0}^+ + 1}}{\sigma_{k_0}^+ - \sigma_{k_0}^-} h(s)\varphi_{k_0}(s)\, ds \neq 0.$$

Such a k_0 exists in view of claim 3; indeed if $c_1^k + \int_0^1 \frac{s^{-\sigma_k^+ + 1}}{\sigma_k^+ - \sigma_k^-} h(s)\varphi_k(s)\, ds = 0$ for all k, then $\varphi_k \equiv 0$ for all k and u would be identically zero, thus giving rise to a contradiction. Moreover, from (72), we have that $k_0 > 1$, and, by claim 3, $\varphi_k \equiv 0$ in $(0,1)$ for all $1 \leqslant k \leqslant k_0 - 1$. Repeating the same arguments we used above to prove (66), it is now possible to show that

$$\lim_{\rho \to 0^+} u(\rho\,\theta)\rho^{-\sigma_{k_0}^+} = \sum_{k=k_0}^{k_0 + m_{k_0} - 1} \left[c_1^k + \int_0^1 \frac{s^{-\sigma_k^+ + 1}}{\sigma_k^+ - \sigma_k^-} h(s)\varphi_k(s)\, ds \right] \psi_k(\theta), \tag{73}$$

where m_{k_0} is the geometric multiplicity of the eigenvalue μ_{k_0}. We notice that the sum at the right hand side is a nontrivial function in $L^2(\mathbb{S}^{N-1})$ which, being $k_0 > 1$, is orthogonal to the first positive eigenfunction ψ_1. Hence the right hand side of (73) changes sign in $\mathbb{S}^{N-1}$. Therefore the limit in (73) implies that u changes sign in a neighborhood of 0, which is in contradiction with the positivity assumption on u. Condition (70) follows and the proof of Proposition 4.1 is now complete. □

Remark 4.2. *Let us consider the function $f(\cdot, u, h, N, a) : (0, r] \to \mathbb{R}$, $s \mapsto f(s, u, h, N, a)$ given in (27). From the proof of proposition 4.1 and suitable change of variables, we have that*

$$f(s, u, h, N, a) = s^{-\sigma}\Psi_u(s) + \int_0^s \frac{t^{1-\sigma} - s^{-2\sigma - N + 2}\, t^{\sigma + N - 1}}{2\sigma + N - 2} h(t)\Psi_u(t)\, dt, \tag{74}$$

where

$$\Psi_u(t) := \int_{\mathbb{S}^{N-1}} u(t\,\theta)\psi_1(\theta)\, dV(\theta).$$

Arguing as we did in the proof of proposition 4.1 to deduce (43), we can infer that

$$\begin{aligned} \Psi_u(t) &= s^{-\sigma} t^{\sigma} \Psi_u(t) + s^{-2\sigma - N + 2} t^{\sigma} \int_0^s \frac{\rho^{\sigma + N - 1}}{-2\sigma - N + 2} h(\rho)\Psi_u(\rho)\, d\rho \\ &\quad + t^{\sigma} \int_t^s \frac{\rho^{1-\sigma}}{2\sigma + N - 2} h(\rho)\Psi_u(\rho)\, d\rho + t^{-\sigma - N + 2} \int_0^t \frac{\rho^{\sigma + N - 1}}{2\sigma + N - 2} h(\rho)\Psi_u(\rho)\, d\rho. \end{aligned} \tag{75}$$

From (74) and (75), direct calculations yield

$$\frac{d}{ds} f(s, u, h, N, a) = 0 \quad \textit{for all } s \in (0, r].$$

Hence the function $f(\cdot, u, h, N, a)$ is constant.

Remark 4.3. *If we let the assumption of positivity of u drop, following the proof of Proposition 4.1, we can still prove a Cauchy's integral type formula for u. More precisely, if $u \in H^1(B(0, R))$ is a weak solution to* (25) *in $B(0, R)$ which changes sign in any neighborhood of* 0, *with a radial* *potential* $h \in L^{\infty}_{\rm loc}(0,R) \cap L^p(0,R)$ *for some* $p > N/2$*, then, following the notation introduced in* (33) *and letting* $k_0 > 1$ *be the smallest index for which*

$$\int_{\mathbb{S}^{N-1}} \bigg(r^{-\sigma^+_{k_0}} u(r\eta) + \int_0^r \frac{s^{1-\sigma^+_{k_0}}}{\sigma^+_{k_0} - \sigma^-_{k_0}} h(s)u(s\,\eta)\,ds - r^{\sigma^-_{k_0}-\sigma^+_{k_0}} \int_0^r \frac{s^{1-\sigma^-_{k_0}}}{\sigma^+_{k_0} - \sigma^-_{k_0}} h(s)u(s\,\eta)\,ds \bigg) \psi_{k_0}(\eta)\,dV(\eta) \neq 0,$$

for any $\theta \in \mathbb{S}^{N-1}$ *and* $r \in (0,R)$ *there holds*

$$\lim_{\rho\to 0^+} u(\rho\,\theta)\rho^{-\sigma^+_{k_0}} = \sum_{\{k:\,\mu_k=\mu_{k_0}\}} \bigg[\int_{\mathbb{S}^{N-1}} \bigg(r^{-\sigma^+_k} u(r\eta) + \int_0^r \frac{s^{1-\sigma^+_k}}{\sigma^+_k - \sigma^-_k} h(s)u(s\,\eta)\,ds - r^{\sigma^-_k-\sigma^+_k} \int_0^r \frac{s^{1-\sigma^-_k}}{\sigma^+_k - \sigma^-_k} h(s)u(s\,\eta)\,ds \bigg) \psi_k(\eta)\,dV(\eta) \bigg] \psi_k(\theta).$$

Without the assumption of radial symmetry of the potential, it is still possible to evaluate the exact behavior near the singularity of the first Fourier coefficient φ_1 (see (29) and (30)).

Lemma 4.4. *Let* $a \in L^{\infty}(\mathbb{S}^{N-1})$ *be such that* $\Lambda_N(a) < 1$*,* $R > 0$*, and* $u \in H^1(B(0,R))$*,* $u \geqslant 0$ *a.e. in* $B(0,R)$*,* $u \not\equiv 0$*, be a weak* H^1*-solution to*

$$-\Delta u(x) = \bigg[\frac{a(x/|x|)}{|x|^2} + q(x) \bigg] u(x) \qquad \text{in } B(0,R),$$

where $q \in L^{\infty}_{\rm loc}\big(B(0,R) \setminus \{0\}\big) \cap L^p(B(0,R))$ *for some* $p > \frac{N}{2}$*. Then, for any* $0 < r < R$*,*

$$\lim_{\rho\to 0^+} \rho^{-\sigma} \int_{\mathbb{S}^{N-1}} u(\rho\,\theta)\psi_1(\theta)\,dV(\theta) = \int_{\mathbb{S}^{N-1}} \bigg(r^{-\sigma} u(r\,\theta) + \int_0^r \frac{s^{1-\sigma}}{2\sigma+N-2} q(s\,\theta)u(s\,\theta)\,ds - r^{-2\sigma-N+2} \int_0^r \frac{s^{N-1+\sigma}}{2\sigma+N-2} q(s\,\theta)u(s\,\theta)\,ds \bigg) \psi_1(\theta)\,dV(\theta). \tag{76}$$

Proof. The proof follows the lines of the first part of the proof of Proposition 4.1. By scaling, it is sufficient to prove (76) for $r = 1$. Let

$$u(x) = u(\rho\,\theta) = \sum_{k=1}^{\infty} \varphi_k(\rho)\psi_k(\theta) \quad \text{and} \quad q(x)u(x) = q(\rho\,\theta)u(\rho\,\theta) = \sum_{k=1}^{\infty} \zeta_k(\rho)\psi_k(\theta)$$

where $\rho = |x| \in (0,1]$, $\theta = x/|x| \in \mathbb{S}^{N-1}$,

$$\varphi_k(\rho) = \int_{\mathbb{S}^{N-1}} u(\rho\,\theta)\psi_k(\theta)\,dV(\theta)\,, \quad \zeta_k(\rho) = \int_{\mathbb{S}^{N-1}} q(\rho\,\theta)u(\rho\,\theta)\psi_k(\theta)\,dV(\theta),$$

and ψ_k is an L^2-normalized eigenfunction of the operator $-\Delta_{\mathbb{S}^{N-1}} - a(\theta)$ on the sphere associated to the k-th eigenvalue μ_k, i.e. satisfying (11). The first Fourier coefficient φ_1 solves

$$\varphi_1''(\rho) + \frac{N-1}{\rho}\varphi_1'(\rho) - \frac{\mu_1}{\rho^2}\varphi_1(\rho) = \zeta_1(\rho) \quad \text{in } (0,1).$$

A direct calculation shows that, for some $c_1^1, c_2^1 \in \mathbb{R}$,

$$\varphi_1(\rho) = \rho^{\sigma_1^+}\left(c_1^1 + \int_\rho^1 \frac{s^{-\sigma_1^+ +1}}{\sigma_1^+ - \sigma_1^-}\zeta_1(s)\,ds\right) + \rho^{\sigma_1^-}\left(c_2^1 + \int_\rho^1 \frac{s^{-\sigma_1^- +1}}{\sigma_1^- - \sigma_1^+}\zeta_1(s)\,ds\right),$$

where $\sigma_1^+ = -\frac{N-2}{2} + \sqrt{\left(\frac{N-2}{2}\right)^2 + \mu_1}$ and $\sigma_1^- = -\frac{N-2}{2} - \sqrt{\left(\frac{N-2}{2}\right)^2 + \mu_1}$. From Theorem 1.2 and standard regularity theory, we deduce that $u(x) \leqslant \text{const}\,|x|^{\sigma_1^+}$ in $B(0,1)$, hence, Hölder's inequality yields

$$\int_0^1 \left|s^{-\sigma_1^+ +1}\zeta_1(s)\right| ds \leqslant \int_0^1 s^{-\sigma_1^+ +1}\left(\int_{\mathbb{S}^{N-1}} |q(s\,\theta)| u(s\,\theta)\psi_1(\theta)\,dV(\theta)\right) ds \tag{77}$$
$$\leqslant \text{const} \int_0^1 s^{1-\frac{N-1}{p}}\left(\int_{\mathbb{S}^{N-1}} s^{\frac{N-1}{p}}|q(s\,\theta)|\,dV(\theta)\right) ds$$
$$\leqslant \text{const}\,\|q\|_{L^p(B(0,1))}\left(\int_0^1 s^{(1-\frac{N-1}{p})\frac{p}{p-1}}\,ds\right)^{1-\frac{1}{p}} < \infty.$$

In a similar way, we obtain that

$$s \mapsto \frac{s^{-\sigma_1^- +1}}{\sigma_1^- - \sigma_1^+}\,\zeta_1(s) \in L^1(0,1). \tag{78}$$

Since $u \in L^{2^*}(B(0,1))$, $\sigma_1^- < \sigma_1^+$, from (77), (78), and the fact that $\rho^{\sigma_1^-}\psi_1(\theta) \notin L^{2^*}(B(0,1))$, we conclude that there must be

$$c_2^1 = -\int_0^1 \frac{s^{-\sigma_1^- +1}}{\sigma_1^- - \sigma_1^+}\,\zeta_1(s)\,ds,$$

hence

$$\rho^{-\sigma_1^+}\varphi_1(\rho) = c_1^1 + \int_\rho^1 \frac{s^{-\sigma_1^+ +1}}{\sigma_1^+ - \sigma_1^-}\,\zeta_1(s)\,ds + \rho^{\sigma_1^- - \sigma_1^+}\int_0^\rho \frac{s^{-\sigma_1^- +1}}{\sigma_1^+ - \sigma_1^-}\,\zeta_1(s)\,ds \quad \text{for any } \rho \in (0,1). \tag{79}$$

Notice that, by standard regularity theory, φ_1 is continuous at $\rho = 1$, thus, letting $\rho \to 1^-$ in (79), we obtain

$$c_1^1 = \varphi_1(1) - \int_0^1 \frac{s^{-\sigma_1^- +1}}{\sigma_1^+ - \sigma_1^-}\,\zeta_1(s)\,ds. \tag{80}$$

Arguing as in (77), we obtain that

$$\rho^{\sigma_1^- - \sigma_1^+}\int_0^\rho \left|\frac{s^{-\sigma_1^- +1}}{\sigma_1^- - \sigma_1^+}\zeta_1(s)\right| ds \leqslant \text{const}\,\rho^{2-\frac{N}{p}}. \tag{81}$$

Since $p > \frac{N}{2}$, (79), (80), and (81) imply that

$$\lim_{\rho\to 0^+} \rho^{-\sigma_1^+}\varphi_1(\rho) = \varphi_1(1) - \int_0^1 \frac{s^{-\sigma_1^- +1}}{\sigma_1^+ - \sigma_1^-}\,\zeta_1(s)\,ds + \int_0^1 \frac{s^{-\sigma_1^+ +1}}{\sigma_1^+ - \sigma_1^-}\,\zeta_1(s)\,ds,$$

and (76) for $r = 1$ follows. The result in the case $r \neq 1$ can be easily obtained just by scaling. □

In order to extend the result of Proposition 4.1 to the case in which the potential is a non radial perturbation of the dipole-type singular term, we will construct a subsolution and a supersolution which solve equations of type (25) and the behavior of which is consequently known in view of Proposition 4.1.

Lemma 4.5. *Let $a \in L^\infty(\mathbb{S}^{N-1})$ be such that $\Lambda_N(a) < 1$, $C \in \mathbb{R}$, and $\varepsilon > 0$. Then, for all*

$$0 < r < \begin{cases} \left[\frac{(N-2)^2}{4C^+}(1-\Lambda_N(a))\right]^{1/\varepsilon}, & \text{if } C > 0, \\ +\infty, & \text{if } C \leqslant 0, \end{cases} \tag{82}$$

and for all $\gamma \in H^{1/2}(\partial B(0,r))$, $\gamma \geqslant 0$, $\gamma \not\equiv 0$, the Dirichlet boundary value problem

$$\begin{cases} -\Delta u(x) = \left[\dfrac{a(x/|x|)}{|x|^2} + C|x|^{-2+\varepsilon}\right] u(x), & \text{in } B(0,r), \\ u\big|_{\partial B(0,r)} = \gamma, & \text{on } \partial B(0,r), \end{cases} \tag{83}$$

admits a unique weak solution $u \in H^1(B(0,r))$. Moreover u is continuous and strictly positive in $B(0,r) \setminus \{0\}$, and there exists a positive constant C' depending on a, C, ε, N, and r, such that

$$\|u\|_{H^1(B(0,r))} \leqslant C' \|\gamma\|_{H^{1/2}(\partial B(0,r))}. \tag{84}$$

In addition, if $\gamma \in W^{2-1/k,k}(\partial B(0,r))$ for some $k > N/2$, then $u \in C^0(\overline{B(0,r)} \setminus \{0\})$.

Proof. For a fixed r satisfying (82) and $\gamma \in H^{1/2}(\partial B(0,r))$, $\gamma \geqslant 0$, $\gamma \not\equiv 0$, let $\tilde{v}$ be the unique $H^1(B(0,r))$-weak solution to the problem

$$\begin{cases} \tilde{v} \in H^1(B(0,r)), \\ -\Delta \tilde{v} = 0, & \text{in } B(0,r), \\ v = \gamma, & \text{on } \partial B(0,r). \end{cases}$$

By classical trace embedding theorems, it follows that

$$\|\tilde{v}\|_{H^1(B(0,r))} \leqslant \mathrm{const}(N,r)\|\gamma\|_{H^{1/2}(\partial B(0,r))}, \tag{85}$$

for some positive constant $\mathrm{const}(N,r)$ depending only on N and r. Let us define the quadratic form $a : H^1_0(B(0,r)) \times H^1_0(B(0,r)) \to \mathbb{R}$ as

$$\mathcal{Q}(w,u) := \int_{B(0,r)} \left[\nabla w(x) \cdot \nabla w(x) - \frac{1}{|x|^2}\big(a(x/|x|) + C|x|^\varepsilon\big) w(x) u(x)\right] dx,$$

and $\Phi \in H^{-1}(B(0,r))$ as

$${}_{H^{-1}}\big\langle \Phi, u \big\rangle_{H^1_0} := \int_{B(0,r)} \left(\frac{a(x/|x|)}{|x|^2} + \frac{C}{|x|^{2-\varepsilon}}\right) \tilde{v}(x) u(x)\, dx.$$

By Hardy's inequality, it is easy to verify that

$$\mathcal{Q}(u,u) \geqslant \left[1 - \Lambda_N(a) - \frac{4C^+ r^\varepsilon}{(N-2)^2}\right] \int_{B(0,r)} |\nabla u(x)|^2\, dx. \tag{86}$$

Since (82) implies that $\left[1 - \Lambda_N(a) - \frac{4C^+ r^\varepsilon}{(N-2)^2}\right] > 0$, we conclude that the bilinear bounded form $\mathcal{Q}$ is coercive. Furthermore, the function $x \mapsto a(x/|x|)|x|^{-2} + C|x|^{-2+\varepsilon}$ belongs to $L^{\frac{2N}{N+2}}(B(0,r))$, hence Φ is a bounded linear functional on $H^1_0(B(0,r))$. From the Lax-Milgram lemma we deduce that there exists a unique $w \in H^1_0(B(0,r))$ such that $\mathcal{Q}(w,u) = {}_{H^{-1}}\langle \Phi, u\rangle_{H^1_0}$ for all $u \in H^1_0(B(0,r))$. In particular w weakly solves

$$\begin{cases} -\Delta w(x) - \dfrac{1}{|x|^2}\Big[a(x/|x|) + C|x|^\varepsilon\Big] w(x) = \left[\dfrac{a(x/|x|)}{|x|^2} + \dfrac{C}{|x|^{2-\varepsilon}}\right]\tilde v(x), & \text{in } B(0,r),\\ w = 0, & \text{on } \partial B(0,r). \end{cases} \tag{87}$$

Testing the above equation with w and using (86), Poincaré's and Hölder's inequalities and (85), we obtain that

$$\|w\|_{H^1(B(0,r))} \leqslant c(a,C,\varepsilon,N,r)\|\tilde v\|_{H^1(B(0,r))} \leqslant c'(a,C,\varepsilon,N,r)\|\gamma\|_{H^{1/2}(\partial B(0,r))},$$

for some positive constants $c(\lambda,C,\varepsilon,N,r)$ and $c'(\lambda,C,\varepsilon,N,r)$ depending on a, C, ε, N, and r. It is now easy to verify that $u := w + \tilde v \in H^1(B(0,r))$ satisfies (84) and is the unique weak solution to (83). Moreover, testing (83) with $-u^- := -\max\{-u,0\}$ and using (86), we obtain that

$$0 = \mathcal{Q}(u^-,u^-) \geqslant \left[1 - \Lambda_N(a) - \frac{4C^+ r^\varepsilon}{(N-2)^2}\right]\int_{B(0,r)} |\nabla u^-(x)|^2\,dx,$$

which, in view of (82), implies that $u^- = 0$ a.e. in $B(0,r)$, i.e. $u \geqslant 0$ a.e in $B(0,r)$. The Strong Maximum Principle allows us to conclude that $u > 0$ in $B(0,r)\setminus\{0\}$, while standard regularity theory for elliptic equations ensures interior continuity of u outside the origin.

If, in addition, we assume that $\gamma \in W^{2-1/k,k}(\partial B(0,r))$ for some $k > N/2$, then $\tilde v \in W^{2,k}(B(0,r)$, and hence $\tilde v \in C^{0,\alpha}(\overline{B(0,r)})$, hence, from elliptic regularity theory applied to (87) outside 0, we obtain that $u \in C^0(\overline{B(0,r)}\setminus\{0\})$. □

Proof of Theorem 1.1. Let $R > 0$ such that $\overline{B(0,R)} \subset \Omega$. Since $q(x) = O(|x|^{-(2-\varepsilon)})$ as $|x| \to 0$ for some $\varepsilon > 0$ and $q \in L^\infty_{\rm loc}\big(\Omega\setminus\{0\}\big)$, there exists a positive constant $\tilde C$ such that $-\tilde C|x|^{-(2-\varepsilon)} \leqslant q(x) \leqslant \tilde C|x|^{-(2-\varepsilon)}$ for a.e. $x \in B(0,R)$. Let us fix $\bar r = \bar r(R,N,q,a,\varepsilon)$, such that $0 < \bar r < \min\Big\{R, \big[\frac{(N-2)^2}{4\tilde C}(1-\Lambda_N(a))\big]^{1/\varepsilon}\Big\}$. We notice that the Maximum Principle implies that $u > 0$ in $\overline{B(0,R)}\setminus\{0\}$, whereas standard elliptic regularity theory yields $u \in W^{2,k}(\overline{B(0,R)}\setminus B(0,s))$ for all $s \in (0,R)$ and some $k > N/2$, and, consequently, u is continuous in $\overline{B(0,R)}\setminus\{0\}$. Hence the function $\gamma_r := u\big|_{\partial B(0,r)}$ belongs to $W^{2-1/k,k}(\partial B(0,r))$ for some $k > N/2$ and is continuous and strictly positive on $\partial B(0,r)$ for all $0 < r \leqslant \bar r$. From Lemma 4.5 we deduce that, for any $0 < r \leqslant \bar r$, there exist $\underline{u}_r \in H^1(B(0,r))$ and $\bar u^r \in H^1(B(0,r))$ continuous and strictly positive in $\overline{B(0,r)}\setminus\{0\}$, weakly satisfying

$$\begin{cases} -\Delta \underline{u}_r(x) = \left[\dfrac{a(x/|x|)}{|x|^2} - \tilde C|x|^{-2+\varepsilon}\right]\underline{u}_r(x), & \text{in } B(0,r),\\ \underline{u}_r\big|_{\partial B(0,r)} = \gamma_r, & \text{on } \partial B(0,r), \end{cases}$$

and

$$\begin{cases} -\Delta \bar u_r(x) = \left[\dfrac{a(x/|x|)}{|x|^2} + \tilde C|x|^{-2+\varepsilon}\right]\bar u_r(x), & \text{in } B(0,r),\\ \bar u_r\big|_{\partial B(0,r)} = \gamma_r, & \text{on } \partial B(0,r). \end{cases}$$

From Proposition 4.1, there exist two constants $A_2 > A_1 > 0$ (depending on $\bar r$, N, q, a, ε, and u) such that

$$A_1|x|^\sigma \leqslant \underline{u}_{\bar r}(x) \quad \text{and} \quad \bar u_{\bar r}(x) \leqslant A_2|x|^\sigma, \qquad \text{for all } x \in B(0,\bar r/2)\setminus\{0\}. \tag{88}$$

Furthermore, for all $0 < r \leqslant \bar r$, $u - \underline{u}_r$ satisfies

$$\begin{cases} -\Delta(u-\underline{u}_r)(x) - \left[\dfrac{a(x/|x|)}{|x|^2} - \tilde C|x|^{-2+\varepsilon}\right](u-\underline{u}_r)(x) \geqslant 0, & \text{in } B(0,r),\\ (u-\underline{u}_r)\big|_{\partial B(0,r)} = 0, & \text{on } \partial B(0,r), \end{cases} \tag{89}$$

while $u - \bar u_r$ satisfies

$$\begin{cases} -\Delta(u-\bar u_r)(x) - \left[\dfrac{a(x/|x|)}{|x|^2} + \tilde C|x|^{-2+\varepsilon}\right](u-\bar u_r)(x) \leqslant 0, & \text{in } B(0,r),\\ (u-\bar u_r)\big|_{\partial B(0,r)} = 0, & \text{on } \partial B(0,r). \end{cases} \tag{90}$$

Testing (89), respectively (90), with $-(u-\underline{u}_r)^-$, respectively $(u-\bar u_r)^+$, and using (86), we obtain that, for any $0 < r \leqslant \bar r$,

$$\underline{u}_r(x) \leqslant u(x) \leqslant \bar u_r(x), \qquad \text{for all } x \in B(0,r)\setminus\{0\}. \tag{91}$$

In particular, from (88) and (91), we deduce that

$$A_1|x|^\sigma \leqslant u(x) \leqslant A_2|x|^\sigma, \qquad \text{for all } x \in B(0,\bar r/2)\setminus\{0\}. \tag{92}$$

Estimate (92) and the continuity of u outside the origin imply that there exists a positive constant C (depending on q, R, Ω, a, ε, and u) such that

$$\frac{1}{C}|x|^\sigma \leqslant u(x) \leqslant C|x|^\sigma \qquad \text{for all } x \in \overline{B(0,R)}\setminus\{0\}. \tag{93}$$

Let us now fix $\delta = \delta(N,a,\varepsilon) > 0$ such that

$$\delta < \min\Big\{\varepsilon, \sqrt{\big(\tfrac{N-2}{2}\big)^2 + \mu_1}\Big\}$$

and set

$$\hat r = \min\bigg\{\bar r, \Big[\tfrac{\delta}{\tilde C}\Big(2\sqrt{\big(\tfrac{N-2}{2}\big)^2+\mu_1} - \delta\Big)\Big]^{1/\varepsilon}, \Big(\tfrac{\min_{\mathbb S^{N-1}}\psi_1}{C}\Big)^{1/\delta}\bigg\},$$

with C given in (93). The function $\hat u$ defined as

$$\hat u(x) = |x|^{\sigma-\delta}\psi_1\big(x/|x|\big) \tag{94}$$

belongs to $H^1(B(0,R))$ and, for all $0 < r \leqslant \hat r$, satisfies

$$\begin{cases} -\Delta\hat u(x) - \dfrac{a(x/|x|)}{|x|^2}\,\hat u(x) = \delta\Big(2\sqrt{\big(\tfrac{N-2}{2}\big)^2+\mu_1} - \delta\Big)|x|^{-2}\,\hat u(x) \geqslant \tilde C\,|x|^{-2+\varepsilon}\,\hat u(x), & \text{in } B(0,r),\\ \hat u(x)\big|_{\partial B(0,r)} = r^{\sigma-\delta}\psi_1(x/r) \geqslant \bar u_r(x), & \text{on } \partial B(0,r). \end{cases}$$

Hence, for all $0 < r \leqslant \hat r$, $\hat u - \bar u_r$ satisfies

$$\begin{cases} -\Delta(\hat u-\bar u_r)(x) - \left[\dfrac{a(x/|x|)}{|x|^2}\, x\cdot\mathbf{d} + \tilde C|x|^{-2+\varepsilon}\right](\hat u-\bar u_r)(x) \geqslant 0, & \text{in } B(0,r),\\ (\hat u-\bar u_r)\big|_{\partial B(0,r)} \geqslant 0, & \text{on } \partial B(0,r). \end{cases}$$

Testing the above equation with $-(\hat u - \bar u_r)^-$ and using (86), we obtain that, for any $0 < r \leqslant \hat r$,

$$\bar u_r(x) \leqslant \hat u(x), \quad \text{for all } x \in B(0,r) \setminus \{0\}. \tag{95}$$

From Proposition 4.1, for any $0 < r \leqslant \hat r$, the functions

$$x \mapsto \frac{\underline{u}_r(x)}{|x|^\sigma \psi_1(x/|x|)} \quad \text{and} \quad x \mapsto \frac{\bar u_r(x)}{|x|^\sigma \psi_1(x/|x|)}$$

have limits as $|x| \to 0$, which, accordingly with (26–27) and taking into account the continuity of functions $\underline{u}_r$ and $\bar u_r$ up to the boundary $|x| = r$, can be computed as

$$\begin{aligned}\underline{L}_r := & \lim_{|x|\to 0} \frac{\underline{u}_r(x)}{|x|^\sigma \psi_1(x/|x|)} \\ = & \int_{\mathbb{S}^{N-1}} \bigg(r^{-\sigma} u(r\eta) - \tilde C \int_0^r \frac{s^{1-\sigma}}{2\sigma + N - 2}\, s^{-2+\varepsilon} \underline{u}_r(s\,\eta)\, ds \\ & + \tilde C r^{-2\sigma - N + 2} \int_0^r \frac{s^{N-1+\sigma}}{2\sigma + N - 2}\, s^{-2+\varepsilon} \underline{u}_r(s\,\eta)\, ds \bigg) \psi_1(\eta)\, dV(\eta),\end{aligned}$$

and

$$\begin{aligned}\bar L_r := & \lim_{|x|\to 0} \frac{\bar u_r(x)}{|x|^\sigma \psi_1(x/|x|)} \\ = & \int_{\mathbb{S}^{N-1}} \bigg(r^{-\sigma} u(r\eta) + \tilde C \int_0^r \frac{s^{1-\sigma}}{2\sigma + N - 2}\, s^{-2+\varepsilon} \bar u_r(s\,\eta)\, ds \\ & - \tilde C r^{-2\sigma - N + 2} \int_0^r \frac{s^{N-1+\sigma}}{2\sigma + N - 2}\, s^{-2+\varepsilon} \bar u_r(s\,\eta)\, ds \bigg) \psi_1(\eta)\, dV(\eta).\end{aligned}$$

From (91) and (93), it follows that

$$\underline{L}_r = r^{-\sigma} \int_{\mathbb{S}^{N-1}} u(r\eta)\psi_1(\eta)\, dV(\eta) + o(1) \quad \text{as } r \to 0.$$

From (95), (94), and the choice of δ, we obtain that

$$\bar L_r = r^{-\sigma} \int_{\mathbb{S}^{N-1}} u(r\eta)\psi_1(\eta)\, dV(\eta) + o(1) \quad \text{as } r \to 0.$$

Hence, from Lemma 4.4, we conclude that, for any R such that $\overline{B(0,R)} \subset \Omega$,

$$\begin{aligned}\lim_{r\to 0} \underline{L}_r = \lim_{r\to 0} \bar L_r = & \int_{\mathbb{S}^{N-1}} \bigg(R^{-\sigma} u(R\,\eta) + \int_0^R \frac{s^{1-\sigma}}{2\sigma + N - 2}\, q(s\,\eta) u(s\,\eta)\, ds \\ & - R^{-2\sigma - N + 2} \int_0^R \frac{s^{N-1+\sigma}}{2\sigma + N - 2}\, q(s\,\eta) u(s\,\eta)\, ds \bigg) \psi_1(\eta)\, dV(\eta).\end{aligned} \tag{96}$$

In view of (91), there holds that, for any $0 < r \leqslant \bar r$,

$$\begin{aligned}\underline{L}_r = & \lim_{|x|\to 0} \frac{\underline{u}_r(x)}{|x|^{a_{\mu_1}} \psi_1(x/|x|)} \leqslant \liminf_{|x|\to 0} \frac{u(x)}{|x|^{a_{\mu_1}} \psi_1(x/|x|)} \\ & \leqslant \limsup_{|x|\to 0} \frac{u(x)}{|x|^{a_{\mu_1}} \psi_1(x/|x|)} \leqslant \lim_{|x|\to 0} \frac{\bar u_r(x)}{|x|^{a_{\mu_1}} \psi_1(x/|x|)} = \bar L_r.\end{aligned}$$

Letting $r \to 0$ and using (96), we complete the proof. □

5. Behavior of solutions to the semilinear problem

The L^q and L^∞ bounds of solutions to dipole-type linear Schrödinger equations with properly summable potentials, derived in Theorems 1.2 and 3.2, allow us to obtain in the semilinear case analogous estimates.

Theorem 5.1. *Let Ω be a bounded domain containing 0, $a \in L^\infty(\mathbb{S}^{N-1})$ such that $\Lambda_N(a) < 1$, and $f : \Omega \times \mathbb{R} \to \mathbb{R}$ such that, for some positive constant C,*

$$\left| \frac{f(x,u)}{u} \right| \leqslant C\left(1 + |u|^{2^*-2}\right) \quad \text{for a.e. } (x,u) \in \Omega \times \mathbb{R}.$$

Then, for any $\Omega' \Subset \Omega$ and for any weak $H^1(\Omega)$-solution u of (7), there holds $\frac{u}{\varphi} \in L^\infty(\Omega')$.

Proof. Let $\Omega' \Subset \Omega$ and $u \in H^1(\Omega)$ be a weak $H^1(\Omega)$-solution to (7). We set

$$V(x) = \frac{f(x, u(x))}{\varphi^{2^*-2}(x)u(x)}$$

and notice that $u \in L^{2^*}(\Omega)$ yields

$$\int_\Omega \varphi^{2^*}(x)|V(x)|^{N/2}\, dx < +\infty.$$

Hence Theorem 3.2 implies that $\frac{u}{\varphi} \in L^q(\varphi^{2^*}, \Omega')$ for all $1 \leqslant q < +\infty$. Since

$$\int_\Omega \varphi^{2^*}(x)|V(x)|^s\, dx \leqslant \text{const}\left(1 + \int_\Omega \varphi^{2^*}(x)\left|\frac{u(x)}{\varphi(x)}\right|^{(2^*-2)s} dx\right),$$

we obtain that $V \in L^s(\varphi^{2^*}, \Omega)$ for all $s \geqslant \frac{N-2}{4}$. The conclusion follows now from Theorem 1.2. ☐

Proof of Theorem 1.3. From Theorem 5.1, it follows that $q(x) = \frac{f(x,u(x))}{u(x)}$ is such that $q \in L^\infty_{\rm loc}(\Omega \setminus \{0\})$ and $q(x) = O(|x|^{-(2-\varepsilon)})$ as $|x| \to 0$ for some $\varepsilon > 0$. Hence the conclusion follows from Theorem 1.1. ☐

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