

An optimal partition problem related to nonlinear eigenvalues *

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Abstract

We extend to the case of many competing densities the results of the paper [7]. More precisely, we are concerned with an optimal partition problem in N -dimensional domains related to the method of nonlinear eigenvalues introduced by Z. Nehari, [16]. We prove existence of the minimal partition and some extremality conditions. Moreover, in the case of two-dimensional domains we give an asymptotic formula near the multiple intersection points. Finally we show some connections between the variational problem and the behavior of competing species systems with large interaction.

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1 Introduction

Let $\Omega \in \mathbb{R}^N$, $N \geq 2$, be a bounded domain and $f : \mathbb{R} \rightarrow \mathbb{R}$ a superlinear function. Set $F(s) = \int_0^s f(t)dt$ and let us define, for $u \in H_0^1(\Omega)$, the functional

$$J^*(u) := \int_{\Omega} \left(\frac{1}{2} |\nabla u(x)|^2 - F(u(x)) \right) dx .$$

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With each open $\omega \in \Omega$ we associate the first nonlinear eigenvalue as:

$$\varphi(\omega) := \inf_{\substack{u \in H_0^1(\omega) \\ u > 0}} \sup_{\lambda > 0} J^*(\lambda u).$$

It is well known that $\varphi(\omega)$ is a critical value of the functional J^* over $H_0^1(\omega)$. Thus it corresponds to (at least) one *positive* solution u to the boundary value problem

$$\begin{cases} -\Delta u(x) = f(u(x)) & x \in \omega \\ u = 0 & x \in \partial\omega ; \end{cases}$$

u will be referred as eigenfunction associated to $\varphi(\omega)$.

In this paper we consider the problem of finding a partition of Ω (in open sets) that achieves

$$\inf \left\{ \sum_{i=1}^k \varphi(\omega_i) : \bigcup_{i=1}^k \bar{\omega}_i = \bar{\Omega}, \omega_i \cap \omega_j = \emptyset \text{ if } i \neq j \right\}. \quad (1)$$

Many free boundary problems can be formulated in terms of optimal partitions, and they are usually studied in the case $k = 2$ components. For instance, we quote [2] for applications to the flow of two liquids in models of jets and cavities. Moreover, optimal partition problems arise in linear eigenvalue theory and various fields of real analysis. For example, the most recent proof of the well known ‘‘monotonicity formula’’ relies upon a problem of optimal partition in two subsets related to the first eigenvalue of the Dirichlet operator on the sphere (see [3]).

Our interest in the variational problem (1) is motivated by the asymptotic analysis of solutions to superlinear variational systems with large competitive interaction. This kind of problems are connected to the study of the spatial segregation of biological species, which move by diffusion, as the interspecific competition rate tends to infinity; concerning the variational and topological approach to this kind of problems, we quote among others [6, 8, 9, 10, 11, 13, 14] and references therein.

The problem (1) in the case of $k = 2$ components has been studied by the authors in [7]. Aim of this paper is to extend part of the results there to the general case $k \geq 3$. In particular we shall establish the existence of an optimal partition $\{\omega_i\}_{i=1}^k$, at first in a relaxed sense. Then, if u_i is any (positive) eigenfunction related to $\varphi(\omega_i)$, we prove that the function $\sum_{i=1}^k u_i$ is lipschitz continuous; as a byproduct, the associated partition is open and hence it is a solution of (1). Furthermore, in dimension $N = 2$, we discuss some qualitative properties both of the eigenfunctions and of the free boundary $\partial\{x \in \Omega : \sum_{i=1}^k u_i(x) > 0\}$. In performing the local analysis at a multiple intersection point, a key role will be played by the already mentioned monotonicity formula in [2] together with some extensions developed in Section 4.

The paper is organized as follows: in Section 2 we shall prove the existence of the minimal partition; in the following Section 3 we shall give the basic extremality conditions fulfilled by the eigenfunctions u_i associated to the optimal partition. Section 4 is devoted to prove some suitable versions of the monotonicity formula that will find the first application in the proof of the lipschitz continuity of the minimizers in Section 5. Sections 6 and 7

contain some results on the local behavior of the eigenfunctions around multiple points of the free boundary. Finally in the Appendix we shall deepen the link between (1) and a class of superlinear variational systems; at first we shall give an existence result and then we shall perform the asymptotic analysis leading to our optimal partition problem.

2 The variational problem

In this section we prove the existence of a minimal partition to problem (1) in a weak sense. In fact, at first we will not find an open partition, but a partition made of sets which are supports of H_0^1 -functions, i.e., the eigenfunctions associated to our problem. Throughout all the paper, Ω will be a bounded domain in \mathbb{R}^N , with the additional property, in the results about points on $\partial\Omega$, of being of class C^2 .

Let f satisfy the following assumptions:

(f_1) $f \in C^1(\mathbb{R})$, $f(-s) = -f(s)$, and there exist positive constants C , p such that for all $s \in \mathbb{R}$

$$|f(s)| \leq C(1 + |s|^{p-1}) \quad 2 < p < 2^*,$$

where $2^* = +\infty$ when $N = 2$ and $2^* = 2N/(N - 2)$ when $N \geq 3$

(f_2) there exists $\gamma > 0$ ($2 + \gamma \leq p$) such that, for all $s \neq 0$,

$$f'(s)s^2 - (1 + \gamma)f(s)s > 0$$

Remark 2.1 It is well known that, when f is not an odd function,

$$\inf_{\substack{u \in H_0^1(\omega) \\ u \neq 0}} \sup_{\lambda > 0} J^*(\lambda u)$$

is achieved by a *one-sign* critical point of J^* (this functional is defined in the Introduction and recalled here below). If the infimum is restricted to the positive functions and then to the negative ones, it gives two possibly different critical levels and two correspondent critical points, one positive and one negative. To fix the ideas, in this paper we consider only positive critical points, i.e. positive values of s . Thus the assumption $f(-s) = -f(s)$ is not truly necessary: we can extend any other f , without loss of generality, to be an odd function.

Moreover, we can allow f to be x -dependent, although for the sake of simplicity we shall always refer to f as a function of s only.

Observe that, in a standard way, from assumptions (f_1) and (f_2) we can obtain the following properties for the primitive F of f :

$$F(s) \leq C(1 + |s|^p), \quad f(s)s - (2 + \gamma)F(s) \geq 0.$$

For $u \in H_0^1(\Omega)$ and $U := (u_1, \dots, u_k) \in (H_0^1(\Omega))^k$ we define the functionals

$$\begin{aligned} J^*(u) &:= \int_{\Omega} \left(\frac{1}{2} |\nabla u(x)|^2 - F(u(x)) \right) dx, \\ J(U) &:= \sum_{i=1}^k J^*(u_i), \end{aligned} \tag{2}$$

observing that

$$(u_i \cdot u_j = 0 \text{ a.e. on } \Omega \text{ for } i \neq j) \Rightarrow J(U) = J^*\left(\sum_{i=1}^k u_i\right).$$

Next we define the Nehari manifolds associated to these functionals:

$$\mathcal{N}(J^*) := \{u \in H_0^1(\Omega) : u \geq 0, u \not\equiv 0, \nabla J^*(u) \cdot u = 0\},$$

$$\mathcal{N}(J) := (\mathcal{N}(J^*))^k,$$

$$\mathcal{N}_0 := \mathcal{N}_J \cap \{u_i \cdot u_j = 0 \text{ a.e. on } \Omega \text{ for } i \neq j\}.$$

With this notation we introduce the problem we want to study in this paper:

$$\begin{aligned} c_0 &:= \inf \left\{ \sum_{i=1}^k \varphi(\omega_i) : \bigcup_{i=1}^k \overline{\omega_i} = \overline{\Omega}, \omega_i \cap \omega_j = \emptyset \text{ if } i \neq j \right\} = \\ &= \inf \left\{ \sum_{i=1}^k \sup_{\lambda_i > 0} J^*(\lambda_i u_i) : u_i \in H_0^1(\Omega), u_i \geq 0, u_i \not\equiv 0, u_i \cdot u_j = 0 \text{ a.e. on } \Omega \text{ for } i \neq j \right\} = \\ &= \inf \{J(U) : U \in \mathcal{N}_0\}. \end{aligned} \tag{3}$$

Observe that in the previous equality the first infimum is intended over all the partitions of Ω into subsets which are supports of $H_0^1(\Omega)$ -functions. In this sense, (3) is a relaxed reformulation of the initial problem (1) (for more details about the equivalent characterizations see [7]). A similar characterization, when $k = 2$ was also exploited in [5] when seeking changing sign solutions to superlinear problems.

Our first result concerns the existence of the optimal partition:

Theorem 2.1 *There exists (at least) a k -uple of functions $U := (u_1, \dots, u_k) \in \mathcal{N}_0$ such that*

$$-\Delta u_i(x) = f(u_i(x)) \quad x \in \text{supp}(u_i)$$

and U and their supports achieve c_0 .

To prove this theorem we need a preliminary lemma:

Lemma 2.1 *Let $u \in \mathcal{N}(J^*)$, $J^*(u) \leq c_0 + 1$. Then there exist positive constants C_1, C_2 such that $\|u\|_{H_0^1(\Omega)} \leq C_1$ and $\|u\|_{L^p} \geq C_2 > 0$.*

Proof: by assumptions we have $u \not\equiv 0$ and

$$\begin{aligned} \int_{\Omega} |\nabla u|^2 - f(u)u &= 0, \\ \int_{\Omega} \frac{1}{2} |\nabla u|^2 - F(u) &\leq c_0 + 1. \end{aligned}$$

Multiplying the first equation by $2 + \gamma$ (see Assumption (f_2)) and subtracting the second we easily obtain that $\int |\nabla u|^2$ is bounded. On the other hand, using the first equation, the superlinear properties of f and the Poincaré inequality we obtain the bound for $\|u\|_{L^p}$, $p < 2^*$. \blacksquare

Proof of Theorem 2.1: let us consider a minimizing sequence $U_n := (u_1^{(n)}, \dots, u_k^{(n)})$ in \mathcal{N}_0 . This means that $u_i^{(n)} \in \mathcal{N}(J^*)$ and hence the previous lemma applies, providing the existence of $(u_1^{(0)}, \dots, u_k^{(0)})$ both H_0^1 -weak limit and L^p -strong limit of a subsequence. Using again the previous lemma and L^p -convergence we deduce that $u_i^{(0)} \not\equiv 0$ for every i , and thus we can find positive constants λ_i , $i = 1, \dots, k$, such that

$$\sup_{\lambda > 0} J^*(\lambda u_i^{(0)}) = J^*(\lambda_i u_i^{(0)}).$$

Now we observe that, by weak convergence, $\|u_i^{(0)}\| \leq \liminf \|u_i^{(n)}\|$, and so

$$\sum_{i=1}^k J^*(\lambda_i u_i^{(0)}) \leq \sum_{i=1}^k J^*(\lambda_i u_i^{(n)}) + o(1) \leq \sum_{i=1}^k J^*(u_i^{(n)}) + o(1) \leq c_0.$$

On the other hand, for every i we have $\lambda_i u_i^{(0)} \in \mathcal{N}(J^*)$, and, by strong L^p -convergence, $u_i^{(0)} u_j^{(0)} = 0$ almost everywhere on Ω . Thus, by definition of c_0 ,

$$\sum_{i=1}^k J^*(\lambda_i u_i^{(0)}) \geq c_0.$$

Hence we have found a k -uple of functions that achieves the infimum. Now using the equivalent characterizations of (3) and standard critical point techniques the theorem follows. \blacksquare

3 The basic extremality conditions

In this section we discuss some basic variational inequalities satisfied by the eigenfunctions.

Lemma 3.1 *Let $U = (u_1, \dots, u_k)$ be as in Theorem 2.1. For a fixed index i define*

$$v_i = u_i - \sum_{\substack{j=1 \\ j \neq i}}^k u_j.$$

Then each v_i satisfies (in distributional sense) the inequality:

$$-\Delta v_i(x) \geq f(v_i(x)) \quad x \in \Omega.$$

Proof: we recall that i is fixed throughout the whole proof of the lemma. We wish to prove that

$$\int_{\Omega} (\nabla v_i \nabla \varphi - f(v_i) \varphi) \geq 0$$

for all $\varphi > 0$, $\varphi \in C_0^1(\Omega)$. Assume by contradiction the existence of $\varphi > 0$ such that the opposite inequality holds. We will obtain a contradiction constructing a k -uple in \mathcal{N}_0 that

decreases the value of 3. Let $\Lambda_i v_i := \lambda_i u_i - \sum_{j \neq i} \lambda_j u_j$ with $|\lambda_j - 1| \leq \delta$ for all j : if δ is small enough we can also assume by continuity that

$$\int_{\Omega} (\nabla \Lambda_i v_i \nabla \varphi - f(\Lambda_i v_i) \varphi) < 0. \quad (4)$$

By the inf-sup characterization of c_0 and by the behavior of the function $J^*(\lambda u)$ for fixed $u > 0$, we can take δ so small that

$$\nabla J^*((1 - \delta)u_j)u_j > 0 \quad \nabla J^*((1 + \delta)u_j)u_j < 0 \quad \forall j. \quad (5)$$

Let us fix $\bar{t} > 0$ small and let us consider a C^1 function $t : (\mathbb{R}^+)^k \rightarrow \mathbb{R}^+$ where $t(\lambda_1, \dots, \lambda_k) = 0$ if for at least one j it happens $|\lambda_j - 1| \geq \delta$, and $t(\lambda_1, \dots, \lambda_k) = \bar{t}$ if $|\lambda_j - 1| \leq \delta/2$ for every j . Next we define the continuous map

$$\Phi(\lambda_1, \dots, \lambda_k) = \lambda_i u_i - \sum_{\substack{j=1 \\ j \neq i}}^k \lambda_j u_j + t(\lambda_1, \dots, \lambda_k) \varphi.$$

Note that Φ^- is a positive function whose support is union of $k - 1$ disjoint connected components, each of them belonging to the support of some u_j . Now we define the function $\tilde{U}(\lambda_1, \dots, \lambda_k) = (\tilde{u}_1, \dots, \tilde{u}_k)$ in such a way that $\tilde{u}_j = \Phi^-(\lambda_1, \dots, \lambda_k)$ restricted to $\text{supp}(u_j)$, $j \neq i$ and $\tilde{u}_i = \Phi^+$. Let us compute $J(\tilde{U})$:

$$\begin{aligned} J(\tilde{U}) &= \int (\frac{1}{2} |\nabla \Lambda_i v_i|^2 + \frac{t^2}{2} |\nabla \varphi|^2) dx + \int (t \nabla \Lambda_i v_i \nabla \varphi - F(\Lambda_i v_i + t \varphi)) dx \leq \\ &J(\lambda_1 u_1, \dots, \lambda_k u_k) + \bar{t} \int (\nabla \Lambda_i v_i \nabla \varphi - f(\Lambda_i v_i) \varphi) dx + o(\bar{t}). \end{aligned}$$

By (4) and taking \bar{t} small enough, this implies

$$J(\tilde{U}(\lambda_1, \dots, \lambda_k)) < J(\lambda_1 u_1, \dots, \lambda_k u_k)$$

if $|\lambda_j - 1| \leq \delta/2$ for every j .

Now, if \bar{t} is small, we can assume that (5) holds for $\tilde{U}(\lambda_1, \dots, \lambda_k)$ instead of $(\lambda_1 u_1, \dots, \lambda_k u_k)$. Thus by continuity there exists (μ_1, \dots, μ_k) such that $|\mu_i - 1| \leq \delta/2$ and

$$\nabla J(\tilde{U}(\mu_1, \dots, \mu_k)) \cdot \tilde{U}(\mu_1, \dots, \mu_k) = 0$$

that means $\tilde{U}(\mu_1, \dots, \mu_k) \in \mathcal{N}_0$. But this is in contradiction with the definition of U as in Theorem 2.1 and the fact that

$$J(\tilde{U}(\mu_1, \dots, \mu_k)) < J(\mu_1 u_1, \dots, \mu_k u_k) \leq J(U) = \inf_{V \in \mathcal{N}_0} J(V).$$

■

Lemma 3.2 *Let $U = (u_i, \dots, u_k)$ be as in Theorem 2.1. Then each u_i satisfies (in distributional sense) the inequality:*

$$-\Delta u_i(x) \leq f(u_i(x)) \quad x \in \Omega.$$

Proof: again i is fixed. Assume by contradiction the existence of $\varphi > 0$, $\varphi \in C_0^1(\Omega)$ such that

$$\int_{\Omega} (\nabla \lambda_i u_i \nabla \varphi - f(\lambda_i u_i) \varphi) > 0 \quad (6)$$

for all λ_i such that $|\lambda_i - 1| \leq \delta$, δ small enough. As in the proof of Lemma 3.1 we can assume δ so small that (5) holds and we consider the function $t(\lambda_i)$ analogous to the one introduced therein. Then we let $\Phi(\lambda_i) := \lambda_i u_i - t(\lambda_i) \varphi$ and we define \tilde{U} with components $\tilde{u}_i = \Phi^+$, $\tilde{u}_j = u_j$ if $j = 1, \dots, k-1$, $j \neq i$ and finally $\tilde{u}_k(x) = u_k(x)$ if $x \in \text{supp}(u_k)$, $\tilde{u}_k(x) = \Phi^-$ if $x \in \text{supp}(u_i) \cap \text{supp}(\Phi^-)(x)$. By computing $J(\tilde{U}(\lambda_i))$ we obtain

$$\begin{aligned} J(\tilde{U}) &= \int \left(\frac{1}{2} |\nabla(\lambda_i u_i - t\varphi)|^2 - F(\lambda_i u_i - t\varphi) \right) dx + \sum_{j \neq i} J^*(u_j) = \\ &= J(u_1, \dots, \lambda_i u_i, \dots, u_k) + \int \frac{t^2}{2} |\nabla \varphi|^2 dx - \bar{t} \int (\nabla \lambda_i u_i \nabla \varphi - f(\lambda_i u_i) \varphi) dx + o(\bar{t}). \end{aligned}$$

By (6) and taking \bar{t} small enough, this implies

$$J(\tilde{U}(\lambda_i)) < J(u_1, \dots, \lambda_i u_i, \dots, u_k)$$

if $|\lambda_i - 1| \leq \delta/2$.

Now a contradiction with the properties of U as in Theorem 2.1 can be obtained by arguing as in the final step of the proof of Lemma 3.1. \blacksquare

4 Monotonicity formulae

Let us now recall the *monotonicity lemma* in [2], Lemma 5.1.

Lemma 4.1 (The monotonicity formula) *Let $(w_1, w_2) \in (H^1(\Omega))^2$ be non negative subharmonic functions in a ball $B(x_0, \bar{r}) \subset \Omega$ (i.e. $-\text{div} \nabla w_i \leq 0$ in distributional sense). Assume that $w_1(x)w_2(x) = 0$ a.e.. Assume that $x_0 \in \partial(\text{supp}(w_i))$ for $i = 1, 2$. Define*

$$\Phi(r) = \prod_{i=1}^2 \frac{1}{r^2} \int_{B(x_0, r)} \frac{|\nabla w_i(x)|^2}{|x - x_0|^{N-2}} dx.$$

Then $\Phi < +\infty$ and it is a non decreasing function in $[0, \bar{r}]$.

The fact that Φ is finite is shown in the proof of Lemma 2.1 of [4]. We will prove an extension of this formula to the case of $h \geq 2$ subharmonic components when Ω is a subset of \mathbb{R}^2 :

Lemma 4.2 *Let $N = 2$ and let $w_1, \dots, w_h \in H^1(\Omega)$ be non negative subharmonic functions in a ball $B(x_0, \bar{r}) \subset \Omega$ (i.e. $-\text{div} \nabla w_i \leq 0$ in distributional sense). Assume that $w_i(x)w_j(x) = 0$ a.e. if $i \neq j$ and that $x_0 \in \partial(\text{supp}(w_j)) \cap \Omega$ for all $j = 1, \dots, h$. Define*

$$\Phi(r) = \prod_{i=1}^h \frac{1}{r^h} \int_{B(x_0, r)} |\nabla w_i(x)|^2 dx.$$

Then Φ is a non decreasing function in $[0, \bar{r}]$.

Proof: by computing $\Phi'(r)$ we obtain

$$\begin{aligned} \Phi'(r) &= -\frac{h^2}{r^{h^2+1}} \prod_{i=1}^h \int_{B(x_0, r)} |\nabla w_i(x)|^2 dx + \\ &+ \frac{1}{r^{h^2}} \sum_{i=1}^h \left[\left(\prod_{j \neq i} \int_{B(x_0, r)} |\nabla w_j(x)|^2 dx \right) \int_{\partial B(x_0, r)} |\nabla w_i(x)|^2 dx \right] = \\ &\Phi(r) \left(-\frac{h^2}{r} + \sum_{i=1}^h \frac{\int_{\partial B(x_0, r)} |\nabla w_i(x)|^2}{\int_{B(x_0, r)} |\nabla w_i(x)|^2} \right). \end{aligned} \quad (7)$$

Since each w_i is positive and subharmonic, testing with w_i on the sphere we obtain, for all i :

$$\int_{B(x_0, r)} |\nabla w_i|^2 \leq \int_{\partial B(x_0, r)} w_i \frac{\partial}{\partial n} w_i.$$

Now we pass in polar coordinates. We write $w_i = w_i(r, \theta)$, and $\partial_r, \partial_\theta$ for the derivatives. We have, by the previous inequality,

$$\begin{aligned} \int_{B(x_0, r)} |\nabla w_i|^2 &\leq r \int_0^{2\pi} w_i \partial_r w_i \leq \\ &\leq r \left(\int_0^{2\pi} w_i^2 \right)^{1/2} \left(\int_0^{2\pi} (\partial_r w_i)^2 \right)^{1/2}. \end{aligned}$$

Now let us introduce $\Gamma_i(r) = \text{supp}(w_i(r, \cdot)) \subset [0, 2\pi]$, $\sum_{i=1}^h |\Gamma_i(r)| = 2\pi$ and note that

$$\inf \frac{\int_{\Gamma_i(r)} |\partial_\theta w_i|^2}{\int_{\Gamma_i(r)} |w_i|^2} \geq \left(\frac{\pi}{|\Gamma_i(r)|} \right)^2.$$

Hence we can go on with the previous chain of inequalities as follows:

$$\begin{aligned} \int_{B(x_0, r)} |\nabla w_i|^2 &\leq r \frac{\Gamma_i(r)}{\pi} \left(\int_0^{2\pi} r (\partial_\theta w_i)^2 \right)^{1/2} \left(\int_0^{2\pi} \frac{1}{r} (\partial_r w_i)^2 \right)^{1/2} \leq \\ &\leq r \frac{\Gamma_i(r)}{2\pi} \left(\int_0^{2\pi} r (\partial_\theta w_i)^2 + \int_0^{2\pi} \frac{1}{r} (\partial_r w_i)^2 \right) \leq \\ &\leq r \frac{\Gamma_i(r)}{2\pi} \int_{\partial B(x_0, r)} \|\nabla w_i\|^2. \end{aligned}$$

We obtain

$$\sum_{i=1}^h \frac{\int_{\partial B(x_0, r)} |\nabla w_i(x)|^2}{\int_{B(x_0, r)} |\nabla w_i(x)|^2} \geq \frac{1}{r} \sum_{i=1}^h \frac{2\pi}{\Gamma_i(r)} \geq \frac{h^2}{r}.$$

Comparing with (7) the thesis follows. ■

Unfortunately, these lemmas do not apply directly in our situation, because the functions u_i 's are not subharmonic, but only subsolutions of a superlinear equation (see Lemma 3.2). This property is not sufficient to guarantee the monotonicity of the function $\Psi(r)$, but only its boundedness as $r \rightarrow 0$.

Lemma 4.3 *Let $U = (u_1, \dots, u_k)$ as in Theorem 2.1 and let $w_i = \sum_{j \in I_i} u_j$, where $I_1 \cup \dots \cup I_h \subset \{1, \dots, k\}$. Assume that $x_0 \in \partial(\text{supp}(w_i))$. Then the following holds:*

1. If $h = 2$ then

$$\prod_{i=1}^2 \frac{1}{r^2} \int_{B(x_0, r)} \frac{|\nabla w_i(x)|^2}{|x - x_0|^{N-2}} dx \leq C. \quad (8)$$

2. If $N = 2$ then for all $h \geq 2$

$$\prod_{i=1}^h \frac{1}{r^h} \int_{B(x_0, r)} |\nabla w_i(x)|^2 dx \leq C. \quad (9)$$

Proof: let us consider a small ball $B(x_0, r)$ centered at x_0 and the eigenvalue problem

$$\begin{cases} -\Delta \varphi(x) = a\varphi(x) & x \in B(x_0, r) \\ \varphi(x) > 0 & x \in B(x_0, r) \end{cases}$$

where $a = \sup\left\{\left|\frac{f(w_i(x))}{w_i(x)}\right|, x \in B(x_0, r)\right\}$. The existence of a positive solution φ to the problem, where φ is radial with respect to x_0 , is ensured if r is small enough. Then let us consider

$$\tilde{u}_i(x) = \frac{u_i(x)}{\varphi(x)} \quad x \in B(x_0, r'), \quad r' < r.$$

By elementary computation it holds

$$-\varphi^2 \Delta \tilde{u}_i - 2\varphi \nabla \tilde{u}_i \nabla \varphi \leq 0$$

that means

$$-\operatorname{div}(\varphi^2(\nabla \tilde{u}_i)) \leq 0.$$

Now consider $w_i = \sum_{I_i} u_i$, where $I_1 \cup \dots \cup I_h \subset \{1, \dots, k\}$ and let $\tilde{w}_i := w_i/\varphi$: then $-\operatorname{div}(\varphi^2(\nabla \tilde{w}_i)) \leq 0$. Let $h = 2$: following the proof of The Monotonicity Lemma 4.1, it is easy to obtain that the function

$$\prod_{i=1}^2 \frac{1}{r^2} \int_{B(x_0, r)} \varphi^2(x) \frac{|\nabla w_i(x)|^2}{|x - x_0|^{N-2}} dx$$

is increasing in the r variable on $[0, r']$.

By this formula and since there exist positive $a < b$ such that $a < \varphi(x) < b$ for all $x \in B(x_0, r')$, it follows in particular that (8) holds.

The same kind of construction together with the application of Lemma 4.2 when $N = 2$, allows to prove (9) for all $h \geq 0$. ■

Finally we state suitable version of the lemma above which holds for x_0 on the boundary of Ω :

Lemma 4.4 *Let $\Omega \subset \mathbb{R}^N$ be a regular subset of class C^2 and let $U = (u_1, \dots, u_k)$ as in Theorem 2.1. If $x_0 \in \partial\Omega$, then*

$$\frac{1}{r^N} \int_{B(x_0, r)} |\nabla U|^2 dx \leq C$$

where C is continuous with respect to $x_0 \in \partial\Omega$.

Proof: (sketch) let us follow the proof of the monotonicity lemma in [2] with the formal substitution $u^+ = u^- = \sum_{i=1}^k u_i$, $\Gamma_i = (\Gamma_i(r) =) \text{supp}(u^\pm) \cap B(x_0, r)$. Then, since by the regularity of the boundary $\frac{|\Gamma_i(r)|}{|\partial B(x_0, r)|} = \frac{1}{2} + o(1)$ as $r \rightarrow 0$, the arguments there allow to prove that

$$\Phi(r) = \prod_{i=1}^2 \frac{1}{r^2} \int_{B(x_0, r)} \frac{|\nabla U|^2}{|x - x_0|^{N-2}} dx$$

is bounded. ■

5 Regularity of U

With some abuse of notation, in the following two sections U will denote both the vector (u_1, \dots, u_k) and the sum of its k components (and the same for V , W , and so on).

Let us define the set of zeroes of U as

$$\mathcal{Z}(U) = \{x \in \Omega : U(x) = 0 \text{ i.e. } u_i(x) = 0 \forall i = 1, \dots, k\}$$

and define the *multiplicity* of $x \in \mathcal{Z}(U)$ as the number $m(x)$:

$$m(x) = \#\{i : \text{meas}(\{u_i > 0\} \cap B(x, r)) > 0 \forall r > 0\}.$$

Note that $m(x) \geq 2$ for all $x \in \mathcal{Z}(U)$: indeed, by Theorem 2.1 and the maximum principle, we know that $u_i(x) > 0$ if $x \in \text{supp}(u_i)$.

Remark 5.1 Let $x \in \mathcal{Z}(U)$ such that $m(x) = 2$ and consider $B(x, r)$ for any $r < d(x, \{y \in \mathcal{Z}(U) : m(y) \geq 3\})$. Then, by Lemma 3.1 it follows that the function $u = u_1 - u_2$ (resp. $u_2 - u_1$) is a solution of $-\Delta u = f(u)$ on $B(x, r)$.

In the following we shall denote by $\mathcal{Z}_3(U) = \{x \in \mathcal{Z}(U) : m(x) \geq 3\}$.

Theorem 5.1 (Local Lipschitz Continuity) *Let U be as in Theorem 2.1. Then U is Lipschitz continuous in the interior of Ω .*

Proof: let Ω' be compactly enclosed in Ω . Consider

$$\Gamma(r) = \sup_{x \in \Omega'} \frac{1}{r^N} \int_{B(x, r)} |\nabla U|^2.$$

Our thesis is equivalent to the boundedness of Γ over Ω' . Assume not; then there are sequences (x_n) in Ω' and $r_n \rightarrow 0$ such that

$$\lim_{n \rightarrow +\infty} \frac{1}{r_n^N} \int_{B(x_n, r_n)} |\nabla U|^2 = +\infty. \quad (10)$$

Note that $d(x_n, \mathcal{Z}_3(U)) \rightarrow 0$ by Remark 5.1.

Claim 0. There is a sequence $z_n \in \mathcal{Z}_3(U)$ such that (10) is satisfied with ball centered at z_n and radius of order r_n .

Let $r > 0$ be fixed and consider $A_r = \{x \in \Omega : d(x, \mathcal{Z}_3(U)) \geq r\}$ (note that, by Remark 5.1, in A_r we can give alternate positive and negative sign to the u_i 's in such a way that the resulting function locally solves our differential equation).

Then let us define

$$\Phi(x) = \frac{1}{r^N} \int_{B(x,r)} |\nabla U(y)|^2 dy = \frac{1}{r^N} \int_{B(0,r)} |\nabla U(x+y)|^2 dy.$$

By elementary computations it turns out that $-\Delta(|\nabla U(x)|^2) \leq 2f'(U(x))|\nabla U(x)|^2$ and thus $-\Delta\Phi \leq a_r\Phi$ on A_r , for some positive constant a_r depending on r .

Up to subsequences, we can assume the existence of $z_0 \in \mathcal{Z}_3(U)$ such that $x_n \in B(z_0, \rho)$ for any $n > N_\rho$. We fix ρ so small that $-\Delta\varphi = a_r\varphi$ has a strictly positive solution φ on $B(z_0, \rho)$, which is radially symmetric with respect to z_0 . Then it holds $-\operatorname{div}(\varphi^2 \nabla \frac{\Phi}{\varphi}) \leq 0$ on $A_{r,\rho} := B(z_0, \rho) \cap A_r$, and by the maximum principle it holds $\max_{A_{r,\rho}} \frac{\Phi}{\varphi} \leq \max_{\partial A_{r,\rho}} \frac{\Phi}{\varphi}$. Thus there exists $C > 0$ (independent of r) such that $\max_{A_{r,\rho}} \Phi \leq C \max_{\partial A_r} \Phi$; from this we obtain that, at fixed n , (10) holds for a choice of x'_n such that $d(x'_n, \mathcal{Z}_3(U)) \asymp r_n$; finally let us consider $z_n \in \mathcal{Z}_3(U)$ such that $|z_n - x'_n| = d(x'_n, \mathcal{Z}_3(U))$. Then (10) holds for balls centered at z_n and radius $r_n + d(x'_n, \mathcal{Z}_3(U)) \asymp r_n$.

Claim 1. There is a sequence (denoted again by r_n) satisfying (10) and moreover

$$\int_{\partial B(z_n, r_n)} |\nabla U|^2 \leq \frac{\gamma}{r_n} \int_{B(z_n, r_n)} |\nabla U|^2$$

where γ only depends on the distance of Ω' to $\partial\Omega$.

Let n be fixed large enough and set $U_n = U$, $z_n = x$ and $r_n = \delta$. For the sake of simplicity we assume that $d(\Omega', \partial\Omega) = 1$. Let $\gamma > N$ and assume by contradiction that

$$r^\gamma \frac{d}{dr} \left(\frac{1}{r^\gamma} \int_{B(x,r)} |\nabla U|^2 \right) = \int_{\partial B(x,r)} |\nabla U|^2 - \frac{\gamma}{r} \int_{B(x,r)} |\nabla U|^2 > 0 \quad (11)$$

for all $\delta < r < 1$. By assumption (10) the function

$$g(r) = \frac{1}{r^N} \int_{B(x,r)} |\nabla U|^2$$

is such that $g(\delta) > g(1)$. Then, if we let

$$\tilde{g}(r) = \frac{1}{r^{\gamma-N}} \int_{B(x,r)} |\nabla U|^2$$

it turns out that $\tilde{g}(\delta) > g(\delta) > g(1) = \tilde{g}(1)$. As a consequence there exists $r \in (\delta, 1)$ such that $\frac{d}{dr} \tilde{g}(r) < 0$, in contradiction with (11).

By the monotonicity lemma and (8), we infer

$$\prod_{i=1}^2 \frac{1}{r^2} \int_{B(z_n, r)} \frac{|\nabla v_i(x)|^2}{|x - z_n|^{N-2}} dx \leq C$$

where $C > 0$ (independent of n) for all $v_1 := u_i, v_2 := \sum_{j \neq i} u_j$, such that $z_n \in \partial \text{supp}(u_i) \cap \partial \text{supp}(u_l)$ for some $l \neq i$. Consequently (since $|x - z_n| \leq r$ for all x)

$$\frac{1}{r^N} \int_{B(z_n, r)} |\nabla v_1(x)|^2 dx \cdot \frac{1}{r^N} \int_{B(z_n, r)} |\nabla v_2(x)|^2 dx \leq C.$$

Then it follows from assumption (10) that there exists only one component, say u_1 , such that

$$\frac{1}{r_n^N} \int_{B(z_n, r_n)} |\nabla u_1(x)|^2 dx \rightarrow \infty, \quad \frac{1}{r_n^N} \int_{B(z_n, r_n)} |\nabla u_i(x)|^2 dx \rightarrow 0 \quad \forall i \neq 1. \quad (12)$$

Now consider the sequence of functions

$$U_n(x) = \frac{1}{L_n r_n} U(z_n + r_n x)$$

defined in $x \in B(0, 1)$, where

$$L_n^2 := \frac{1}{r_n^N} \int_{B(z_n, r_n)} |\nabla U(x)|^2 dx \quad (\rightarrow \infty).$$

Then $\int_{B(0,1)} |\nabla U_n|^2 = 1$ and, by Claim 1, $\int_{\partial B(0,1)} |\nabla U_n|^2$ is bounded too. Then there exists $U_0 \in H^1(B(0, 1))$ such that, up to a subsequence, $U_n \rightharpoonup U_0$ weakly and $U_n \rightarrow U_0$ strongly in $H^{1/2}(\partial B(0, 1))$. Moreover, by (12) we deduce that $\int_{B(0,1)} |\nabla u_i^{(n)}|^2 \rightarrow 0$ for all indices $i \geq 2$ and thus $u_i^{(n)} \rightarrow 0$ strongly in $H^1(B(0, 1))$ for all $i \geq 2$.

Claim 2. $U_0(x) \neq 0$.

Let us remark that, by the very definition, for $x \in B(0, 1)$ it holds

$$-\Delta u_i^{(n)}(x) = \frac{r_n}{L_n} (-\Delta u_i(z_n + r_n x)) \leq \frac{r_n}{L_n} f(u_i(z_n + r_n x)) \leq a r_n^2 u_i^{(n)}(x),$$

where $a = \sup f(u_i(x))/u_i(x)$. Let us multiply by $u_i^{(n)}$ and integrate on $B(0, 1)$:

$$\int_{B(0,1)} |\nabla u_i^{(n)}|^2 \leq \int_{\partial B(0,1)} u_i^{(n)} \frac{\partial}{\partial \nu} u_i^{(n)} + r_n^2 \int_{B(0,1)} a (u_i^{(n)})^2.$$

Let $i = 1$ and then pass to the limit as $n \rightarrow \infty$: then the left-hand side converges to 1. Furthermore, since $r_n \rightarrow 0$ and by the strong convergence in $L^p(B(0, 1))$ ($p > 1$) and in $L^2(\partial B(0, 1))$ we obtain $\int_{\partial B(0,1)} u_1^{(0)} \frac{\partial}{\partial \nu} u_1^{(0)} \geq 1$ and thus $u_1^{(0)} > 0$ on a set of positive measure.

Claim 3. $u_1^{(0)}$ is harmonic and thus $\{x \in \Omega : U_0(x) > 0\} = B(0, 1)$.

Let us recall that for all $x \in B(0, 1)$, it holds

$$-\Delta u_i^{(n)}(x) \leq ar_n^2 u_i^{(n)}(x)$$

and analogously, setting $v_i^{(n)}(x) = u_i^{(n)}(x) - \sum_{i \neq j} u_j^{(n)}(x)$, we know that

$$-\Delta v_i^{(n)}(x) \geq -ar_n^2 |v_i^{(n)}(x)|.$$

Then, passing to the limit and taking into account that $r_n \rightarrow 0$, $L_n \rightarrow \infty$ and $u_i^{(n)} \rightarrow 0$ in $H^1(B) \forall i \geq 2$, we finally obtain $\Delta v_1^{(0)} = \Delta u_1^{(0)} = 0$.

Claim 4. $v_1^{(n)} \geq v_1^{(0)} + o(1)$ where $o(1) \rightarrow 0$ in the H^1 -topology.

Since

$$-\Delta(v_1^{(n)} - v_1^{(0)}) \geq 0$$

then we can write $v_1^{(n)} - v_1^{(0)} = a_n + b_n$ where a_n is such that $-\Delta a_n = 0$ in $B(0, 1)$ with boundary conditions $a_n = v_1^{(n)} - v_1^{(0)}$, and b_n vanishes on the boundary and satisfies $-\Delta b_n \geq 0$. Then, by the maximum principle, $b_n > 0$ in $B(0, 1)$; furthermore $v_1^{(n)} - v_1^{(0)} \rightarrow 0$ in $H^{\frac{1}{2}}(\partial B(0, 1))$ implies $a_n \rightarrow 0$ in $H^1(B(0, 1))$ and now the result follows.

Final step. Now the thesis follows since the convergence is uniform on almost every circle $\partial B(0, r)$. Fix such a radius r : then there exists N such that $v_1^{(n)} > 0$ in $\partial B(0, r)$ for all $n > N$. Since $v_1^{(n)}$ is supersolution by Lemma 3.1, then we can compute

$$0 = v_1^{(n)}(0) \geq C_r \int_{B(0, r)} v_1^{(n)}(x) dx > 0,$$

a contradiction. ■

Furthermore, if Ω is sufficiently regular, then U inherits regularity up to the boundary, as stated in the following

Theorem 5.2 (Global Lipschitz Continuity) *Let $\Omega \subset \mathbb{R}^N$ be a regular set of class C^2 and let U be as in Theorem 2.1. Then U is Lipschitz continuous in $\bar{\Omega}$.*

Proof: in the following we set U defined on \mathbb{R}^N by setting $u_{k+1} \equiv 0$ on \mathbb{R}^N . Arguing by contradiction we assume that there are sequences (x_n) in Ω and $r_n \rightarrow 0$ such that

$$\lim_{n \rightarrow +\infty} \frac{1}{r_n^N} \int_{B(x_n, r_n)} |\nabla U|^2 = +\infty \quad (13)$$

with $d(x_n, \partial\Omega) \rightarrow 0$ by the local Lipschitz continuity.

With the same kind of arguments as in the proof of Theorem 5.1 (see Claim 0 there) we can assume that $x_n \in \partial\Omega$ and, up to subsequences, $x_n \rightarrow x_0 \in \partial\Omega$. But now a contradiction with (13) comes by directly applying Lemma 4.4 to x_0 . ■

6 Further properties in dimension $N = 2$.

Aim of this section is to give a detailed description of the local features of the free boundary in the two-dimensional case. The first property of the free boundary concerns with the local behavior of $\mathcal{Z}(U)$ about the points of multiplicity 2.

Lemma 6.1 *Let $N = 2$, U be as in Theorem 2.1 and let $x_0 \in \mathcal{Z}(U)$ such that $m(x_0) = 2$. Then $\nabla U(x_0) \neq 0$ and $\mathcal{Z}(U)$ is locally a C^1 -curve through x_0 .*

Proof: let $x_0 \in \partial\{u_1 > 0\} \cap \partial\{u_2 > 0\}$, so that there exists $B(x_0)$ such that $B(x_0) \cap \text{supp}(u_i) = \emptyset$ for all $i > 2$. It follows by Remark 5.1 and by standard regularity results for PDEs that $u = u_1 - u_2$ is a $C^1(B(x_0))$ -function and $\text{supp}(u_i) \cap B(x_0)$ is open, $i = 1, 2$. By the Dini Theorem, if $\nabla U(x_0) \neq 0$ then the set $\{x \in B(x_0) : u = 0\}$ is a regular curve. Hence let by contradiction $\nabla U(x_0) = 0$; u is C^1 and satisfies the equation $-\Delta u = a(x)u$ (with $a(x) = f(u(x))/u(x)$). Following the same argument of the proof of Lemma 4.3 we consider a positive and regular function φ such that $-\Delta\varphi = a(x)\varphi$ on $B(x_0)$ (such a function clearly exists when B is sufficiently small). So we have that u/φ solves the second order equation

$$-\text{div} \left(\varphi^2 \nabla \left(\frac{u}{\varphi} \right) \right) = 0.$$

But this means that u/φ satisfies all the assumptions necessary to apply the main theorem in [1], which says that the null level set of u near x_0 is made up by a finite number of curves starting from x_0 . Obviously in our situation such number must be even. Now recall that each $\text{supp}(u_i)$ is connected in Ω : by a geometrical argument we can see that the null level set is made up by (two semi-curves joining in) one C^1 -curve. But again applying [1] we have $\nabla U(x_0) \neq 0$, a contradiction. ■

Now let us consider a point $x_0 \in \mathcal{Z}(U)$ with multiplicity higher than 2. Then x_0 is a singular point for U as stated in the following:

Lemma 6.2 *Let $N = 2$ and let U be as in Theorem 2.1. If $x_0 \in \mathcal{Z}(U)$ is such that $m(x_0) \geq 3$, then $|\nabla U(x)| \rightarrow 0$ as $x \rightarrow x_0$.*

Proof: assume by contradiction the existence of $x_n \rightarrow x_0$ and $r_n \rightarrow 0$ such that

$$\lim_{n \rightarrow +\infty} \frac{1}{r_n^2} \int_{B(x_n, r_n)} |\nabla U|^2 = C \quad (14)$$

for some positive C . The arguments here follow the line in the proof of Theorem 5.1. First, we can assume that (14) holds with balls of radii r_n and center at $x_n \in \mathcal{Z}_3(U)$ (as in Claim 0 therein). Again, with an argument similar to that in Claim 1 and exploiting the lipschitz continuity of U , we can assume

$$\int_{\partial B(x_n, \rho_n)} |\nabla U|^2 \leq \frac{3}{\rho_n} \int_{B(x_n, \rho_n)} |\nabla U|^2$$

where $\rho_n = (2L/C)r_n$ and L is the Lipschitz constant in $\Omega' = \{x \in \Omega : d(x, \partial\Omega) \geq d(x_0, \partial\Omega)/2\} \supset \{x_n\}$. (We denote ρ_n again by r_n .)

By the extended monotonicity lemma (Lemma 4.2) and the corresponding remark (9) with $h = 3$, there exists $C > 0$ (independent of n) such that

$$\prod_{i=1}^3 \frac{1}{r^3} \int_{B(x_n, r)} |\nabla v_i(x)|^2 dx \leq C$$

for all $v_1 := u_i$, $v_2 := u_j$ and $v_3 := \sum_{h \notin \{i, j\}} u_h$, such that $x_n \in \partial \text{supp}(u_i) \cap \partial \text{supp}(u_j) \cap \partial \text{supp}(u_l)$ for some $l \notin \{i, j\}$. Then, due to assumption (14), we deduce that there exist at most two components, say u_1 and u_2 , such that

$$\lim_{n \rightarrow \infty} \frac{1}{r_n^2} \int_{B(x_n, r_n)} |\nabla u_i(x)|^2 dx > 0, \quad i = 1, 2 \quad \frac{1}{r_n^2} \int_{B(x_n, r_n)} |\nabla u_i(x)|^2 dx \rightarrow 0 \quad \forall i \geq 3. \quad (15)$$

Now consider the sequence of functions

$$U_n(x) = \frac{1}{L_n r_n} U(x_n + r_n x)$$

defined in $x \in B(0, 1)$ where

$$L_n^2 := \frac{1}{r_n^N} \int_{B(z_n, r_n)} |\nabla U(x)|^2 dx \quad (\rightarrow \infty).$$

Then $\int_{B(0,1)} |\nabla U_n|^2 = 1$ and, by Claim 1, $\int_{\partial B(0,1)} |\nabla U_n|^2$ is bounded too. Then there exists $U_0 \in H^1(B(0, 1))$ such that, up to a subsequence, $U_n \rightharpoonup U_0$ weakly and $U_n \rightarrow U_0$ strongly in $H^{1/2}(\partial B(0, 1))$. And by (15) we deduce that $\int_{B(0,1)} |\nabla u_i^{(n)}|^2 \rightarrow 0$ for all indices $i \geq 3$ and thus $u_i^{(n)} \rightarrow 0$ strongly in $H^1(B(0, 1))$ for all $i \geq 3$. Now two cases may occur: either $\int_{B(0,1)} |\nabla u_1^{(n)}|^2 \rightarrow 1$ (resp. $\int_{B(0,1)} |\nabla u_2^{(n)}|^2 \rightarrow 1$), or $\int_{B(0,1)} |\nabla u_i^{(n)}|^2 \rightarrow \alpha_i > 0$ for both $i = 1, 2$, with $\alpha_1 + \alpha_2 = 1$. In the first case we obtain the thesis by following exactly the same reasoning of Theorem 5.1 from Claim 2 on.

Here below, assume that the second case hold.

Claim 1. $U_0(x) \not\equiv 0$.

Let us note that for all $x \in B(0, 1)$, it holds

$$-\Delta u_i^{(n)}(x) = \frac{r_n}{L_n} (-\Delta u_i(z_n + r_n x)) \leq \frac{r_n}{L_n} f(u_i(z_n + r_n x)) \leq a r_n^2 u_i^{(n)}(x),$$

where again $a = \sup f(u_i(x))/u_i(x)$. Let us multiply by $u_i^{(n)}$ and integrate on $B(0, 1)$:

$$\int_{B(0,1)} |\nabla u_i^{(n)}|^2 \leq \int_{\partial B(0,1)} u_i^{(n)} \frac{\partial}{\partial \nu} u_i^{(n)} + r_n^2 \int_{B(0,1)} a (u_i^{(n)})^2.$$

When $i = 1$ or $i = 2$, in passing to the limit as $n \rightarrow \infty$ the left-hand side converges to $\alpha_i > 0$. Furthermore $\int_{\partial B(0,1)} u_i^{(0)} \frac{\partial}{\partial \nu} u_i^{(0)} \geq 1$ and thus $u_i^{(0)} > 0$ for $i = 1, 2$, both on a set of positive measure.

Claim 2. $v_1^{(0)} = u_1^{(0)} - u_2^{(0)}$ (resp. $v_2^{(0)}$) is harmonic.

Let us recall that for all $x \in B(0,1)$, it holds

$$-\Delta u_i^{(n)}(x) \leq \alpha r_n^2 u_i^{(n)}(x)$$

and analogously, setting $v_i^{(n)}(x) = u_i^{(n)}(x) - \sum_{i \neq j} u_j^{(n)}(x)$, we know that

$$-\Delta v_i^{(n)}(x) \geq -\alpha r_n^2 |v_i^{(n)}(x)|.$$

Then, passing to the limit and taking into account that $r_n \rightarrow 0$, $L_n \rightarrow C$ and $u_i^{(n)} \rightarrow 0$ in $H^1(B)$, $i \geq 3$, we obtain $\Delta u_1^{(0)} - \Delta u_2^{(0)} = \Delta v_1^{(0)} \geq 0$ and $\Delta u_2^{(0)} - \Delta u_1^{(0)} = \Delta v_2^{(0)} \geq 0$, giving $\Delta u_1^{(0)} = \Delta u_2^{(0)}$ and finally the proof of the claim.

Rightly in the same way as in Theorem 5.1 it can be proven:

Claim 3. $v_i^{(n)} \geq v_i^{(0)} + o(1)$, $i = 1, 2$ where $o(1) \rightarrow 0$ in the H^1 -topology.

Now, if $v_1^{(0)}(0) \neq 0$, then either $v_1^{(0)} > 0$ or $v_2^{(0)} > 0$ in $B(0, \bar{r})$ for some $\bar{r} > 0$ and we can obtain the final contradiction as in the Final Step in the proof of Theorem 5.1.

Thus, assuming that $0 \in \mathcal{Z}(v_1^{(0)})$, standard results on harmonic functions (see [12]) imply that $v_1^{(0)}(r, \theta) \sim r^p \cos(p(\theta + \theta_0))$ for some $p \geq 1$. Thus, by Claim 3 and a diagonal process, we can assume that, for n large enough, $u_i^{(n)}$ ($i = 1, 2$) is larger than a fixed positive constant m_i on some $\omega_{n,i} = \{(\rho, \theta) : r < \rho < R, \alpha_{n,i} < \theta < \beta_{n,i}\} \subset \text{supp}(u_i^{(n)})$ (we can think for instance $\beta_{n,1} < \alpha_{n,2}$); note that $\alpha_{n,2} - \beta_{n,1} \rightarrow 0$ as $n \rightarrow \infty$. Now, since 0 is a zero of U_n with multiplicity $m(0) \geq 3$, there exists a third component, say $u_3^{(n)}$, and a continuous path $\gamma_n : [0, 1] \rightarrow B(0, R)$ such that $\gamma_n(0) = 0$, $\gamma_n(1) \in \partial B(0, R)$, $\gamma_n(t) \in \{\beta_{n,1} < \theta < \alpha_{n,2}\}$, $u_3^{(n)}(\gamma_n(t)) > 0$ for all $t \in (0, 1]$. Therefore, denoting with $\tilde{\omega}_{n,i}$, $i = 1, 2$, the connected component of $\text{supp}(u_i^{(n)})$ that contains $\omega_{n,i}$ we have that $\tilde{\omega}_{n,1}$ and $\tilde{\omega}_{n,2}$ are locally disjoint. More precisely, setting $S_n = \{(\rho, \theta) : r < \rho < R, \alpha_{n,1} < \theta < \beta_{n,2}\}$ we obtain that $\tilde{\omega}_{n,i} \cap S_n$, $i = 1, 2$, have positive distance.

Now we define the sequence of affine transformations (T_n) given by

$$T_n(x) = r_n x_n + z_n$$

For $x \in T_n(S_n)$ we define

$$\begin{aligned} w_i^{(n)}(x) &:= u_i(x) & x \in T_n(\tilde{\omega}_{n,i}) & & i = 1, 2 \\ w_i^{(n)}(x) &:= -u_i(x) & x \in \text{supp}(u_i) & & i = 3, \dots, k-1 \\ w_k^{(n)}(x) &:= \begin{cases} -u_k(x) & x \in \text{supp}(u_k) \\ -u_i(x) & x \in \text{supp}(u_i) \setminus T_n(\tilde{\omega}_{n,i}) \end{cases} & & & i = 1, 2. \end{aligned}$$

With the usual abuse of notation the symbol W_n denotes both the vector $(w_1^{(n)}, \dots, w_k^{(n)})$ and the sum of its components.

Claim 4. $-\Delta W_n \geq f(W_n)$ in $T_n(S_n)$.

Let n be fixed and let us drop the dependence on n . Assume by contradiction the existence of $\varphi > 0$, $\varphi \in C_0^1(\Sigma_n)$ such that

$$\int_{T(S)} (\nabla W \nabla \varphi - f(W)\varphi) < 0.$$

We proceed as in the proof of Lemma 3.1, deforming U over $T(S)$, projecting it on the Nehari manifold and obtaining a new k -uple that decreases the infimum (3).

We define $\Lambda W := \lambda_1 w_1 + \lambda_2 w_2 + \sum_{j=3}^k \lambda_j w_j$ with $|\lambda_j - 1| \leq \delta$ for all j : if δ is small enough we have by assumption

$$\int_{T(S)} (\nabla(\Lambda W) \nabla \varphi - f(\Lambda W)\varphi) < 0.$$

Again we can take δ so small that

$$\nabla J^*((1 - \delta)u_j)u_j > 0 \quad \nabla J^*((1 + \delta)u_j)u_j < 0 \quad \forall j.$$

Let us fix $\bar{t} > 0$ and let us consider the C^1 -function $t : (\mathbb{R}^+)^k \rightarrow \mathbb{R}^+$ already introduced in the proof of Lemma 3.1; then define the continuous map

$$\Phi(\lambda_1, \dots, \lambda_k)(x) = \lambda_1 w_1(x) + \lambda_2 w_2(x) + \sum_{j=3}^k \lambda_j w_j(x) + t(\lambda_1, \dots, \lambda_k)\varphi(x) \quad x \in T(S).$$

Note that

$$\text{supp}(((\Phi(\lambda_1, \dots, \lambda_k))^-) \subset \text{supp}(\Lambda W)^- \tag{16}$$

and

$$\text{supp}((\Lambda W)^+) \subset \text{supp}(((\Phi(\lambda_1, \dots, \lambda_k))^+).$$

and, if \bar{t} is small enough, the right-hand side contains at least two disjoint components ω_i^* , $i = 1, 2$, that are supersets of $T(\tilde{\omega}_1)$ and $T(\tilde{\omega}_2)$ respectively.

We are ready to define the partition of Ω that will provide the desired contradiction.

We can define a map $U^*(\lambda_1, \dots, \lambda_k)(x)$ setting $U^*(\lambda_1, \dots, \lambda_k)(x) := |\Phi(x)(\lambda_1, \dots, \lambda_k)|$ when $x \in T(S)$ and $U^*(\lambda_1, \dots, \lambda_k)(x) := \sum \lambda_i u_i(x)$ otherwise. Now consider all the connected components of $\text{supp}(U^*)$: with each of them we want to associate an index i . Such association must be continuous with respect to the parameters λ_i .

This procedure will define the supports of the components of U^* . Every connected component of $\text{supp}(U^*)$ that is not completely contained in $T(S)$ naturally inherits an index by the original partition. Also every connected component of $\text{supp}(\Phi^-)$ completely contained in $T(S)$ inherits an index by the original partition using (16) and recalling that W and U differ only in the sign. Finally, we have to prescribe an index to the connected component of $\text{supp}(\Phi^+)$ completely contained in $T(S)$. To this aim we consider the maximal function

$$\bar{\Phi}(x) := \sup_{|\lambda_j - 1| \leq \delta} \Phi(\lambda_1, \dots, \lambda_k)(x).$$

It is clear that every connected component of $\text{supp}(\Phi^+)$ is contained in a connected component of $\text{supp}(\bar{\Phi}^+)$, and therefore it will be sufficient to prescribe the index law for them. Those not completely contained in $T(S)$ naturally inherit an index by the original partition. We put the other ones in $\text{supp}(u_1^*)$. It is easy to see that U^* is continuous.

Now we compute $J(U^*)$ and, with arguments similar to those in the proof of Lemma 3.1, we obtain a contradiction with Theorem 2.1.

Final Step. Now fix $t_n > 0$ and $R > r_n > r$ in such a way that, if we set $y_n = \gamma_n(t_n)$ then it holds $\int_{\partial B(y_n, r_n)} W_n > 0$. (This is certainly possible since $\partial B(y_n, r_n) \subset \omega_{n,1} \cup \omega_{n,2}$ except for a small arc of total length less than $2R(\alpha_{n,2} - \beta_{n,1}) \rightarrow 0$ as $n \rightarrow \infty$.) This immediately leads to a contradiction thanks to Claim 4. Indeed $W_n(y_n) = -u_3^{(n)}(y_n) < 0$ and since W_n is superharmonic,

$$W_n(y_n) \geq C_{r_n} \int_{\partial B(y_n, r_n)} W_n > 0,$$

a contradiction. ■

7 Local properties of the free boundary in dimension $N = 2$

We start this section by proving an asymptotic formula describing the behavior of $\sum u_i$ in the neighborhood of an isolated multiple point. Next we show that all the multiple points are indeed isolated.

Lemma 7.1 *Let $N = 2$ and let U be as in Theorem 2.1. Let $x_0 \in \mathcal{Z}(U)$ with $m(x_0) = h \geq 3$, and assume that x_0 is isolated in $\mathcal{Z}_3(U)$. Then there exist $h \leq p \leq 2h - 2$ and $\theta_0 \in (-\pi, \pi]$ such that*

$$U(r, \theta) = r^{\frac{p}{2}} |\cos(\frac{p}{2}(\theta + \theta_0))| + o(r^{\frac{p}{2}})$$

as $r \rightarrow 0$, where (r, θ) denotes polar coordinates around x_0 .

Proof: since by assumption x_0 is isolated in $\mathcal{Z}_3(U)$, there is $B = B(x_0, \rho)$ such that $\mathcal{Z}_3(U) \cap B = \{x_0\}$ and $\text{supp}(u_i) \cap \partial B = \cup_{j \in I_i} b_j$, where b_j are consecutive arcs on ∂B and I_i is finite for all i . Choosing a slightly smaller radius we can suppose that the intersection of ∂B with \mathcal{Z} is transversal. Let $M = \sum_i |I_i|$ and assume that M is even: then define a function $v(r, \theta)$ such that $|v(r, \theta)| = |u(r, \theta)|$ and $\text{sign}(v(r, \theta)) = (-1)^j$ if $(\rho, \theta) \in b_j$, $j = 1, \dots, M$. Note that the resulting function is alternately positive and negative on the consecutive (with respect to θ) local components of U . If on the contrary M is odd, we define $|v(r, \theta)| = |u(r^2, 2\theta)|$ and we prescribe an alternating sign to the local components of $u(r^2, 2\theta)$. It is worthwhile noticing that the resulting function v is of class C^1 in $\tilde{B} = B(\rho^2, x_0)$, provided ρ is small enough. Indeed we can assume that none of the supports is completely contained in B and, therefore, $\text{supp}(u_i) \cap B$ is simply connected for every i . In this way each connected component of $B \setminus u^{-1}(0)$ corresponds to two components of $\tilde{B} \setminus v^{-1}(0)$ to which we give opposite sign. In both the even and

the odd cases v is of class C^1 and it solves an equation of type $-\Delta v = a(x)v$ in $B \setminus \{x_0\}$ (resp. $\tilde{B} \setminus \{x_0\}$), where $a \in L^\infty$ is given by $f(v)/v$ and $r^2 f(v)/v$ respectively. Moreover, by Lemma 6.2, we know that $\nabla v(x_0) = 0$: this implies that v is in fact solution of the equation on the whole of B (resp. \tilde{B}) and thus it is of class $C^{2,\alpha}$. Now, following the same argument of the proof of Lemma 4.3, we consider a positive and regular function φ such that $-\Delta\varphi = a\varphi$ on B , if r is small enough. The result is that

$$-\operatorname{div} \left(\varphi^2 \nabla \left(\frac{v}{\varphi} \right) \right) = 0.$$

Then, we can apply to v/φ the asymptotic formula of Hartman and Winter as recalled in [12]. To complete the proof, let us define p in the following way: let d denote the *local* multiplicity of x_0 as critical point of v ; define $p = 2d$ in the even case and $p = d$ in the odd case. In this way p represents the number of connected components of $\cup \operatorname{supp}(u_i)$ in a ball centered in x_0 . By elementary geometrical considerations, taking into account that the $\operatorname{supp}(u_i)$'s are connected, one can easily obtain the desired bound on p . ■

As we mentioned before, the relevance of Lemma 7.1 lays on the fact that, as we are going to prove, all the multiple points are isolated.

Theorem 7.1 *\mathcal{Z}_3 consists of isolated points.*

The last part of this section is devoted to the proof of Theorem 7.1 through a sequence of intermediate results.

To start with, we investigate the structure of both the sets of the double and that of the multiple intersection points \mathcal{Z}_3 . We denote by $\mathcal{Z}^2 = \mathcal{Z} \setminus \mathcal{Z}_3$ the set of the points having multiplicity two. As an easy consequence of Lemma 6.1 we can state the following result:

Lemma 7.2 *Let Ω' be open, compactly included in Ω . Assume that $\overline{\Omega'} \cap \mathcal{Z}_3 = \emptyset$. Then $\mathcal{Z}^2 \cap \Omega'$ is a finite disjoint union of C^1 -arcs.*

Let ω_i denote the supports $\operatorname{supp}(u_i)$; we take an index pair (i, j) such that $\partial\omega_i, \partial\omega_j$ do intersect and we consider

$$\begin{aligned} \Gamma_{i,j} &= \partial\omega_i \cap \partial\omega_j \cap \mathcal{Z}^2 \\ \omega_{i,j} &= \omega_i \cup \omega_j \cup \Gamma_{i,j}. \end{aligned}$$

Proposition 7.1 *Each $\omega_{i,j}$ is open and $\pi_1(\omega_{i,j})$ is finitely generated.*

Proof: at first we prove that $\omega_{i,j}$ is open. Indeed, let x_0 be a point of multiplicity two: then, thanks to Lemma 6.1, $\Gamma_{i,j}$ is locally a regular arc. Hence x_0 is in the interior of $\omega_{i,j}$. Since both ω_i and ω_j are open, we obtain the first claim. Now we consider a loop γ in $\omega_{i,j}$. Let Ω'_γ denote the bounded region delimited by γ . Assuming that Ω'_γ contains some multiple intersection points, we deduce, from the connectedness of the supports, that at least one third support, say ω_h , lies in Ω'_γ . Since the total number of the supports is k we easily complete the proof, taking into account that the boundary of Ω is assumed to be regular. ■

Proposition 7.2 *Each $\Gamma_{i,j}$ consists in a finite union of C^1 -arcs.*

Proof: as we showed in the previous proposition, $\omega_{i,j}$ has a finite number of “holes” (closed sets that we will name H_α , $\alpha \in A$). We consider also the connected components of $\partial\Omega$ as holes. Take a connected component $\tilde{\Gamma} \subset \Gamma_{i,j}$ as a parameterized curve. There are only two possibilities: either the curve connects two different holes, or it is doubly asymptotic to the same hole H_α . In the last case we easily deduce that $\tilde{\Gamma} \cup H_\alpha$ disconnects the plane. Being ω_i and ω_j connected, one easily sees that actually $\tilde{\Gamma} = \Gamma_{i,j}$. Otherwise, we observe, reasoning in a similar way, that only two connected components of $\Gamma_{i,j}$ can connect the same pair of holes. In every case we obtain a finite number of connected components. ■

An straightforward consequence of the above discussion is the following:

Lemma 7.3 *The set \mathcal{Z}_3 has a finite number of connected components.*

Now we will need the following definition of adjacent supports:

Definition 7.1 *We say that ω_i and ω_j are adjacent if*

$$\Gamma_{i,j} \neq \emptyset.$$

Let us list some basic properties:

- 1 Every ω_i is adjacent to some other ω_j . This follows from the Boundary Point Lemma.
- 2 Let us pick k points $x_i \in \omega_i$, $i = 1, \dots, k$. If ω_i and ω_j are adjacent, $i < j$, then there exists a smooth arc γ_{ij} with $\gamma_{ij}(0) = x_i$, $\gamma_{ij}(1) = x_j$ lying in $\omega_i \cup \omega_j \cup \{y_{ij}\}$, for some $y_{ij} \in \mathcal{Z}_2$.
- 3 We can choose the arcs γ_{ij} in a manner that they are mutually disjoint, except for the extreme points.

Construction of an auxiliary function v

We call \mathcal{G} the graph induced by the arcs γ_{ij} and their endpoints. There are many possibilities:

- If \mathcal{G} has no loops we can prescribe a sign to each vertex in such a way that to adjacent supports there correspond opposite signs. Therefore, defining $v(x) = \pm u(x)$, taking the correct sign rule we obtain a C^1 -function.

- Otherwise, let us define an order relation between loops, according whether one is contained in the interior region of the other. Let us select a *minimal* loop γ (no other loops are contained in its interior region). If the number of vertex of γ is even we can manage to assign a sign law to all the subset of \mathcal{G} contained in the interior of γ so that adjacent supports have opposite sign. Finally define $v(x) = \pm u(x)$, according to this law.

If the number of vertex of γ is odd, we wish to “double” the loop, by considering its square root (in complex sense). To this aim, we can assume that it contains, in its interior, at least an element of \mathcal{Z}_3 (if not, we can perform a conformal inversion exchanging the inner with the outer points).

Take this point as the origin and define new ω_i 's by taking the complex square roots of the old ones. In this way the new loop γ will have an even number of edges. A little problem may be caused by those supports which are not simply connected; to overcome this, we may operate suitable cuts (having care of not disconnecting the supports). In this way we can assume that all the ω_i involved in this procedure are simply connected; with some abuse, we shall call Ω also the cut domain. The new subgraph can now carry a sign law which is compatible with the adjacency relation. Defining, according to this sign law, $v(r, \theta) = \pm u(r^2, 2\theta)$, we obtain again a function of class C^1 , excepted possibly at the cuts.

Lemma 7.4 *The points of \mathcal{Z}_3 lying in the interior of a minimal loop γ are isolated.*

Proof: by construction, $-\Delta v = a(x)v$, where either $a = f(v)/v$, or $a = r^2 f(v)/v$, in $\Omega \setminus \mathcal{Z}_3$. Moreover, by its construction, v is of class C^1 in Ω . We are going to prove that v is actually a solution of $-\Delta v = a(x)v$, in a distributional sense, over the whole of Ω . This will complete the proof of the lemma; indeed, reasoning as in the proof of Lemma 7.1, we would deduce that the origin, which was arbitrarily chosen among the points of \mathcal{Z}_3 in the interior region of γ , is an isolated critical point of v ; moreover one can easily see that the multiple point of the interior part of γ is unique.

To carry over this program, we need some more results:

Lemma 7.5 *It holds*

$$\mathcal{Z}_3 \subset \partial \mathcal{Z}^2.$$

Proof: assume not, then there would be an element y_0 of \mathcal{Z}_3 having a positive distance d from the set $\mathcal{Z} \setminus \mathcal{Z}_3$ of double intersection points. Let $r < d/2$: then the ball $B(y_0, r)$ intersects at least three supports; therefore there exist, say, $x \in \omega_i$ and $z_0 \in \mathcal{Z}_3$ such that $\rho = d(x, z_0) = d(x, \mathcal{Z}_3) < d(x, \mathcal{Z} \setminus \mathcal{Z}_3)$. Now, since v solves a linear elliptic PDE and does not change sign in ω_i , the ball $B(x, \rho)$ being tangent from the interior of ω_i to \mathcal{Z}_3 in z_0 , we infer from the Boundary Point Lemma that $\nabla v(z_0) \neq 0$, in contrast with Lemma 6.2. ■

Lemma 7.6 *Let $\mathcal{Z}_3^\varepsilon = \mathcal{Z}^2 \cap B_\varepsilon(\mathcal{Z}_3)$. For every $\varepsilon > 0$ there exists $\delta > 0$ such that*

$$\int_{\mathcal{Z}_3^\varepsilon} |\nabla v| < \delta.$$

Proof: by testing the equation with the test function $\varphi = 1$ and integrating over the set $u_i > \alpha$ we obtain the bound, independent of α and i ,

$$\int_{\partial\{u_i > \alpha\}} |\nabla v| < C$$

and therefore, passing to the limit as $\alpha \rightarrow 0$,

$$\int_{\partial\{u_i > 0\}} |\nabla v| < C.$$

The assertion then follows from Lemma 7.5, together with Lemma 7.3. ■

Proposition 7.3 v solves $-\Delta v = a(x)v$ in the whole of Ω .

Proof: in our settings, one easily sees that \mathcal{Z}_3 is connected and that it is the limit set of the $\Gamma_{i,j}$'s, for the pairs of indices involved in the loop γ . Thanks to Lemma 7.5, for any ε we can take a neighborhood $\mathcal{V}_\varepsilon \subset B_\varepsilon(\mathcal{Z}_3)$ of \mathcal{Z}_3 , in such a way that the boundary $\partial\mathcal{V}_\varepsilon$ is the union of a finite number of arcs of \mathcal{Z}^2 and supplementary union of pieces of total length smaller than $C\varepsilon$. Let φ be a test function. We write

$$\begin{aligned} & \int_{\Omega} (\nabla v \nabla \varphi - a(x)v\varphi) = \\ &= \int_{\Omega \setminus \mathcal{V}_\varepsilon} (\nabla v \nabla \varphi - a(x)v\varphi) + \int_{\mathcal{V}_\varepsilon} (\nabla v \nabla \varphi - a(x)v\varphi) \leq \\ & \leq C \int_{\partial\mathcal{V}_\varepsilon} |\nabla v| + C \int_{\mathcal{V}_\varepsilon} (|\nabla v| + |v|). \end{aligned}$$

Let $\delta > 0$: we can find $\varepsilon > 0$ such that Lemma 7.6 holds. Moreover, from Lemma 6.2, we can assume that ε is taken so small that $\sup_{\mathcal{V}_\varepsilon} (|\nabla v| + |v|) < \delta$. Hence the above integral is bounded by $C\delta$. Since δ was arbitrarily chosen we obtain that v solves the equation in a distributional sense. Usual regularity arguments allow us to complete the proof. ■

End of the Proof of Lemma 7.4: now the proof of the Lemma follows from Proposition 7.3 using the same arguments of Lemmas 6.1 and 7.1, namely reducing to a function in the kernel of a divergence-type operator and applying the results of [1] and [12] about the number of critical points and the regularity of the level sets of solutions to second order differential equations. ■

Proof of Theorem 7.1: recalling Lemma 7.3, we argue by induction over the number h of connected components of the set \mathcal{Z}_3 . If $h = 1$ then, by Lemma 7.4 there is at most one minimal loop of the adjacency relation. If there is one, then Lemma 7.4 gives the desired assertion. If there are none, as we already mentioned, the auxiliary function v solves the equation globally and, by the above mentioned regularity result, we obtain the thesis. Now, let the Theorem be true for h and assume that \mathcal{Z}_3 has $h + 1$ connected components. Again, if the adjacency relation has no loops we are done. Otherwise, we apply Lemma 7.4 to treat those connected components contained in the interior of the minimal loop and the inductive hypothesis to treat all those contained in the outer region. ■

8 Appendix: Systems with large interactions

8.1 Assumptions and main results

In this section we shall analyze the connection between the optimal partition problem and the limit case of competitive systems with large interaction. We assume that the functions f and H satisfy the following set of assumptions. For easier understanding, the reader may think to the model case $f(s) = s^{p-1}$, $H(s_1, \dots, s_k) = \sum_{i \neq j} s_i^q s_j^q$, with $2 < 2q < p < 2^*$.

(f₁) $f \in C^1(\mathbb{R})$, $f(-s) = -f(s)$, and there exist positive constants C , p such that for

all $s \in \mathbb{R}$

$$|f(s)| \leq C(1 + |s|^{p-1}) \quad 2 < p < 2^*,$$

where $2^* = +\infty$ when $N = 2$ and $2^* = 2N/(N - 2)$ when $N \geq 3$

(f_2) there exists $\gamma > 0$ ($2 + \gamma \leq p$) such that, for all $s \neq 0$,

$$f'(s)s^2 - (1 + \gamma)f(s)s > 0$$

(h_1) H is a function of class $C^1(\mathbb{R}^k)$, $C^2(\mathbb{R}^k \setminus \bigcup_{i=1}^k \{s_i = 0\})$ and even with respect to each of its variables; there exists a constant β , $0 < \beta < \gamma$ such that

$$|H(s_1, \dots, s_k)| \leq C(1 + \sum |s_i|)^{2+\beta}$$

(h_2) there exists $0 < \alpha \leq \beta$ such that the Hessian matrix

$$\left(\frac{\partial^2 H(s_1, \dots, s_k)}{\partial s_i \partial s_j} s_i s_j - (1 + \alpha) \delta_{ij} \frac{\partial H(s_1, \dots, s_k)}{\partial s_i} s_i \right)_{i,j}$$

is negative semidefinite for $s_i \neq 0$ (in particular, it has non-positive diagonal terms). Here $\delta_{i,j}$ denotes the standard Kronecker symbol

(h_3)

$$\begin{aligned} H(s_1, \dots, s_k) &\geq 0, \quad \frac{\partial H(s_1, \dots, s_k)}{\partial s_i} s_i \geq 0 \\ H(s_1, \dots, s_k) &= 0 \quad \text{if at least } k-1 \text{ variables are } 0 \\ \frac{\partial H(s_1, \dots, s_k)}{\partial s_i} &= 0 \quad \text{if } s_i = 0 \text{ or } s_j = 0 \forall j \neq i \end{aligned}$$

Let Ω be a bounded open set in \mathbb{R}^N , $N \geq 2$ and consider the class of problems with parameter $\varepsilon > 0$

$$\begin{cases} -\Delta u_i(x) = f(u(x)) - \frac{1}{\varepsilon} \frac{\partial}{\partial u_i} H(u_1(x), \dots, u_k(x)) & \text{in } \Omega \\ u_i(x) > 0 & \text{in } \Omega \\ u_i = 0 & \text{on } \partial\Omega \end{cases} \quad (17)$$

with associated action functionals

$$I_\varepsilon(U) := \int_\Omega \left(\sum_{i=1}^k \left[\frac{1}{2} |\nabla u_i(x)|^2 - F(u_i(x)) \right] + \frac{1}{\varepsilon} H(U(x)) \right) dx. \quad (18)$$

We will prove

Theorem 8.1 *Let f satisfy (f_1), (f_2) and H satisfy (h_1)–(h_3). Then for all $\varepsilon > 0$ there exists a solution $U^\varepsilon = (u_1^\varepsilon, \dots, u_k^\varepsilon) \in (H_0^1(\Omega))^k$ to problem (17) such that*

$$I_\varepsilon(U^\varepsilon) = c_\varepsilon := \inf_{\substack{u_i \in H_0^1(\Omega), u_i > 0 \\ \forall i=1, \dots, k}} \sup_{\substack{\lambda_i > 0 \\ \forall i=1, \dots, k}} I_\varepsilon(\lambda_1 u_1, \dots, \lambda_k u_k).$$

Moreover there exists $U = (u_1, \dots, u_k) \in (H_0^1(\Omega))^k$ such that $u_i^\varepsilon \rightarrow u_i$ in $H_0^1(\Omega)$ as $\varepsilon \rightarrow 0$, $i = 1, \dots, k$. Each positive function u_i solves

$$-\Delta u_i(x) = f(u_i(x)) \quad x \in \text{supp}(u_i)$$

and

$$c_0 = J(U) = \inf_{\substack{u_i \in H_0^1(\Omega), u_i > 0 \\ u_j u_i = 0 \ \forall i \neq j}} \sup_{\substack{\lambda_i > 0 \\ \forall i=1, \dots, k}} J(\lambda_1 u_1, \dots, \lambda_k u_k) = \inf_{U \in \mathcal{N}_0} J(U).$$

where c_0 and J are defined in (3) and (2), respectively.

The proof of Theorem 8.1 will be divided into two propositions. We will first consider the problem (17) with $\varepsilon = 1$, and we will give a general existence result. Next, we will deduce the existence result for every $\varepsilon > 0$ and we will study the asymptotic behavior.

8.2 A class of elliptic systems

Aim of this section is to prove the existence of a k -uple $(u_1, \dots, u_k) \in (H_0^1(\Omega))^k$ where each u_i is positive and solve (17) when $\varepsilon = 1$. For easier notation we will write

$$\begin{aligned} U &:= (u_1, \dots, u_k), & u_i &\in H_0^1(\Omega) \\ \Lambda &:= (\lambda_1, \dots, \lambda_k), & \lambda_i &\in \mathbb{R} \\ \Lambda U &:= (\lambda_1 u_1, \dots, \lambda_k u_k) \end{aligned}$$

and so on.

We seek solutions of (17) as critical points of the following functional

$$I(U) := \int_{\Omega} \left(\sum_{i=1}^k \left[\frac{1}{2} |\nabla u_i(x)|^2 - F(u_i(x)) \right] + H(U(x)) \right) dx.$$

In particular we are interested in the search of *minimal* solutions of (17), that is, critical points of I achieving the lower value of I when restricted to the critical set $\mathcal{N}(I)$:

$$\mathcal{N}(I) := \left\{ U \in \mathcal{T} : \frac{\partial I(U)}{\partial u_i} \cdot u_i = 0 \ \forall 1 \leq i \leq k \right\},$$

where

$$\mathcal{T} := \{U \in (H_0^1(\Omega))^k : u_i \not\equiv 0 \ \forall 1 \leq i \leq k\}.$$

Setting

$$c := \inf_{U \in \mathcal{N}(I)} I(U)$$

we are going to prove

Proposition 8.1 *Under assumptions (f₁)–(h₃) there exists a critical point for I , say $U^c \in \mathcal{N}(I)$, such that $I(U^c) = c$. Moreover, $u_i^c > 0$ for every i .*

The proof of the proposition will be divided into three steps; the first consists in studying the behavior of I when restricted to the hyperplane generated by a fixed k -uple U with nontrivial components:

Lemma 8.1 *Let $U \in \mathcal{T}$ and consider*

$$\Phi(U) := \sup_{\lambda_i > 0} I(\Lambda U).$$

Then there exists a unique k -uple of positive numbers $\Lambda(U)$ such that

(i) $\Phi(U) = I(\Lambda(U)U)$;

(ii) $\Lambda(U)U \in \mathcal{N}(I)$;

(iii) *there exist $\gamma, \gamma_1 > 0$ such that if $U \in \mathcal{N}(I)$ then $\|u_i\|_{L^p} \geq \gamma$ and $\|u_i\| \geq \gamma_1$ for every i ;*

(iv) $\Phi(\cdot)$ *is l.s.c. with respect to the weak convergence in \mathcal{T} ;*

(v) *the map $U \mapsto \Lambda(U)$ is continuous from \mathcal{T} to $(\mathbb{R}^+)^k$.*

Proof: fix $U \in \mathcal{T}$ and define $\Psi_k(\Lambda) := I(\Lambda U)$. As a function of k real variables, Ψ_k is of class C^2 on \mathbb{R}^k but the coordinate hyperplanes (and C^1 everywhere), and even with respect to each of its variables. We are interested in studying the set of critical points of Ψ_k , and we will argue by induction on k (this is the reason for the explicit dependence on k of the notation). To do that, a crucial remark is the following: let $\bar{\Lambda}$ a critical point of Ψ having exactly h nonzero components; then, with some abuse of notations, we can see $\bar{\Lambda}$ as a critical point of Ψ_h with all nonzero components. To this aim, we observe that $H(s_1, \dots, s_{k-1}, 0)$ satisfies the same assumptions of H in the case of $k - 1$ components.

It is standard to prove that all the critical points are isolated, and hence it make sense to consider local degrees of $\nabla \Psi_k$ for the value 0. Although H is not C^2 , exploiting the positivity properties of H (assumption (h3)) and an homotopy argument, one can prove

$$i(\nabla \Psi_k, \bar{\Lambda}, 0) = i(\nabla \Psi_h, \bar{\Lambda}, 0),$$

where i denotes the topological index (topological local degree), and $\bar{\Lambda}$ is a critical point of Ψ_k with h nonzero components (this descend from the fact that a critical point on a coordinate hyperplane is a minimum along directions orthogonal to the hyperplane). By means of topological degree computations, we will show that there are exactly 2^k positive local maxima for Ψ_k , one in each octant. Since Ψ_k is negative out of a large ball, this will finally prove the uniqueness of $\Lambda(U)$ and assertion (i).

Let $\bar{\Lambda}$ be a critical point of Ψ_k ; then

$$\bar{\lambda}_i \int_{\Omega} \left(|\nabla u_i|^2 - \frac{1}{\bar{\lambda}_i^2} f(\bar{\lambda}_i u_i) \bar{\lambda}_i u_i + \frac{1}{\bar{\lambda}_i^2} H_{u_i}(\bar{\Lambda} U) \bar{\lambda}_i u_i \right) = 0 \quad \forall i : \bar{\lambda}_i \neq 0. \quad (19)$$

Claim 1. If $\bar{\Lambda}$ (critical point of Ψ_k) has exactly h nonzero components, then

$$i(\nabla \Psi_k, \bar{\Lambda}, 0) = (-1)^h.$$

Moreover, if $h = k$, then $\bar{\Lambda}$ is a local maximum.

By the previous arguments, we have to compute the local degree of $\bar{\Lambda}$ seen as a critical point of Ψ_h having all nonzero components. Therefore we can consider $\bar{\lambda}_i \neq 0$, $i = 1, \dots, h$, and we observe that Ψ_h is of class C^2 near $\bar{\Lambda}$. Clearly, if we show that the Hessian quadratic form associated to Ψ_h in $\bar{\Lambda}$ is negative semidefinite, the claim will follow.

We estimate the second derivatives $(\Psi_h)_{\lambda_i \lambda_i}$ by exploiting first (f_2) and then the identities (19)

$$\begin{aligned} (\Psi_h)_{\lambda_i \lambda_i}(\bar{\Lambda}) &= \int_{\Omega} \left(|\nabla u_i|^2 - \frac{1}{\bar{\lambda}_i^2} f'(\bar{\lambda}_i u_i) \bar{\lambda}_i^2 u_i^2 + \frac{1}{\bar{\lambda}_i^2} H_{u_i u_i}(\bar{\Lambda} U) \bar{\lambda}_i^2 u_i^2 \right) \\ &\leq \int_{\Omega} \left(|\nabla u_i|^2 - \frac{1+\alpha}{\bar{\lambda}_i^2} f(\bar{\lambda}_i u_i) \bar{\lambda}_i u_i + \frac{1}{\bar{\lambda}_i^2} H_{u_i u_i}(\bar{\Lambda} U) \bar{\lambda}_i^2 u_i^2 \right) \\ &= -\alpha \int_{\Omega} |\nabla u_i|^2 + \frac{1}{\bar{\lambda}_i^2} \int_{\Omega} (H_{u_i u_i}(\bar{\Lambda} u) \bar{\lambda}_i^2 u_i^2 - (1+\alpha) H_{u_i}(\bar{\Lambda} U) \bar{\lambda}_i u_i). \end{aligned}$$

Now we compute the mixed derivatives

$$(\Psi_k)_{\lambda_i \lambda_j}(\bar{\Lambda}) = \frac{1}{\bar{\lambda}_i \bar{\lambda}_j} H_{u_i u_j}(\bar{\Lambda} U) \bar{\lambda}_i u_i \bar{\lambda}_j u_j.$$

Thus we can bound the Hessian form from above with

$$\left(\frac{\partial^2 \Psi_h}{\partial \lambda_i \partial \lambda_j} \right)_{i,j} (a_1, \dots, a_h) \leq -\alpha \sum_{i=1}^h \left(a_i^2 \int_{\Omega} |\nabla u_i|^2 \right) + Q_H(\bar{\Lambda} U)(a_1, \dots, a_h)$$

where the quadratic form Q_H is negative definite by assumption (h_2) . Thus the claim follows.

Claim 2. There are exactly 2^k critical points with nonzero components (and thus maxima), one in each of the octants.

We argue by induction on k . First, for $k = 1$, by a direct analysis, exploiting the superlinear behavior of f as in (f_2) it is easy to show that Ψ_1 has a local minimum at the origin and two maxima, one for λ positive and one for λ negative.

Then we suppose that the claim is true for every $h < k$. By the growth assumptions on F and H , $\Psi_k(\Lambda) \rightarrow -\infty$ as $|\Lambda| \rightarrow \infty$, implying that Ψ_k must have at least a local maximum in each of the octants (remember that Ψ_k is symmetric). Moreover, since Ψ_k turns out to be concave outside a suitably large ball B_R , we have

$$\deg(\nabla \Psi_k, B_R, 0) = (-1)^k.$$

On the other hand, let us count all the possible critical points of Ψ_k , ordered by number of nonzero components: by the superquadraticity property of F we know that Ψ_k has one local minimum at the origin, providing a local degree $+1$; for $h = 1$, by the inductive assumption there are exactly two critical points of index -1 for any of the k components, and hence $2k$ critical points of index -1 on the axes; in general, for $h < k$, we have $2^h \binom{k}{h}$ critical points with h nonzero components, each with local degree $(-1)^h$. Finally the other possible critical points are local maxima by Claim 1, and thus the local degree at each

of them is $(-1)^k$. Let n be the number of such maxima; by the excision property of the degree it must hold:

$$(-1)^k = \deg(\nabla\Psi_k, B_R, 0) = n(-1)^k + \sum_{h=0}^{k-1} 2^h \binom{k}{h} (-1)^h$$

that gives $n = 2^k$, proving the claim.

Now we drop the dependence on k and we define $\Lambda(U)$ as the unique (by the evenness of the function) maximum of Ψ with all positive components, and we will prove that it satisfies (ii)-(v).

Assertion (ii) follows by noting that $\Lambda(U)$ is (the unique) positive solution of (19). Now we take $U \in \mathcal{N}(I)$, so that $\lambda_i(U) = 1$ for every i , and by (19) it holds

$$0 = \int_{\Omega} (|\nabla u_i|^2 - f(u_i)u_i + H_{u_i}(U)u_i) \geq \int_{\Omega} (|\nabla u_i|^2 - f(u_i)u_i) \quad (20)$$

by the positivity of H . Now (iii) follows by using the subcritical assumption (f_1) and the Sobolev inequality.

In order to prove the weak l.s.c. property of Φ , let U_n weakly converge to $U_0 \in \mathcal{T}$. Then $\|U_0\| \leq \liminf \|U_n\|$; on the other side, due to the subcritical growth of F (assumption (f_1)) and thanks to the compact embedding $H_0^1(\Omega) \subset L^q$ for all $2 < q < 2^*$, it holds

$$\begin{aligned} \Phi(U_0) &= I(\Lambda(U_0)U_0) \leq \\ &\leq I(\Lambda(U_0)U_n) + o(1) \leq \Phi(U_n) + o(1), \end{aligned}$$

and passing to the limit

$$\Phi(U_0) \leq \liminf_{n \rightarrow \infty} \Phi(U_n)$$

as required.

Finally (v) holds as a consequence of the Implicit Function Theorem applied to (19). \blacksquare

It is worthwhile noticing that, defining the following value

$$d := \inf_{U \in \mathcal{T}} \Phi(U), \quad (21)$$

it turns out by the lemma that $d = c$. This remark will be useful in proving that the problem of minimizing I on the Nehari manifold $\mathcal{N}(I)$ has a solution:

Lemma 8.2 *There exists $(U^c) \in \mathcal{N}(I)$ such that $I(U^c) = c$.*

Proof: let $(U_n) \subset \mathcal{N}(I)$ be a minimizing sequence and assume $I(U_n) \leq c + 1$. Now, computing: $(\alpha + 2)I(U_n) - \nabla I(U_n) \cdot U_n$ with $\alpha > 0$ as in assumption (h_2) , we have

$$\begin{aligned} (\alpha + 2)c &\geq \frac{\alpha}{2} \int_{\Omega} \sum |\nabla u_i^{(n)}|^2 + \\ &+ \int_{\Omega} \sum [f(u_i^{(n)})u_i^{(n)} - (2 + \alpha)F(u_i^{(n)})] + \\ &+ \int_{\Omega} \sum [(2 + \alpha)H(U_n) - \sum H_{u_i}(U_n)u_i^{(n)}] \end{aligned}$$

where by assumptions (f_2) and (h_2) the three terms in square brackets are positive. We have

$$\|U_n\|^2 \leq \frac{2(2+\alpha)}{\alpha}(c+1) \quad (22)$$

and thus the sequence (U_n) is bounded. Hence, up to a subsequence, there exists $\bar{U} \in (H_0^1(\Omega))^k$ weak limit of (U_n) . It is easy to verify that $\bar{U} \in \mathcal{T}$; indeed by Lemma 8.1 (iii), it holds $\|u_i^{(n)}\|_{L^p} \geq \gamma$ for all $n \in \mathbb{N}$, and, by the L^p -strong convergence of U_n to \bar{U} , we also obtain $\|(\bar{u})_i\|_{L^p} \geq \gamma$.

Now we can apply Lemma 8.1 to \bar{U} : by the weak l.s.c. property of Φ as in (iv), we have $\Phi(\bar{U}) \leq c$; furthermore, by definition of d as in (21) it holds $\Phi(\bar{U}) \geq d \equiv c$. Thus

$$c \equiv \sup_{\lambda_i > 0} I(\Lambda \bar{U})$$

and by applying Lemma 8.1 again, we obtain the existence of a k -uple $\bar{\Lambda}$ such that (due to (i)) $c \equiv I(\bar{\Lambda} \bar{U})$ and (due to (ii)) $\bar{\Lambda} \bar{U} \in \mathcal{N}(I)$. Setting $U^c := \bar{\Lambda} \bar{U}$ we finally conclude the proof. \blacksquare

Lemma 8.3 *If $U^c \in \mathcal{N}(I)$ and $I(U^c) = c$, then U^c is a critical point of I .*

Proof: assume by contradiction that $U^c \in \mathcal{N}(I)$, $I(U^c) = c$, but $\nabla I(U^c) \neq 0$. Then there exists $\rho > 0$ and $\delta > 0$ (we may assume $4\delta < \gamma_1$ as in Lemma 8.1 (iii)) such that

$$U \in B_\delta(U^c) \implies \|\nabla I(U)\| \geq \rho.$$

By the quantitative deformation lemma (see, for instance, [17]) we derive the existence of a continuous map $\eta : (H_0^1(\Omega))^k \rightarrow (H_0^1(\Omega))^k$ and of a constant $\nu > 0$ such that:

- (i) $\eta(U) = U$ for all U such that $\|U - U^c\| > 4\delta$ or $|I(U) - c| > 2\nu$;
- (ii) $I(\eta(U)) \leq I(U)$ for all $U \in (H_0^1(\Omega))^k$;
- (iii) $U \in B_{2\delta}(U^c) \wedge I(U) \leq c + \nu \implies I(\eta(U)) \leq c - \nu$.

Let us consider the deformation under this map of the hypersurface $\Gamma(T) := TRU^c$, where $T = (t_1, \dots, t_k)$ and R is a number fixed in such a way that $(R-1)\gamma_1 > 4\delta$. By the monotonicity properties of η and Lemma 8.1, we have that $\sup_{t_i > 0} \eta(\Gamma(T)) < c$. Now we claim that $\eta \circ \Gamma$ intersects the manifold $\mathcal{N}(I)$ and thus, by definition of c it holds $\sup \eta(\Gamma(T)) \geq c$, a contradiction. To prove the claim, let us consider the map:

$$\begin{aligned} \mathcal{H} : [0, 1]^k &\rightarrow (\mathbb{R}^+)^k \\ T &\mapsto \Lambda(\eta(\Gamma(T))) - 1 \end{aligned}$$

which is continuous by Lemma 8.1 (v). By construction it holds (see Lemma 3.1 in [7] for details):

- if $t_i = 0$, then $\mathcal{H}_i = +\infty$.

- if $t_i = 1$, then $\mathcal{H}_i < 0$.

Thus we are in condition to apply Miranda Theorem [15], and we find \bar{T} such that $\mathcal{H}(\bar{T}) = (0, 0)$; this implies $\eta(\Gamma(\bar{T})) \in \mathcal{N}(I)$ and ends the proof. \blacksquare

Now combining Lemma 8.2 and Lemma 8.3 we have the existence of a k -uple U^c critical point of I such that $I(U^c) = c$. What is left to show, in order to obtain by this result the full of Proposition 8.1, is that any u_i^c is strictly positive. To this aim we note that $|U^c| = (|u_1^c|, \dots, |u_k^c|) \in \mathcal{N}(I)$ and $I(U^c) = c = I(|U^c|)$ by definition of F and H . Then by Lemma 8.3 we know that $|U^c|$ is a critical point of I and so its components must be smooth by the standard regularity theory for elliptic PDEs. In this way we obtain the equalities $u_i^c = |u_i^c| \geq 0$ for every i ; now the strict inequalities come by the strong maximum principle. With this final remark the proof of Proposition 8.1 is complete.

8.3 Systems with large interaction

Let us fix a function $H : \mathbb{R}^k \rightarrow \mathbb{R}$ satisfying (h_1) – (h_3) . For all fixed $\varepsilon > 0$, let us define I_ε , the energy functional associated to system (17) as in (18). By Proposition 8.1 we know that there exists a k -uple of positive functions $U^\varepsilon = (u_1^\varepsilon, \dots, u_k^\varepsilon)$ which realizes

$$c_\varepsilon := \inf_{\substack{u_i \in H_0^1(\Omega) \\ u_i \neq 0}} \sup_{\lambda_i > 0} I_\varepsilon(\Lambda U).$$

In this section we are going to show the connection between the optimal partition problem and the solutions of (17). Let us recall the definition of the number (3):

$$c_0 := \inf_{\substack{u_i \in H_0^1(\Omega) \\ u_i \neq 0}} \sup_{\lambda_i > 0} J(\Lambda U) \\ \int u_i^2 u_j^2 = 0, i \neq j$$

where

$$J(U) := \int_{\Omega} \sum_{i=1}^k \left(\frac{1}{2} |\nabla u_i(x)|^2 - F(u_i(x)) \right) dx.$$

As a first remark we note that

$$c_\varepsilon \leq c_0 \tag{23}$$

indeed I_ε coincides with J on k -uples of function having disjoint support (see Assumption (h_3)).

We are going to prove

Proposition 8.2 *Assume that f and H satisfy (f_1) – (h_3) . Then*

1. for every $\varepsilon > 0$, c_ε is a critical level of I_ε , associated with a positive critical point $U^\varepsilon \in (H_0^1(\Omega))^k$;
2. there exists a positive $U^0 \in (H_0^1(\Omega))^k$ such that, up to a subsequence, $U^\varepsilon \rightarrow U^0$ in $(H_0^1(\Omega))^k$ and U^0 achieves c_0 .

Proof of part 1. Let $\varepsilon > 0$ be fixed. By applying Proposition 8.1 with $\frac{1}{\varepsilon}H$ instead of H (they clearly satisfy the same assumptions), we immediately obtain the existence of a positive solution U^ε . It is easy to prove that there exists $\gamma_2 > 0$, independent of ε , such that $\|U^\varepsilon\| < \gamma_2$. To this end it suffices to recall inequality (22) in the proof of Lemma 8.2 that provides, for every $\varepsilon > 0$,

$$\|U^\varepsilon\|^2 \leq \frac{2(2+\alpha)}{\alpha}(c_\varepsilon + 1).$$

Since $c_\varepsilon = c_\varepsilon \leq c_0$ the required estimate follows by setting $\gamma_2 := \frac{2(2+\alpha)}{\alpha}(c_0 + 1)$. Henceforth there exists, up to a subsequence, a weak nonnegative limit $U^0 \in (H_0^1(\Omega))^k$:

$$U^\varepsilon \rightharpoonup U^0 \quad \varepsilon \rightarrow 0. \quad (24)$$

Moreover it holds $u_i^0 \not\equiv 0$ for every i ; indeed, as consequence of inequality (20) in the proof of Lemma 8.1,(iii), we know that $\|u_i^\varepsilon\|_{L^p} \geq \gamma$, for some $\gamma > 0$ independent of ε . By the compact embedding $H_0^1(\Omega) \hookrightarrow L^p$ we obtain $\|u_i^0\|_{L^p} \geq \gamma$, giving the result. ■

The second part of Proposition 8.2 will be essentially provided in the following crucial lemma:

Lemma 8.4 *Let U^0 be the weak limit of U^ε when taking a subsequence $\varepsilon = \varepsilon_n \rightarrow 0$ as in (24). Then it holds*

$$(i) \quad U^0 \in \mathcal{N}(J);$$

$$(ii) \quad \|u_i^\varepsilon - u_i^0\| \rightarrow 0 \text{ for every } i;$$

$$(iii) \quad \frac{1}{\varepsilon} \int_{\Omega} H(U^\varepsilon) \rightarrow 0;$$

$$(iv) \quad c_\varepsilon \rightarrow c_0.$$

Proof: let us first note that, since $U^0 \in \mathcal{T}$, then we can find a k -uple Λ^0 of positive numbers such that $\Lambda^0 U^0 \in \mathcal{N}(J)$. By summing up the following equalities

$$\begin{aligned} \frac{\partial}{\partial u_i} I_\varepsilon(U^\varepsilon) \cdot u_i^\varepsilon &= 0 \\ \frac{\partial}{\partial u_i} J(\Lambda^0 U^0) \cdot \lambda_i^0 u_i^0 &= 0 \end{aligned}$$

and then passing to the limit, we obtain, for every i ,

$$\int_{\Omega} \left(\frac{f(\lambda_i^0 u_i^0)}{\lambda_i^0 u_i^0} - \frac{f(u_i^0)}{u_i^0} \right) (u_i^0)^2 + \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \int_{\Omega} H_{u_i}(U^\varepsilon) u_i^\varepsilon \leq 0.$$

Since $f(t)/|t|$ is increasing by (f_2) and the limit of the integral term is non-negative, we obtain that $\lambda_i^0 \leq 1$ for every i . Let us now make explicit the fact that $\Lambda^0 U^0 \in \mathcal{N}(J)$:

$$\begin{aligned} \int_{\Omega} (\lambda_i^0)^2 |\nabla u_i^0|^2 - \int_{\Omega} f(\lambda_i^0 u_i^0) \lambda_i^0 u_i^0 &= 0 \quad \forall i \\ \sum \left\{ \frac{(\lambda_i^0)^2}{2} \int_{\Omega} |\nabla u_i^0|^2 - \int_{\Omega} F(\lambda_i^0 u_i^0) \right\} &\geq c_0. \end{aligned}$$

By multiplying the second inequality with $(2 + \alpha)$, $\alpha > 0$ chosen as in assumption (h_2) , and then subtracting all the first equalities, we obtain

$$(2 + \alpha)c_0 \leq \sum \left\{ \frac{(\lambda_i^0)^2 \alpha}{2} \int_{\Omega} |\nabla u_i^0|^2 + \int_{\Omega} [f(\lambda_i^0 u_i^0) \lambda_i^0 u_i^0 - (2 + \alpha)F(\lambda_i^0 u_i^0)] \right\}. \quad (25)$$

On the other hand, we observe that the weak convergence of $U^\varepsilon \in \mathcal{N}(I_\varepsilon)$ to U^0 implies

$$\begin{aligned} \int_{\Omega} |\nabla u_i^0|^2 - \int_{\Omega} f(u_i^0) u_i^0 + \lim_{\varepsilon \rightarrow 0} \varepsilon^{-1} \int_{\Omega} H_{u_i}(U^\varepsilon) u_i^\varepsilon &\leq 0 \quad \forall i \\ \lim_{\varepsilon \rightarrow 0} c_\varepsilon &\geq \sum \left\{ \frac{1}{2} \int_{\Omega} |\nabla u_i^0|^2 - \int_{\Omega} F(u_i^0) \right\} + \lim_{\varepsilon \rightarrow 0} \varepsilon^{-1} \int_{\Omega} H(U^\varepsilon). \end{aligned} \quad (26)$$

As before, multiplying the last inequality by $(2 + \alpha)$ and then subtracting all the firsts

$$\begin{aligned} (2 + \alpha) \lim_{\varepsilon \rightarrow 0} c_\varepsilon &\geq \sum \left\{ \frac{\alpha}{2} \int_{\Omega} |\nabla u_i^0|^2 + \int_{\Omega} [f(u_i^0) u_i^0 - (2 + \alpha)F(u_i^0)] \right\} + \\ &\quad + \lim_{\varepsilon \rightarrow 0} \varepsilon^{-1} \int_{\Omega} [(2 + \alpha)H(U^\varepsilon) - \sum H_{u_i}(U^\varepsilon) u_i^\varepsilon]. \end{aligned}$$

Now, since by construction $c_0 \geq c_\varepsilon$ for all $\varepsilon > 0$, we can compare this inequality with (25); to shorten notation we introduce $\mathcal{F}(t) := f(t)t - (2 + \alpha)F(t)$, noting that $\mathcal{F}(t)$ is increasing for $t > 0$ by (f_2) . It turns out

$$\begin{aligned} \sum \left\{ \frac{\alpha}{2} \int_{\Omega} ((\lambda_i^0)^2 - 1) |\nabla u_i^0|^2 + \int_{\Omega} [\mathcal{F}(\lambda_i^0 u_i^0) - \mathcal{F}(u_i^0)] \right\} &\geq \\ &\geq \lim_{\varepsilon \rightarrow 0} \varepsilon^{-1} \int_{\Omega} [(2 + \alpha)H(U^\varepsilon) - \sum H_{u_i}(U^\varepsilon) u_i^\varepsilon]. \end{aligned}$$

By assumptions (h_2) , (h_3) it turns out that the term at the r.h.s. is non-negative: now since each λ_i^0 is not greater than 1, we conclude that in fact $\lambda_i^0 = 1$ for every i . This proves assertion (i) and, in turn, implies that:

$$\lim_{\varepsilon \rightarrow 0} \varepsilon^{-1} \int_{\Omega} [(2 + \alpha)H(U^\varepsilon) - \sum H_{u_i}(U^\varepsilon) u_i^\varepsilon] = 0. \quad (27)$$

From this the strong convergence of each u_i^ε to its limit easily follows. Indeed, if we assume for contradiction that $\lim \int_{\Omega} |\nabla u_i^\varepsilon|^2 > \int_{\Omega} |\nabla u_i^0|^2$, then all the above inequalities become strict, leading a contradiction in (27). Thus (ii) is proved.

Let us now prove assertion (iii) . To this aim we deduce by (25) and $\lambda_i^0 = 1$, the relation $\int_{\Omega} |\nabla u_i^0|^2 = \int_{\Omega} f(u_i^0) u_i^0$. Back to (26) we obtain that

$$\lim_{\varepsilon \rightarrow 0} \varepsilon^{-1} \int_{\Omega} H_{u_i}(U^\varepsilon) u_i^\varepsilon = 0.$$

Now (27) becomes

$$\lim_{\varepsilon \rightarrow 0} \varepsilon^{-1} \int_{\Omega} H(U^\varepsilon) = 0$$

as desired. Furthermore, we deduce by (iii) that

$$\lim_{\varepsilon \rightarrow 0} c_\varepsilon = J(U^0) \leq c_0$$

where the last inequality follows by the definition of c_0 and (i) . Now it is enough to recall that $\lim_{\varepsilon \rightarrow 0} c_\varepsilon \geq c_0$ by (23) to show that indeed $\lim_{\varepsilon \rightarrow 0} c_\varepsilon = c_0$, proving (iv) . \blacksquare

Proof of Proposition 8.2, part 2. In order to conclude the proof of Proposition 8.2 it suffices to show that, for $i \neq j$, u_i^0 and u_j^0 have disjoint support: this is consequence of property (iii) and of the strong convergence of U^ε to its weak limit. Indeed this implies

$$\int_{\Omega} H(U^0) = 0 :$$

by assumption (h_3) this means $u_i^0(x) \cdot u_j^0(x) = 0$ a.e. $x \in \Omega$ and proves $\text{supp}(u_i^0) \cap \text{supp}(u_j^0) = \emptyset$ for every $i \neq j$. Moreover by (iv) we also obtain $J(U^0) = c_0$, finally proving the assertion. ■

As a final remark, note that, if we pass to the limit in (17) as $\varepsilon \rightarrow 0$, by exploiting the strong convergence of the solutions U^ε to the k -uple U^0 and (iii), we realize that

$$-\Delta u_i^0(x) = f(u_i^0(x)) \quad x \in \text{supp}(u_i^0) \quad \forall i.$$

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