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## On a class of optimal partition problems related to the Fučík spectrum and to the monotonicity formulae

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**Abstract.** In this paper we give a unified approach to some questions arising in different fields of nonlinear analysis, namely: (a) the study of the structure of the Fučík spectrum and (b) possible variants and extensions of the monotonicity formula by Alt–Caffarelli–Friedman [1]. In the first part of the paper we present a class of optimal partition problems involving the first eigenvalue of the Laplace operator. Beside establishing the existence of the optimal partition, we develop a theory for the extremality conditions and the regularity of minimizers. As a first application of this approach, we give a new variational characterization of the first curve of the Fučík spectrum for the Laplacian, promptly adapted to more general operators. In the second part we prove a monotonicity formula in the case of many subharmonic components and we give an extension to solutions of a class of reaction–diffusion equation, providing some Liouville–type theorems.

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### 1. Introduction and statement of the results

Let  $\Omega \subset \mathbb{R}^N$  be a connected, open bounded domain with regular boundary  $\partial\Omega$ . For any open  $\omega \subset \Omega$ , let  $\lambda_1(\omega)$  denote the first eigenvalue of the Laplace operator in  $H_0^1(\omega)$ , namely

$$\lambda_1(\omega) = \min_{\substack{u \in H_0^1(\omega) \\ u \neq 0}} \frac{\int_{\omega} |\nabla u(x)|^2 dx}{\int_{\omega} |u(x)|^2 dx}.$$

For a fixed  $p > 0$ , let us consider the following class of optimal partition problems

$$(1) \quad \inf_{\mathcal{P}_k} \frac{1}{k} \sum_{i=1}^k (\lambda_1(\omega_i))^p,$$

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s where the minimization is taken over the class of partitions in  $k$  disjoint, connected, open subsets of  $\Omega$

$$(2) \mathcal{P}_k := \{(\omega_1, \dots, \omega_k) \subset \Omega : \omega_i \text{ is open and connected, } \omega_i \cap \omega_j = \emptyset \text{ if } i \neq j\} .$$

(As the reader can see, here and throughout all the paper, the word “partition” does not necessarily mean that  $\bigcup_{i=1}^k \bar{\omega}_i = \bar{\Omega}$ .) In this paper we shall investigate different aspects of problem (1) and of the analogous optimal partition problem on the spheres of  $\mathbb{R}^N$ .

The motivation of our interest in problems of this type is that they are behind different and relevant questions of nonlinear analysis. For instance, the proof of the monotonicity formula by [1], relies on the determination of the value

$$\inf_{\omega \subset S^{N-1}} \gamma(\lambda_1(\omega)) + \gamma(\lambda_1(S^{N-1} \setminus \omega)),$$

where  $\gamma$  denotes a suitable function (namely,  $\gamma(s) = \sqrt{(N/2 - 1)^2 + s}$ ), and  $S^{N-1}$  denotes the boundary of the unit ball in  $\mathbb{R}^N$ . Similar optimal partition problems arise in connection with the phenomenon of the spatial segregation in reaction–diffusion systems, as shown in [8, 9].

In the present paper, we develop a theory for (1) and we give applications in two independent directions. First, by studying problem (1) for partitions of connected domains of  $\mathbb{R}^N$ , we obtain a new variational characterization of the first curve of the Fučík spectrum of the Laplacian, that can be promptly adapted to the case of the  $p$ -Laplacian and even to more general notions of spectrum. On the other hand, by exploiting partitions of  $S^{N-1}$ , we shall prove some monotonicity formulae related to [1], for the case of many components.

A crucial tool will be the theory already developed by the authors in [7], concerning the regularity of solutions and the properties of the free boundary, in connection with certain classes of optimal partition problems involving nonlinear eigenvalues. More recently, in [8], the theory is shown to apply in the more general context of  $k$ -tuple of functions with mutually disjoint supports and belonging to functional classes characterized by suitable differential inequalities.

Let us now describe the structure of the paper and our main results with some details.

The first part of the paper is devoted to the study of optimal partition problems of the general type of (1). A different class of optimal partition problems, with an area constraint, has been studied in [4, 5]. To begin with, we seek the best partition of  $\Omega$  with respect to the infimum defined in (1). Then, our main goal is to derive the extremality sufficient conditions and to prove some qualitative properties both of the optimal sets  $\omega_i$ 's and of the eigenfunctions associated to  $\lambda_1(\omega_i)$ . Our first result in this direction is:

**Theorem 1.1.** *There exists  $(\omega_1, \dots, \omega_k) \in \mathcal{P}_k$  achieving (1). Furthermore, if  $\phi_1, \dots, \phi_k$  are the associated eigenfunctions normalized in  $L^2$ , then, there exist  $a_i \in \mathbb{R}$  such that the functions  $u_i = a_i \phi_i$  verify in  $\Omega$  the differential inequalities*

$$1. \quad -\Delta u_i \leq \lambda_1(\omega_i) u_i,$$

$$2. \quad -\Delta \left( u_i - \sum_{j \neq i} u_j \right) \geq \lambda_1(\omega_i) u_i - \sum_{j \neq i} \lambda_1(\omega_j) u_j.$$

As a consequence, the main results of [8] apply, providing the regularity of the minimizing  $k$ -tuple  $U = (u_i)_i$  and some qualitative features of the interfaces  $\partial\omega_i \cap \partial\omega_j$ , together with an asymptotic expansion of  $U$  at multiple intersection points in dimension 2. Finally, we extend all the previous considerations to the limiting case in (1) (as  $p \rightarrow \infty$ ):

$$(3) \quad \inf_{(\omega_i) \in \mathcal{P}_k} \max_{i=1, \dots, k} \{a_i \lambda_1(\omega_i)\},$$

where  $a_i$  are given positive weights. This last problem is connected to the Fučík spectrum  $\mathcal{F}$  of the Laplace operator with Dirichlet boundary conditions, which was defined in [11, 18] as the set of pairs  $(\lambda, \mu)$  such that the problem

$$-\Delta u = \lambda u^+ - \mu u^-, \quad u \in H_0^1(\Omega),$$

has a non zero solution (here we use the standard notation  $u^\pm := \max\{\pm u, 0\}$ ).

In fact, to each solution of (3) there corresponds an element of  $\mathcal{F}$ , provided either  $k = 2$  or the boundary of the supports  $\partial\omega_i$  do not have multiple intersection points. In this way we can give a new variational characterization of the first curve of the Fučík spectrum, in terms of an optimal partition of eigenvalues. More precisely, let

$$c(r) := \inf_{(\omega_i) \in \mathcal{P}_2} \max\{r \lambda_1(\omega_1), \lambda_1(\omega_2)\}.$$

Then we have

**Theorem 1.2.** *For all  $r > 0$ , there exists  $u \in H_0^1(\Omega)$  such that  $(\{u^+ > 0\}, \{u^- > 0\})$  achieves  $c(r)$ . Furthermore, the pair*

$$(\lambda_1(\{u^+ > 0\}), \lambda_1(\{u^- > 0\})) = (r^{-1}c(r), c(r))$$

*belongs to the Fučík spectrum and it represents the first nontrivial intersection between  $\mathcal{F}$  and the line of slope  $r$ .*

Variational formulations of the first curve have been given as min–max or constrained minimum, starting from the one–dimensional periodic problem in [17]; the general case  $N \geq 1$  has been first studied in the paper of De Figuereido and Gossez [16]. More recently, new characterizations were proposed in [10] and [19], covering the Fučík spectrum of the  $p$ -laplacian; theoretical and numerical studies have been carried out in [2, 3, 19]. In our opinion, our characterization is of interest from both the theoretical and the computational points of view. Indeed, it admits straightforward extensions to many other nonlinear operators, such as the  $p$ -laplacian and it could be easily modified in order to apply to general boundary conditions; moreover it can be easily implemented numerically using, for instance, a steepest descent method. A formulation related to our Theorem 1.2 was given in [19], but could not cover the full first curve of the spectrum.

Finally, as far as the dimension 1 is concerned, we can characterize an infinity of curves belonging to the Fučík spectrum, in the following way:

**Theorem 1.3.** *Let  $N = 1$ ,  $k \geq 1$  and define, for all  $r > 0$*

$$c_{k+1}(r) = \inf_{a=t_0 < t_1 < \dots < t_k < t_{k+1}=b} \max_i \{r\lambda_1(t_{2i+1} - t_{2i}), \lambda_1(t_{2i+2} - t_{2i+1})\}.$$

*Then the pair*

$$(r^{-1}c_{k+1}(r), c_{k+1}(r))$$

*belongs to  $\mathcal{F}$ .*

The second part of the paper is devoted to the study of the monotonicity formulae. The *monotonicity lemma* was originally stated by Alt, Caffarelli and Friedman in [1] in the following way:

**Lemma 1.1 (The monotonicity formula).** *Let  $(w_1, w_2) \in (H^1(\Omega))^2$  be non negative, continuous, subharmonic functions in a ball  $B(x_0, \bar{r}) \subset \Omega$  (i.e.  $-\Delta w_i \leq 0$  in distributional sense). Assume that  $w_1(x)w_2(x) = 0$ . Assume that  $x_0 \in \partial(\text{supp}(w_i)) \cap \Omega$  for  $i = 1, 2$ . Define*

$$\Phi(r) = \prod_{i=1}^2 \frac{1}{r^2} \int_{B(x_0, r)} \frac{|\nabla w_i(x)|^2}{|x - x_0|^{N-2}} dx.$$

*Then  $\Phi$  is a non decreasing function in  $[0, \bar{r}]$ .*

Since its very first publication, the monotonicity formula was shown to be a powerful tool in proving many local results in the theory of free boundaries. Our first objective consists in developing a variant of Lemma 1.1 for the case of many subharmonic densities having mutually disjoint supports. To this aim we consider the optimal partition value

$$\beta(k, N) := \inf_{\mathcal{P}(k, N)} \frac{2}{k} \sum_{i=1}^k \sqrt{\lambda_1(\omega_i)},$$

where the minimization is taken over all possible partitions in  $k$  disjoint parts of the unit sphere  $S^{N-1}$ . We shall prove:

**Lemma 1.2.** *Let  $\Omega \subset \mathbb{R}^N$ ,  $N \geq 2$ . Let  $w_1, \dots, w_h \in H^1(\Omega)$  be non negative subharmonic functions in a ball  $B(x_0, \bar{r}) \subset \Omega$  (i.e.  $-\Delta w_i \leq 0$  in distributional sense). Assume that  $w_i(x)w_j(x) = 0$  a.e. if  $i \neq j$  and that  $x_0 \in \partial(\text{supp}(w_j)) \cap \Omega$  for all  $j = 1, \dots, h$ . Define*

$$\Phi(r) = \prod_{i=1}^h \frac{1}{r^{\beta(h, N)}} \int_{B(x_0, r)} |\nabla w_i(x)|^2 dx.$$

*Then  $\Phi$  is a non decreasing function in  $[0, \bar{r}]$ .*

In the recent years, several papers have shown the existence of a strong connection between some free boundary problems and the spatially segregated limits of competition–diffusion systems, as the interaction rates tend to infinity. This asymptotic study has been carried out in [12–15, 20, 21]. The link with the optimal partitions was examined by the authors in [8, 9]. This motivates the interest of extending the monotonicity lemma to the case of an arbitrary number of densities whose supports need not to be mutually disjoint, but, instead, satisfy a system of competition–diffusion equations. In particular, as a prototype we consider the following system:

$$(4) \quad \begin{cases} -\Delta u_i(x) = -u_i(x) \sum_{j \neq i} a_{ij} u_j(x), & x \in \mathbb{R}^N, \\ u_i(x) \geq 0, & x \in \mathbb{R}^N, \end{cases}$$

for all  $i = 1, \dots, k$ , where  $a_{ij} > 0$ . Then we shall prove the following monotonicity result:

**Lemma 1.3.** *Let  $N \geq 2$  and let  $(u_1, \dots, u_k)$  be a solution of (4) such that  $u_i > 0$  for all  $i$ . Let  $h \leq k$  be any integer, let  $h' < \beta(h, N)$  and define*

$$\Phi(r) = \prod_{i=1}^h \frac{1}{r^{h'}} \int_{B(0,r)} \left( |\nabla u_i(x)|^2 + u_i^2(x) \sum_{\substack{1 \leq j \leq k \\ j \neq i}} a_{ij} u_j(x) \right) dx .$$

Then there exists  $r' = r(h')$  such that  $\Phi$  is an increasing function in  $[r', \infty)$ .

The above perturbed monotonicity formula will turn out to be the key point in proving an a priori growth estimate on the non trivial solutions of (4). The subsequent Liouville type result will be exploited, in a forthcoming paper [9], in order to prove the uniform hölder-continuity of solutions of competition–diffusion systems, when the interspecific competition rate tends to infinity.

## 2. Optimal partition problems involving linear eigenvalues

Let  $p$  be a positive real number; let  $k \geq 2$  be a fixed integer. In this section we consider the problem of finding a partition of  $\Omega$  that achieves

$$(5) \quad \inf_{(\omega_i) \in \mathcal{P}_k} \sum_{i=1}^k (\lambda_1(\omega_i))^p,$$

where  $\mathcal{P}_k$  is the set of all the possible partitions of  $\Omega$  in  $k$  connected, open subsets (see (2)). Let us recall that  $\lambda_1(\omega)$  denotes the first eigenvalue of  $-\Delta$  in  $H_0^1(\omega)$ ; the associated eigenspace is one dimensional (if  $\omega$  is connected), and the associated eigenfunction does not change its sign.

As a first step of our investigation, we relax problem (5) in the following way. For any measurable  $\omega \subset \Omega$ , let  $\lambda_1(\omega)$  denote

$$\lambda_1(\omega) := \inf \left\{ \frac{\int_{\omega} |\nabla u(x)|^2 dx}{\int_{\omega} |u(x)|^2 dx} : u \in H_0^1(\Omega), u = 0 \text{ a.e. on } \Omega \setminus \omega, u \not\equiv 0 \right\} .$$

When  $\omega$  is open, then  $\lambda_1(\omega)$  is the classical first eigenvalue of the Laplace operator in  $H_0^1(\omega)$ . For this reason, with a slight abuse of notation, we shall name, for any arbitrary measurable set  $\omega$ ,  $H_0^1(\omega) := \{u \in H_0^1(\Omega), u = 0 \text{ a.e. on } \Omega \setminus \omega\}$  (incidentally, we observe that possibly  $H_0^1(\omega) \equiv \{0\}$ , and consequently we define in a standard way  $\lambda_1(\omega) = +\infty$ ). Following this line, let us introduce the class of relaxed partitions

$$\mathcal{P}_k^* := \{(\omega_1, \dots, \omega_k) \subset \Omega : \omega_i \text{ is measurable, } \omega_i \cap \omega_j = \emptyset \text{ if } i \neq j\}$$

and the relaxed minimization problem

$$(6) \quad \inf_{(\omega_i) \in \mathcal{P}_k^*} \sum_{i=1}^k (\lambda_1(\omega_i))^p,$$

Clearly the value in (6) is smaller or equal to the one in (5). Nevertheless, at the end of this section the reader will see that (5) and (6) are in fact equivalent.

By taking into account the variational characterization of  $\lambda_1$ , the infimum value in (6) becomes

$$(7) \quad \inf_{\substack{u_i \in H_0^1(\Omega) \setminus \{0\} \\ u_i \cdot u_j = 0}} \sum_{i=1}^k \left( \frac{\int_{\Omega} |\nabla u_i(x)|^2 dx}{\int_{\Omega} |u_i(x)|^2 dx} \right)^p.$$

We have

*Remark 2.1.* There exists a  $k$ -tuple  $(\phi_1, \dots, \phi_k)$ , with  $\|\phi_i\|_2 = 1$ , that achieves the value (7). Moreover, if we set  $\omega_i = \{x \in \Omega : \phi_i(x) > 0\}$ , then  $\sum (\lambda_1(\omega_i))^p$  achieves (6). Indeed, we can minimize the functional

$$\mathcal{E}(u_1, \dots, u_k) = \sum_{i=1}^k \left( \frac{\int_{\Omega} |\nabla u_i(x)|^2 dx}{\int_{\Omega} |u_i(x)|^2 dx} \right)^p,$$

among  $k$ -tuples of  $H_0^1$  functions, subject to the constraint that  $u_i \geq 0$  for all  $i$  and  $u_i \cdot u_j = 0$  if  $i \neq j$ . Since the above functional is weakly lower semicontinuous, and the constraint is locally weakly compact, the direct method of the calculus of variations applies. This immediately provides the existence of a  $k$ -tuple of functions  $(\phi_i)$  (that we can obviously assume normalized in  $L^2$ ), having disjoint supports, which achieves (7). As a consequence, letting  $\omega_i = \{\phi_i > 0\}$ , we find a solution of (6).

The following part of the section (Sects. 2.1, 2.2) is devoted to the study of the properties of the minimizers of (6) in the case  $p \neq 1$  (the case  $p = 1$  will be considered in Sect. 3 through a limiting procedure). Finally, in Sect. 2.3 we will show the equivalence between (5) and (6).

### 2.1. An auxiliary variational problem (for $p \neq 1$ )

In this section we prove the equivalence of our minimal partition problem (6) with a min–max value for a certain functional defined on  $(H_0^1)^k$ . Let us start the description of the appropriate variational setting with some definitions: first, let  $p \neq 1$  be fixed and let  $q$  be the dual exponent of  $p$ ,  $q = \frac{p}{p-1}$ . Note that  $q \in (-\infty, 0) \cup (1, \infty)$ . For  $u \in H_0^1(\Omega)$  and  $U := (u_1, \dots, u_k) \in (H_0^1(\Omega))^k$  we define the functionals

$$\begin{aligned} J^*(u) &:= \frac{1}{2} \int_{\Omega} |\nabla u(x)|^2 dx - \frac{1}{2q} \left( \int_{\Omega} u(x)^2 dx \right)^q \\ J(U) &:= \sum_{i=1}^k J^*(u_i). \end{aligned}$$

We are interested in studying  $J$  restricted to the  $k$ -tuples of functions where the different components have disjoint support, hence we define

$$\mathcal{H} = \{U := (u_1, \dots, u_k) \in (H_0^1(\Omega))^k : u_i \cdot u_j = 0, \text{ a.e. on } \Omega \text{ for } i \neq j\}.$$

Note that  $U \in \mathcal{H}$  implies  $J(U) = \sum_{i=1}^k J^*(u_i) = J^*(\sum_{i=1}^k u_i)$ .

Next we define the Nehari manifolds associated to  $J^*$  and  $J$

$$\begin{aligned} \mathcal{N}(J^*) &:= \{u \in H_0^1(\Omega) : u \geq 0, u \neq 0, \nabla J^*(u) \cdot u = 0\} \\ \mathcal{N}_0 &:= (\mathcal{N}(J^*))^k \cap \mathcal{H}, \end{aligned}$$

and the value

$$(8) \quad c_q := \inf \{J(U) : U \in \mathcal{N}_0\}.$$

We shall prove the following

**Theorem 2.1.** *Let either  $q < 0$  or  $q > 1$ , and let  $\frac{1}{p} + \frac{1}{q} = 1$ . Then*

$$(9) \quad c_q \equiv \inf_{(\omega_i) \in \mathcal{P}_k^*} \frac{q-1}{2q} \sum_{i=1}^k \lambda_1(\omega_i)^p.$$

Moreover, there exists a  $k$ -tuple of functions  $U := (u_1, \dots, u_k) \in \mathcal{N}_0$  such that  $U$  achieve (8) and each  $u_i$  solves

$$(10) \quad -\Delta u_i(x) = \left( \int_{\Omega} u_i^2 \right)^{q-1} u_i(x) \quad \text{on the set where } u_i(x) > 0.$$

In particular, for all  $i$ , there holds

$$(11) \quad \lambda_1(\{u_i > 0\}) \equiv \left( \int_{\Omega} u_i^2 \right)^{q-1}.$$

*Proof.* First, let  $u \neq 0$ ,  $\omega := \{u > 0\}$  and consider  $J^*$  restricted to the line  $t \mapsto tu$ , namely  $g(t) = J^*(tu)$ . It turns out that, for  $t \neq 0$ ,  $g'(t_u) = 0$  iff

$$t_u^{2q-2} = \frac{\int_{\omega} |\nabla u(x)|^2 dx}{\int_{\omega} |u(x)|^2 dx},$$

and thus

$$J^*(t_u u) = \frac{q-1}{2q} \left( \frac{\int_{\omega} |\nabla u(x)|^2 dx}{\int_{\omega} |u(x)|^2 dx} \right)^{\frac{q}{q-1}}$$

As a consequence

$$(12) \quad \inf_{\substack{u \in H_0^1(\omega) \\ u \neq 0}} J^*(t_u u) = \frac{1}{2p} \lambda_1(\omega)^p.$$

Now some differences are induced by the value of  $q$ . In the case  $q > 1$ , the function  $g$  has a local minimum at the origin and  $\lim_{t \rightarrow \infty} g(t) = -\infty$ , hence  $t_u$  is a local maximum. As a consequence the value  $c_q$  has the equivalent characterization

$$c_q = \inf_{U \in \mathcal{H}} \sum_{i=1}^k J^*(t_{u_i} u_i) = \inf_{U \in \mathcal{H}} \max_{\{(t_1, \dots, t_k) : t_i > 0\}} \sum_{i=1}^k J^*(t_i u_i), \quad \text{if } q > 1.$$

On the other side, if  $q < 0$  the corresponding function  $g(t)$  is such that  $\lim_{t \rightarrow \infty} g(t) = \lim_{t \rightarrow 0} g(t) = \infty$  and hence  $c_q$  is the global infimum of  $J$

$$c_q = \inf_{U \in \mathcal{H}} J(U), \quad \text{if } q < 0.$$

Taking into account these characterizations for the value  $c_q$  and (12), we immediately have (9).

Now that we have proved the equivalence between the value  $c_q$  and the value (6), the existence of a minimizer  $U$  for (8) follows from the existence of a minimal partition for problem (6), as we discussed in Remark 2.1. Then, standard critical point techniques prove that  $U$  is a critical point for  $J$ , i.e.  $\nabla J(U) \equiv 0$  in distributional sense. By computing this means

$$\int_{\Omega} \nabla u_i \nabla v - \left( \int_{\Omega} u_i^2 \right)^{q-1} \int_{\Omega} u_i v = 0 \quad \forall v \in H_0^1(\Omega)$$

and proves (10). Finally, (11) is obtained in light of (10).  $\square$

## 2.2. The extremality conditions

In this section we prove that the extremality condition stated in Theorem 1.1, are verified by the solutions of the auxiliary problem (8). For easier notation we define, for every  $i$ ,

$$\widehat{u}_i := u_i - \sum_{j \neq i} u_j.$$

**Lemma 2.1.** *Let  $U = (u_1, \dots, u_k)$  be as in Theorem 2.1 and  $(\omega_1, \dots, \omega_k)$  be the corresponding supports. Then the following differential inequalities hold in  $\Omega$*

1.  $-\Delta u_i \leq \lambda_1(\omega_i)u_i$ ,
2.  $-\Delta \hat{u}_i \geq \lambda_1(\omega_i)u_i - \sum_{j \neq i} \lambda_1(\omega_j)u_j$ .

*Proof.* The argument is different according to the case that  $q > 1$  or  $q < 0$  and it mimics the proof in [7] for the case of nonlinear eigenvalues. For the reader's convenience we report the proofs adapted to the actual setting.

*The case  $q < 0$ .*

Let us prove 1. We argue by contradiction, assuming the existence of an index  $j$  such that the claim does not hold; that is, there exists  $0 \leq \phi \in C_c^\infty(\Omega)$  such that

$$(13) \quad \int_{\Omega} [\nabla u_j \nabla \phi - \lambda_1(\omega_j)u_j \phi] dx > 0.$$

For  $t > 0$  very small we define a new test function  $V = (v_1, \dots, v_k)$ , belonging to  $\mathcal{H}$ , as follows:

$$v_i = \begin{cases} u_i & \text{if } i \neq j, \\ (u_i - t\phi)^+ & \text{if } i = j. \end{cases}$$

We claim that  $V$  lowers the value of the functional  $J$ . We introduce  $G(s) = \frac{1}{2q}s^q$  and compute as follows

$$\begin{aligned} J(V) - J(U) &= \int_{\Omega} \frac{1}{2} (|\nabla(u_j - t\phi)^+|^2 - |\nabla u_j|^2) \\ &\quad - G\left(\int_{\Omega} ((u_j - t\phi)^+)^2\right) + G\left(\int_{\Omega} (u_j)^2\right) \\ &\leq \int_{\Omega} \frac{1}{2} (|\nabla(u_j - t\phi)|^2 - |\nabla u_j|^2) + 2tG'\left(\int_{\Omega} (u_j)^2\right) \int_{\Omega} u_j \phi + o(t) \\ &\leq -t \int_{\Omega} \left[ \nabla u_j \nabla \phi - 2G'\left(\int_{\Omega} (u_j)^2\right) u_j \phi \right] + o(t). \end{aligned}$$

Note that the last expression, when  $t$  is sufficiently small, is negative by (13), since  $\lambda_1(\omega_j) \equiv 2G'\left(\int_{\Omega} (u_j)^2\right)$ ; hence, choosing  $t$  sufficiently small, we obtain the contradiction

$$J(V) - J(U) < 0.$$

In order to prove 2., let  $j$  and  $0 < \phi \in C_c^\infty(\Omega)$  such that

$$\int_{\Omega} \left[ \nabla \hat{u}_j \nabla \phi - \left( \lambda_1(\omega_j)u_j - \sum_{i \neq j} \lambda_1(\omega_i)u_i \right) \phi \right] dx < 0.$$

Again, we show that the value of the functional can be decreased by replacing  $U$  with an appropriate new test function  $V$ . To this aim we consider the positive and negative parts of  $\hat{u}_j + t\phi$  and we notice that, obviously,

$$\{(\hat{u}_j + t\phi)^- > 0\} \subset \{(\hat{u}_j)^- > 0\} = \cup_{i \neq j} \{u_i > 0\}.$$

Let us define  $V = (v_1, \dots, v_k)$  in the following way:

$$v_i = \begin{cases} (\hat{u}_j + t\phi)^+, & \text{if } i = j, \\ (\hat{u}_j + t\phi)^- \chi_{\{u_i > 0\}}, & \text{if } i \neq j. \end{cases}$$

Here and below  $\chi_A$  denotes the characteristic function of the set  $A$ . We compute as follows

$$\begin{aligned} J(V) - J(U) &= \sum_{i=1}^k \int_{\Omega} \frac{1}{2} (|\nabla v_i|^2 - |\nabla u_i|^2) dx - G \left( \int_{\Omega} (v_i)^2 \right) + G \left( \int_{\Omega} (u_i)^2 \right) \\ &= \int_{\Omega} \frac{1}{2} (|\nabla \hat{u}_j + t\phi|^2 - |\nabla \hat{u}_j|^2) dx \\ &\quad - G \left( \int_{\Omega} ((\hat{u}_j + t\phi)^+)^2 \right) + G \left( \int_{\Omega} (u_j)^2 \right) \\ (14) \quad &\quad - \sum_{i \neq j} \left[ G \left( \int_{\{u_i > 0\}} ((\hat{u}_j + t\phi)^-)^2 \right) - G \left( \int_{\Omega} (u_i)^2 \right) \right] \\ &= t \int_{\Omega} \nabla \hat{u}_j \nabla \phi - 2tG' \left( \int_{\Omega} (u_j)^2 \right) \int_{\Omega} u_j \phi \\ &\quad + \sum_{i \neq j} 2tG' \left( \int_{\Omega} (u_i)^2 \right) \int_{\Omega} u_i \phi + o(t) \\ &= t \int_{\Omega} \left( \nabla \hat{u}_j \nabla \phi - (\lambda_1(\omega_j)u_j - \sum_{i \neq j} \lambda_1(\omega_i)u_i)\phi \right) + o(t). \end{aligned}$$

For  $t$  small enough we find  $J(V) < J(U)$ , a contradiction.

*The case  $q > 1$ .* The idea of the proof is analogous, but a new difficulty arises due to the fact that we can use only test functions belonging to the Nehari manifold  $\mathcal{N}_0$ . Let us show the new argument in the proof of the inequality 2. Assume by contradiction that the assertion does not hold for a certain index  $i$  and thus the existence of  $0 < \phi \in C_c^\infty$  such that

$$\int_{\Omega} \left[ \nabla \hat{u}_j \nabla \phi - \left( \lambda_1(\omega_j)u_j - \sum_{i \neq j} \lambda_1(\omega_i)u_i \right) \phi \right] dx < 0.$$

We will obtain a contradiction constructing a  $k$ -tuple in  $\mathcal{N}_0$  that decreases the value of  $c_q$ . Let  $A_j \hat{u}_j := \lambda_j u_j - \sum_{i \neq j} \lambda_i u_i$  with  $|\lambda_i - 1| \leq \delta$  for all  $i$ : if  $\delta$  is small enough we can also assume by continuity that

$$(15) \quad \int_{\Omega} \left[ \nabla A_j \hat{u}_j \nabla \phi - \left( \lambda_1(\omega_j)\lambda_j u_j - \sum_{i \neq j} \lambda_1(\omega_i)\lambda_i u_i \right) \phi \right] dx < 0.$$

By the inf-sup characterization of  $c_q$  and by the behavior of the function  $J^*(\lambda u)$  for fixed  $u > 0$ , we can take  $\delta$  so small that

$$(16) \quad \nabla J^*((1 - \delta)u_j)u_j > 0, \quad \nabla J^*((1 + \delta)u_j)u_j < 0, \quad \forall j.$$

Let us fix  $\bar{t} > 0$  small and let us consider a  $C^1$  function  $t : (\mathbb{R}^+)^k \rightarrow \mathbb{R}^+$  where  $t(\lambda_1, \dots, \lambda_k) = 0$  if for at least one  $j$  it happens  $|\lambda_j - 1| \geq \delta$ , and  $t(\lambda_1, \dots, \lambda_k) = \bar{t}$  if  $|\lambda_j - 1| \leq \delta/2$  for every  $j$ . Next we define the continuous map

$$\Phi(\lambda_1, \dots, \lambda_k) = \lambda_i u_i - \sum_{j \neq i} \lambda_j u_j + t(\lambda_1, \dots, \lambda_k) \phi.$$

Note that  $\Phi^-$  is a positive function whose support is union of  $k-1$  disjoint connected components, each of them belonging to the support of some  $u_j$ . Now we define the function  $\tilde{U}(\lambda_1, \dots, \lambda_k) = (\tilde{u}_1, \dots, \tilde{u}_k)$  as

$$\tilde{u}_i = \begin{cases} (A_j \hat{u}_j + t(\lambda_1, \dots, \lambda_k) \phi)^+, & \text{if } i = j, \\ (A_j \hat{u}_j + t(\lambda_1, \dots, \lambda_k) \phi)^- \chi_{\{u_i > 0\}}, & \text{if } i \neq j. \end{cases}$$

Let us compute  $J(\tilde{U})$ : in complete analogy with the calculations in (14) we have

$$\begin{aligned} J(\tilde{U}) &= \int_{\Omega} \left( \frac{1}{2} |\nabla(A_j \hat{u}_j)|^2 + \frac{t^2}{2} |\nabla \phi|^2 \right) dx + t \int_{\Omega} \nabla(A_j \hat{u}_j) \nabla \phi \\ &\quad - G \left( \int_{\Omega} ((\lambda_j u_j + t\phi)^+)^2 \right) - \sum_{i \neq j} G \left( \int_{\{u_i > 0\}} ((A_i \hat{u}_i + t\phi)^-)^2 \right) \leq \\ &\leq J(\lambda_1 u_1, \dots, \lambda_k u_k) \\ &\quad + \bar{t} \int_{\Omega} \left( \nabla(A_j \hat{u}_j) \nabla \phi - (\lambda_1(\omega_j) \lambda_j u_j - \sum_{i \neq j} \lambda_1(\omega_i) \lambda_i u_i) \phi \right) + o(\bar{t}). \end{aligned}$$

By (15) and taking  $\bar{t}$  small enough, this implies

$$J(\tilde{U}(\lambda_1, \dots, \lambda_k)) < J(\lambda_1 u_1, \dots, \lambda_k u_k)$$

if  $|\lambda_j - 1| \leq \delta/2$  for every  $j$ .

Now, if  $\bar{t}$  is small, we can assume that (16) holds for  $\tilde{U}(\lambda_1, \dots, \lambda_k)$  instead of  $(\lambda_1 u_1, \dots, \lambda_k u_k)$ . Thus, by Miranda's fixed point theorem [22], there exists  $(\mu_1, \dots, \mu_k)$  such that  $|\mu_i - 1| \leq \delta/2$  and

$$\nabla J(\tilde{U}(\mu_1, \dots, \mu_k)) \cdot \tilde{U}(\mu_1, \dots, \mu_k) = 0,$$

that means  $\tilde{U}(\mu_1, \dots, \mu_k) \in \mathcal{N}_0$ . But this is in contradiction with the definition of  $U$  as in Theorem 2.1 and the fact that

$$J(\tilde{U}(\mu_1, \dots, \mu_k)) < J(\mu_1 u_1, \dots, \mu_k u_k) \leq J(U) = \inf_{V \in \mathcal{N}_0} J(V).$$

With this the proof of the inequality 2 is done. Let us now briefly sketch the proof of the last inequality, namely

$$-\Delta u_i \leq \lambda_1(\omega_i) u_i$$

for all  $i$ . As usual assume by contradiction the existence of  $\phi > 0$ ,  $\phi \in C_c^\infty(\Omega)$  such that

$$(17) \quad \int_{\Omega} [\nabla(\lambda_i u_i) \nabla \phi - \lambda_1(\omega_i) \lambda_i u_i \phi] dx > 0$$

for all  $\lambda_i$  such that  $|\lambda_i - 1| \leq \delta$ ,  $\delta$  small enough. As in the proof of the inequality 2, we can assume  $\delta$  small enough to satisfy (16), and we consider the function  $t(\lambda_i)$  analogous to the one introduced therein. Then we let  $\tilde{\Phi}(\lambda_i) := \lambda_i u_i - t(\lambda_i) \phi$  and we define  $\tilde{U}$  with components  $\tilde{u}_i = \Phi^+$ ,  $\tilde{u}_j = u_j$  if  $j = 1, \dots, k-1$ ,  $j \neq i$  and finally  $\tilde{u}_k(x) = u_k(x)$  if  $x \in \{u_k > 0\}$ ,  $\tilde{u}_k(x) = \Phi^-$  if  $x \in \{u_i > 0\} \cap \{\Phi(x) < 0\}$ . By computing  $J(\tilde{U}(\lambda_i))$ , taking into account (17) and choosing  $\bar{t}$  small enough we obtain

$$J(\tilde{U}(\lambda_i)) < J(u_1, \dots, \lambda_i u_i, \dots, u_k) \quad |\lambda_i - 1| \leq \delta/2.$$

Now a contradiction with the properties of  $U$  as in Theorem 2.1 can be obtained by arguing as in the final step of the proof of the inequality 2.  $\square$

### 2.3. Regularity results

Aim of this section is to prove Theorem 1.1 in the case  $p \neq 1$  (as already said, the case  $p = 1$  is treated in the following section).

The differential inequalities obtained in Lemma 2.1 allows the application of the regularity theory developed by the authors in [8]. To be more precise, consider a  $k$ -tuple of  $H_0^1$  functions  $(v_1, \dots, v_k)$ ; we set  $\omega_i := \{v_i > 0\}$ ,  $f_i(s) := \lambda_1(\omega_i) s$ ,  $\hat{v}_i := v_i - \sum_{j \neq i} v_j$  and

$$\hat{f}(x, \hat{v}_i) := \lambda_1(\omega_i) v_i - \sum_{j \neq i} \lambda_1(\omega_j) v_j.$$

With these notations, if  $(u_1, \dots, u_k)$  is as in Theorem 2.1, then Lemma 2.1 says that  $(u_1, \dots, u_k) \in \mathcal{S}$ , where

$$\mathcal{S} := \left\{ (v_1, \dots, v_k) \in (H^1(\Omega))^k : \begin{array}{l} v_i \geq 0, v_i \cdot v_j = 0 \text{ if } i \neq j \\ -\Delta v_i \leq f_i(x, v_i), -\Delta \hat{v}_i \geq \hat{f}(x, \hat{v}_i) \end{array} \right\}.$$

This class of functions have been introduced by the authors in [8], where a number of qualitative properties for its elements are obtained. We collect those properties in the following theorem, referring to [8] for its proof. We first need a definition:

**Definition 2.1.** The multiplicity of a point  $x \in \Omega$  with respect to the  $k$ -tuple  $(v_1, \dots, v_k)$  is

$$m(x) = \# \{i : \text{meas}(\{v_i > 0\} \cap B_r(x)) > 0, \forall r > 0\}.$$

**Theorem 2.2.** Let  $(v_1, \dots, v_k) \in \mathcal{S}$ , and let  $\omega_i := \{v_i > 0\}$ . Then,  $V := \sum_{i=1}^k v_i$  verifies the following properties.

1. The function  $V$  is Lipschitz continuous in the interior of  $\Omega$ ; if  $\partial\Omega$  is regular, then  $V$  is Lipschitz up to the boundary. In particular,  $\omega_i = \{v_i > 0\}$  is an open set.

2. Let  $x \in \Omega$  such that  $m(x) = 2$ . Then,

$$\lim_{\substack{y \rightarrow x \\ v_i(y) > 0}} \nabla v_i(y) = - \lim_{\substack{y \rightarrow x \\ v_j(y) > 0}} \nabla v_j(y).$$

3. In dimension  $N = 2$ , the set  $\{m(x) \geq 3\}$  consists in a finite numbers of points where  $\nabla V$  is identically zero.

4. In dimension  $N = 2$ , let  $x \in \Omega$  such that  $m(x) = h$ . Then,  $V$  admits a local expansion around  $x$  of the following form:

$$V(r, \theta) = r^{\frac{h}{2}} \left| \cos \left( \frac{h}{2}(\theta + \theta_0) \right) \right| + o(r^{\frac{h}{2}})$$

as  $r \rightarrow 0$ , where  $(r, \theta)$  denotes a system of polar coordinates around  $x$ .

5. In dimension  $N = 2$ , the set  $\{m(x) = 2\}$  consists in a finite number of  $C^1$ -arcs ending either at points with higher multiplicity, or at the boundary  $\partial\Omega$ .

By the above discussion,  $(u_1, \dots, u_k)$  shares all these properties. In particular, this implies that the partition consisting of its supports, besides belonging to  $\mathcal{P}_k^*$ , is an element of  $\mathcal{P}_k$ .

**Lemma 2.2.** *Let  $(\omega_i)$  be the  $k$ -tuple provided in Remark 2.1. Then  $\omega_i$  is open for every  $i$ . Moreover we can assume, without loss of generality, that each  $\omega_i$  is connected (that is,  $(\omega_i) \in \mathcal{P}_k$  and it is a solution of (5)).*

*Proof.* By Theorem 2.1,  $\omega_i = \{u_i > 0\}$ . Hence the application of Theorem 2.2,(1) provides that each  $\omega_i$  is open. Assume that, for some  $i$ ,  $\omega_i$  is not connected, and let  $\{\alpha_j\}_{j \in J}$  denote its connected (open) components. We observe that, for every  $M > 0$ ,  $\#\{j : \lambda_1(\alpha_j) \leq M\} < \infty$  (indeed  $\Omega$  is bounded and  $\lim_{|\omega| \rightarrow 0} \lambda_1(\omega) = \infty$ ). Then  $\lambda_1(\omega_i) = \min_j \{\lambda_1(\alpha_j)\} = \lambda_1(\alpha_h)$ . Replacing  $u_i$  with  $u_i|_{\alpha_h}$ , we obtain a  $k$ -tuple of functions with open, connected supports, that again achieves (7).  $\square$

*Proof of Theorem 1.1 (case  $p \neq 1$ ).* Let  $(u_1, \dots, u_k)$  the functions provided by Theorem 2.1 and let  $\omega_i = \{u_i > 0\}$ . Clearly,  $u_i = a_i \phi_i$ , where  $\phi_i$  denotes the positive eigenfunction associated to  $\lambda_1(\omega_i)$ , normalized in  $L^2$ , and  $a_i = \lambda_1^{1/2(q-1)}(\omega_i)$ . As shown in Lemma 2.1, each  $u_i$  satisfies the required differential inequalities. Therefore, Theorem 2.2 applies, implying that each  $\omega_i$  is open. Finally, also Lemma 2.2 applies, and the theorem follows.  $\square$

A remarkable consequence of the above results is the equivalence between the original problem (5) and the relaxed one (6):

$$(18) \quad \inf_{(\omega_i) \in \mathcal{P}_k} \sum_{i=1}^k (\lambda_1(\omega_i))^p \equiv \inf_{(\omega_i) \in \mathcal{P}_k^*} \sum_{i=1}^k (\lambda_1(\omega_i))^p.$$

Up to now, this is true only when  $p \neq 1$ . The discussion in the following section will trivially imply that it holds also in the case  $p = 1$ .

*Remark 2.2.* Let us conclude this section with a remark about the generality of the theory so far developed. Actually, the procedure leading to the proof of Theorem 2.1 and, consequently, to Theorem 1.1, can be trivially adapted to study, for instance, nonhomogeneous optimal partition problems. Namely, let  $m_i, n_i \in L^\infty$  such that  $\inf_{x \in \Omega} \{m_i(x), n_i(x)\} > 0$ ; finally let  $a_i \in \mathbb{R}^+$  and  $q \in \mathbb{R}, r \in \mathbb{R}$  such that  $q > 1$  and  $r \geq 2$ . By defining the first weighted–eigenvalue as

$$\lambda_1(\omega_i) = \min_{\substack{u \in H_0^1(\omega_i) \\ u \neq 0}} \frac{\int_{\omega_i} n_i(x) |\nabla u(x)|^r dx}{\int_{\omega_i} m_i(x) |u(x)|^r dx},$$

we consider the problem of finding a partition of  $\Omega$  in  $k$  open sets that achieves

$$(19) \quad c_q \equiv \inf_{(\omega_i) \in \mathcal{P}_k} \frac{q-1}{rq} \sum_{i=1}^k [a_i \lambda_1(\omega_i)]^{\frac{q}{q-1}}.$$

Then, with obvious changes in the functional setting, namely by redefining

$$J_i(u) = \frac{a_i}{r} \int_{\Omega} n_i(x) |\nabla u(x)|^r dx - \frac{1}{rq} \left( \int_{\Omega} m_i(x) |u(x)|^r dx \right)^q$$

the whole procedure applies to problem (19). Note that this includes the remarkable case of the  $r$ -laplacian, and will be crucial in connection with the analysis of the Fučík spectrum developed in the last section.

### 3. The limiting cases $p = 1, p = \infty$

In this section we study the asymptotic behavior of the solutions to the problem (8) both as  $p \rightarrow 1$  and  $p \rightarrow \infty$ . This analysis will provide existence and regularity results, analogous to those obtained in the previous section, for two remarkable optimal partition problems that we cannot directly treat with the above techniques. Let us start our description by observing that the eigenvalues corresponding to (5) are uniformly bounded in  $p$ :

**Lemma 3.1.** *Let  $p > 0, p \neq 1$  and let  $(\omega_{1,p}, \dots, \omega_{k,p}) \in \mathcal{P}_k$  achieving (5). Then there exist  $0 < m < M < \infty$  such that*

$$m \leq \lambda_1(\omega_{i,p}) \leq M \quad \forall i, \forall p.$$

*Proof.* The bound from below depends on the monotonicity of the first eigenvalue with respect to the inclusion

$$\omega \subset \Omega \implies \lambda_1(\omega) \geq \lambda_1(\Omega).$$

Hence it suffices letting  $m = \lambda_1(\Omega)$ . The bound from above simply follows by the minimality of the optimal partition.  $\square$

In the following we shall denote  $\lambda_{i,p} := \lambda_1(\omega_{i,p})$ . Let us also recall that, if  $u_{i,p}$  are the eigenfunctions associated to  $c_q$ , then by Theorem 2.1

$$(20) \quad \left( \int_{\Omega} u_{i,p}^2 \right)^{q-1} = \lambda_{i,p} = \left( \int_{\Omega} |\nabla u_{i,p}|^2 \right)^{\frac{q-1}{q}},$$

for all  $i$ .

### 3.1. Proof of Theorem 1.1 (case $p = 1$ )

Let

$$(21) \quad c_1 = \inf_{(\omega_i) \in \mathcal{P}_k} \sum_{i=1}^k \lambda_1(\omega_i),$$

and let  $p \rightarrow 1$ , hence  $q \rightarrow \infty$ : since  $\lambda_{i,p}$  is uniformly bounded in  $p$ , we obtain by (20) that  $u_{i,p}$  is bounded in  $H^1$ . Therefore, there exist  $u_i \in H^1$  such that (up to a subsequence)  $u_{i,p} \rightharpoonup u_i$  weakly in  $H^1$ . Since the convergence is also almost everywhere, then, calling  $\omega_i := \{u_i > 0\}$ , we have that the  $(\omega_i)$ 's are disjoint. We claim that  $(\omega_i)$  is a solution of (21) and that the corresponding  $u_i$  satisfy suitable extremality conditions.

To start with, let us observe that, by virtue of Lemma 3.1, there exist  $\mu_i \in [m, M]$  such that, up to a subsequence,  $\lim_{p \rightarrow 1} \lambda_{i,p} = \mu_i$ . By weak convergence, the differential inequalities for  $u_{i,p}$  pass to the limit, namely

$$-\Delta u_i \leq \mu_i u_i \quad \text{and} \quad -\Delta \hat{u}_i \geq \mu_i u_i - \sum_{j \neq i} \mu_j u_j.$$

This allows to prove that the weak convergence is indeed strong. To this aim, let us test the inequality  $-\Delta \hat{u}_i \geq \mu_i u_i - \sum_{j \neq i} \mu_j u_j$  with  $u_i$ : then

$$\int_{\Omega} |\nabla u_i|^2 \geq \mu_i \int_{\Omega} u_i^2$$

On the other side, by  $-\Delta u_{i,p} \leq \lambda_{i,p} u_{i,p}$  tested with  $u_{i,p}$  it holds

$$\int_{\Omega} |\nabla u_{i,p}|^2 \leq \lambda_{i,p} \int_{\Omega} u_{i,p}^2.$$

By gluing the two previous inequalities when passing to the limit we obtain

$$\int_{\Omega} |\nabla u_i|^2 \geq \limsup \int_{\Omega} |\nabla u_{i,p}|^2.$$

This finally provides  $u_{i,p} \rightarrow u_i$  in  $H^1$ . Furthermore, by the variational characterization of the first eigenvalue we have  $\mu_i \equiv \lambda_1(\omega_i)$ .

As a consequence of this analysis we have that  $c_q \rightarrow c_1$  and that  $(\omega_i)$  achieves  $c_1$ ; furthermore it holds

$$-\Delta u_i \leq \lambda_1(\omega_i) u_i \quad \text{and} \quad -\Delta \hat{u}_i \geq \lambda_1(\omega_i) u_i - \sum_{j \neq i} \lambda_1(\omega_j) u_j,$$

as required. But now we are in a position to apply the already mentioned regularity theory (Theorem 2.2, Lemma 2.2), providing  $(\omega_i) \in \mathcal{P}_k$  and concluding the proof.

### 3.2. The case $p = \infty$ .

Let  $p \rightarrow \infty$  (hence  $q \rightarrow 1$ ). By virtue of the basic property

$$\lim_{p \rightarrow \infty} \left( \sum_{i=1}^k a_i^p \right)^{\frac{1}{p}} = \max\{a_1, \dots, a_k\},$$

where  $a_i$  are positive numbers, we shall succeed in recovering our existence and regularity results for a partition achieving

$$(22) \quad c_\infty := \inf_{(\omega_i) \in \mathcal{P}_k} \max_{i=1, \dots, k} \lambda_1(\omega_i).$$

Indeed we are going to prove

**Theorem 3.1.** *There exists  $U \in \mathcal{S}$  such that  $(\{u_1 > 0\}, \dots, \{u_k > 0\})$  achieves the value (22).*

*Proof.* Let  $p \rightarrow \infty$ ; note that in this case we do not know if  $u_{i,q}$  is  $H^1$ -bounded. But if we define

$$v_{i,q} = \frac{u_{i,q}}{\left(\sum \lambda_{i,q}\right)^{\frac{q}{q-1}}},$$

then (20) ensures  $\|v_{i,q}\|_{H^1} \leq 1$  for all  $i$  and  $q$ . Hence  $v_{i,q}$  admit a weak limit (up to a subsequence) and all the analysis developed in the previous case still holds. In particular, if we call  $u_i$  the  $H^1$  limit of  $v_{i,q}$ , again the extremality conditions hold true and consequently  $(\{u_i > 0\})$  belongs to  $\mathcal{P}_k$ . Let us now prove that  $(\{u_i > 0\})$  is indeed a solution for (22). By the strong convergence of the  $v_{i,q}$ 's and the above mentioned basic property we know that

$$c_q^{\frac{1}{p}} = \left( \sum_{i=1}^k \lambda_{i,q}^p \right)^{\frac{1}{p}} \rightarrow \max\{\lambda_1(\{u_1 > 0\}), \dots, \lambda_1(\{u_k > 0\})\} := M_k,$$

hence it is enough to prove that  $M_k \equiv c_\infty$ . To this aim let  $(\tilde{\omega}_1, \dots, \tilde{\omega}_k) \in \mathcal{P}_k^*$  a  $k$ -tuple of disjoint sets achieving

$$\inf_{\mathcal{P}_k^*} \max_{i=1, \dots, k} \lambda_1(\omega_i) \leq c_\infty$$

(such a  $k$ -tuple exists by arguments similar to the ones in Remark 2.1). Then, as  $p \rightarrow \infty$ , it holds

$$c_q^{\frac{1}{p}} \leq \left( \sum_{i=1}^k \lambda_1(\tilde{\omega}_i)^p \right)^{\frac{1}{p}} \rightarrow \max\{\lambda_1(\tilde{\omega}_1), \dots, \lambda_1(\tilde{\omega}_k)\} \leq c_\infty.$$

This implies

$$M_k = \lim_{p \rightarrow \infty} c_q^{\frac{1}{p}} \leq c_\infty.$$

By the minimality of  $c_\infty$  the opposite inequality  $c_\infty \leq M_k$  is immediate, thus  $M_k \equiv c_\infty$  as claimed.  $\square$

#### 4. The first curve of the Fučík spectrum

Let us consider the problem

$$(23) \quad -\Delta u = \lambda u^+ - \mu u^-$$

in  $\Omega \subset \mathbb{R}^N$  with boundary condition  $u = 0$  on  $\partial\Omega$ ; the Fučík spectrum of  $-\Delta$  on  $H_0^1(\Omega)$  is defined as

$$\mathcal{F} = \{(\lambda, \mu) \in \mathbb{R}^2 : \text{problem (23) has a non-trivial (weak) solution}\}.$$

As already discussed, this object has been argument of a quite extensive literature devoted to study its structure and its connections with the solvability of nonlinear related problems (see references in the introduction). In particular, it is known by [16] that, besides the pairs of equal eigenvalues and the semi-lines  $(\lambda_1(\Omega), t), (t, \lambda_1(\Omega))$  for all  $t > 0$ ,  $\mathcal{F}$  contains a first nontrivial curve  $C_1$  through  $(\lambda_2(\Omega), \lambda_2(\Omega))$ , which extends to infinity. The objective of this section is to give a new description of  $C_1$  in the variational setting developed in the present paper. To this aim, let  $r$  be a positive number; we introduce the problem of finding a partition achieving

$$(24) \quad c(r) := \inf_{(\omega_i) \in \mathcal{P}_2} \max\{r\lambda_1(\omega_1), \lambda_1(\omega_2)\}.$$

Our characterization is given by Theorem 1.2 as stated in the introduction. We recall the result:

**Theorem 1.2.** *For all  $r > 0$ , there exists  $u \in H_0^1(\Omega)$  such that  $(\{u^+ > 0\}, \{u^- > 0\})$  achieves  $c(r)$ . Furthermore, the pair*

$$(25) \quad (\lambda_1(\{u^+ > 0\}), \lambda_1(\{u^- > 0\})) = (r^{-1}c(r), c(r))$$

*belongs to the Fučík spectrum  $\mathcal{F}$  and it represents the first (nontrivial) intersection between  $\mathcal{F}$  and the line of slope  $r$ .*

As a consequence we have a variational characterization of the second eigenvalue of the Laplacian in  $H_0^1(\Omega)$ :

**Corollary 4.1.**

$$\lambda_2(\Omega) \equiv \inf_{\omega \in \Omega} \max\{\lambda_1(\omega), \lambda_1(\Omega \setminus \omega)\}.$$

*Proof.* Note that, for the choice  $r = 1$ , in view of (26), it holds that  $c(1)$  is an eigenvalue corresponding to a sign-changing eigenfunction. Hence  $c(1) \geq \lambda_2(\Omega)$ . On the other hand, it follows from the property of  $c(1)$  of being the *first* intersection with the Fučík spectrum  $\mathcal{F}$  and the well-known fact  $(\lambda_2(\Omega), \lambda_2(\Omega)) \in \mathcal{F}$  that  $c(1) = \lambda_2(\Omega)$ . From this the thesis follows.  $\square$

*Proof of Theorem 1.2.* Let  $q > 1$  be fixed: by Remark 2.2, with the choice  $k = 2$ ,  $m_i(x) = n_i(x) = 1$  and  $a_1 = r, a_2 = 1$ , we immediately obtain the existence of a pair  $(u_{1,q}, u_{2,q})$  whose supports  $(\omega_{1,q}, \omega_{2,q})$  achieve the value

$$c_q \equiv \inf_{(\omega_i) \in \mathcal{P}_2} \frac{q-1}{2q} \left( (r\lambda_1(\omega_1))^{\frac{q}{q-1}} + \lambda_1(\omega_2)^{\frac{q}{q-1}} \right).$$

Now we choose a sequence  $q \rightarrow 1$  and we follow the limiting procedure in Sect. 3.2 and the arguments therein. We thus obtain, when passing to the limit as  $q \rightarrow 1$ , the existence of a pair of  $H_0^1(\Omega)$ -functions  $(u_1, u_2)$  with the following properties. First,  $U = (u_1, u_2) \in \mathcal{S}$ , and  $(\{u_1 > 0\}, \{u_2 > 0\})$  achieves the value

$$\inf_{(\omega_i) \in \mathcal{P}_2} \max\{r\lambda_1(\omega_1), \lambda_1(\omega_2)\}.$$

Let us now define the function  $u := u_1 - u_2$ , that is,  $u_1 = u^+$  and  $u_2 = u^-$ . Since  $U \in \mathcal{S}$ , then properties (1) and (2) of Theorem 2.2 hold. Hence  $u$  is regular and it is a nontrivial solution of

$$-\Delta u = \lambda_1(\{u^+ > 0\})u^+ - \lambda_1(\{u^- > 0\})u^-$$

on  $\Omega$ . We have

$$(26) \quad r\lambda_1(\{u^+ > 0\}) = \lambda_1(\{u^- > 0\}) = c(r).$$

Indeed, assume by contradiction that  $c(r) = r\lambda_1(\{u^+ > 0\})$  and  $r\lambda_1(\{u^+ > 0\}) - \lambda_1(\{u^- > 0\}) = m > 0$ . Let  $x \in \partial\{u^+ > 0\} \cap \partial\{u^- > 0\}$ . Since  $u$  is regular, we can choose  $\rho > 0$  small enough in such a way that, by the monotonicity and continuity properties of the first eigenvalue, there holds

$$r\lambda_1(\{u^+ > 0\}) > r\lambda_1(\{u^+ > 0\} \cup B(x, \rho)) > r\lambda_1(\{u^+ > 0\}) - \frac{m}{4}$$

and

$$\lambda_1(\{u^- > 0\}) < \lambda_1(\{u^- > 0\} \setminus B(x, \rho)) < \lambda_1(\{u^- > 0\}) + \frac{m}{4}.$$

In this way we have a new partition  $(\{u^+ > 0\} \cup B(x, \rho), \{u^- > 0\} \setminus B(x, \rho))$  which lowers the value  $c(r)$ , a contradiction. As a consequence, we finally obtain that the pair  $(r^{-1}c(r), c(r))$  belongs to  $\mathcal{F}$ .

We are left to prove that (25) is in fact the *first* nontrivial intersection of the spectrum with the line of slope  $r$ . First we observe that, by the monotonicity of the first eigenvalue with respect to the inclusion,  $c(r) > \lambda_1(\Omega)$  for every  $r > 0$ . Assume by contradiction the existence of a pair  $(r^{-1}\mu, \mu) \in \mathcal{F}$  such that  $\mu < c(r)$ . This means the existence of  $v \in H_0^1$  with  $v^\pm \neq 0$  such that  $-\Delta v = r^{-1}\mu v^+ - \mu v^-$ : testing the equation with  $v^\pm$  we have

$$r^{-1}\mu = r^{-1} \frac{\int_\Omega |\nabla v^-|^2}{\int_\Omega |v^-|^2} = \frac{\int_\Omega |\nabla v^+|^2}{\int_\Omega |v^+|^2}.$$

Since  $(\{v^+ > 0\}, \{v^- > 0\})$  is an admissible partition of  $\Omega$ , it must hold

$$c(r) \leq \max \left\{ r \frac{\int_\Omega |\nabla v^+|^2}{\int_\Omega |v^+|^2}, \frac{\int_\Omega |\nabla v^-|^2}{\int_\Omega |v^-|^2} \right\} = \mu,$$

a contradiction.  $\square$

Hence, by denoting

$$C_1 := \{(r^{-1}c(r), c(r)) \in \mathbb{R}^2, r > 0\},$$

we have that  $C_1$  is indeed the first nontrivial curve of the Fučík spectrum. We wish to emphasize that our variational characterization of  $C_1$  immediately provides the main feature of the first curve and of the eigenfunctions associated to each element of  $C_1$  (see [16]). In particular, just by reading the definition of  $c(r)$  we can prove the following

**Proposition 4.1.**

- (a)  $C_1$  is a continuous and strictly decreasing curve, symmetric with respect to the diagonal.
- (b)  $C_1 \subset \{(x, y) \in \mathbb{R}^2 : x > \lambda_1(\Omega), y > \lambda_1(\Omega)\}$  and it is asymptotic to the lines  $\lambda_1(\Omega) \times \mathbb{R}$  and  $\mathbb{R} \times \lambda_1(\Omega)$ .
- (c) Any eigenfunction associated to  $(x, y) \in C_1$  admits exactly two nodal domains (Courant nodal domain theorem).

*Proof.* (a) The symmetry of  $C_1$  can be derived by interchanging the role of  $\omega_1$  and  $\omega_2$ . The continuity is immediate by the definition of  $c(r)$  as in (24); the monotonicity of the curve is equivalent to the fact that  $r_1 > r_2 \Rightarrow c(r_1) > c(r_2)$ ; but this directly follows once again by the definition of  $c(r)$ .

(b) The first part is given by the fact that  $c(r) > \lambda_1(\Omega)$ . Proving the existence of the asymptotes is equivalent to show that  $c(r) \rightarrow \lambda_1(\Omega)$  as  $r \rightarrow 0$ . To this aim, let  $\varepsilon > 0$  be fixed and consider a small ball of radius  $\rho$ , such that  $\lambda_1(\Omega \setminus B(x, \rho)) \leq \lambda_1(\Omega) + \varepsilon$ . Let us consider the partition made up by  $\omega_1 = B(x, \rho)$  and  $\omega_2 = \Omega \setminus B(x, \rho)$ , then choose  $r > 0$  small enough in such a way that  $r\lambda_1(B(x, \rho)) < \lambda_1(\Omega) + \varepsilon$ : it turns out that  $c(r) \leq \max\{r\lambda_1(\omega_1), \lambda_1(\omega_2)\} \leq \lambda_1(\Omega) + \varepsilon$ . We have thus proved that  $\forall \varepsilon > 0$  there exists  $r = r_\varepsilon > 0$  such that  $\lambda_1(\Omega) < c(r_\varepsilon) < \lambda_1(\Omega) + \varepsilon$ , concluding the proof.

(c) The nodal property is already true by definition of  $c(r)$  and the procedure of partitioning  $\Omega$  exactly in two connected subsets (see Lemma 2.2). □

*4.1. Further results.*

In this section let us develop some extensions of the previous techniques. The first applies to the search of further elements of the Fučík spectrum. Then, we show how to recover the case of more general operators.

For  $1 \leq h \leq k$ , let us define the numbers

$$c_{h,k}(r) := \inf_{(\omega_i) \in \mathcal{P}_k} \max\{r\lambda_1(\omega_1), \dots, r\lambda_1(\omega_h), \lambda_1(\omega_{h+1}), \dots, \lambda_1(\omega_k)\}.$$

We know by all the previous discussion that the infima above are attained and that a suitable choice of the eigenfunctions  $u_i$  corresponding to the optimal partition  $\omega_i$ , satisfy the extremality conditions stated in Theorem 1.1. Moreover we have that

$$c_{h,k}(r) \equiv \lambda_1(\omega_j) = r\lambda_1(\omega_i), \quad 1 \leq i \leq h, j \geq h + 1.$$

Now assume that the interfaces  $\Gamma_{i,j} = \partial\omega_i \cap \partial\omega_j$  consist only of points of multiplicity two and moreover that  $\Gamma_{i,j} = \emptyset$  unless  $i \in \{1, \dots, h\}$  and  $j \in \{h + 1, \dots, k\}$ . Let us consider the function

$$u = u_1 - u_{h+1} + u_2 - u_{h+2} + \dots$$

Then, by property (2) in Theorem 2.2, we have that  $u$  is regular and thus it is a solution of the equation

$$-\Delta u = r^{-1}c_{h,k}(r)u^+ - c_{h,k}(r)u^-,$$

in the whole of  $\Omega$ . Therefore, under the above topological assumptions on the interfaces  $\Gamma_{i,j}$ , we have proved that

$$(r^{-1}c_{h,k}(r), c_{h,k}(r)) \in \mathcal{F},$$

providing a new nontrivial element of the spectrum. This may happen in several practical situations, as shown by the numerical experiments in [19]. For instance, this is always the case when working in 1–dimensional domains. This actually proves the existence of a sequence of curves of the Fučík spectrum, as stated in Theorem 1.3.

Let us conclude by pointing out some possible generalizations of the above results. Actually, thanks to the discussion developed in Remark 2.2, we already know that the abstract setting leading to the proof of Theorem 2.2, applies to more general problems. As a consequence, we can prove results analogous to Theorem 1.2 and 1.3, which describe nontrivial elements of some possible generalization of the spectrum. In particular, we can characterize a first nontrivial curve of elements in the spectrum of the  $p$ –Laplacian (see [14] for the definition and a comparison) just replacing the notion of first eigenvalue with the one related to the new operator, namely

$$\lambda_1(\omega) := \min_{\substack{u \in W_0^{1,p}(\omega) \\ u \neq 0}} \frac{\int_{\omega} |\nabla u(x)|^p dx}{\int_{\omega} |u(x)|^p dx}.$$

Another interesting application consists in the characterization of the first curve of elements for a generalized notion of spectrum in presence of positive weights  $p, q$ . Namely, Theorem 1.2 applies to describe the set of  $(\lambda, \mu)$  such that

$$-\Delta u = \lambda p(x)u^+ - \mu q(x)u^-$$

has a nontrivial solution. In this case we have the natural replacement of the definition of  $\lambda_1$  with the corresponding weighted ones

$$\lambda_1(\omega_1) = \min_{\substack{u \in H_0^1(\omega_1) \\ u \neq 0}} \frac{\int_{\omega_1} |\nabla u(x)|^2 dx}{\int_{\omega_1} p(x)|u(x)|^2 dx},$$

$$\lambda_1(\omega_2) = \min_{\substack{u \in H_0^1(\omega_2) \\ u \neq 0}} \frac{\int_{\omega_2} |\nabla u(x)|^2 dx}{\int_{\omega_2} q(x)|u(x)|^2 dx}.$$

Finally, let us only mention that the whole theory can be easily modified in order to apply to general boundary conditions besides the Dirichlet case.

### 5. Monotonicity Formulae

Consider the general problem of minimizing

$$(27) \quad \beta(k, N) := \inf_{\mathcal{P}(k, N)} \frac{2}{k} \sum_{i=1}^k \sqrt{\lambda_1(\omega_i)}$$

where  $S^{N-1}$  denotes the boundary of the unit ball in  $\mathbb{R}^N$  and

$$\mathcal{P}(k, N) := \left\{ (\omega_1, \dots, \omega_k) \subset S^{N-1} : \omega_i \text{ is open and connected,} \right. \\ \left. \omega_i \cap \omega_j = \emptyset \text{ if } i \neq j \right\}.$$

In this section we are concerned with the properties of  $\beta(k, N)$  and with its relation with some extensions of the monotonicity formula. First of all, it can be proved that  $\beta(k, N)$  is achieved by a partition containing only open and connected sets of  $S^{N-1}$ . This directly comes by Remark 2.1, Theorem 2.2, and Lemma 2.2, where the results are obtained for partitions of domains in  $\mathbb{R}^N$ . The proof of this fact for partitions of  $S^{N-1}$  can be recovered (possibly through local charts) in a straightforward way.

Let us now concentrate on the value of  $\beta$ : when there are only two parts, the optimal partition is achieved by the equator-cut sphere (see [23]) and hence

$$\beta(2, N) = 2\sqrt{N-1},$$

(thus, in particular, for  $k = 2$  and  $N = 2$  our Lemma 1.2 exactly gives the result in [6]). When  $k \geq 3$  the only exact value of  $\beta$  we can give refers to the dimension  $N = 2$  and reads

$$\beta(k, 2) = k,$$

as can be found in [7]. Nevertheless, for  $k \geq 3$ , we are going prove that

$$\beta(k, N) > 2\sqrt{N-1}$$

in any dimension larger than 2, as a consequence of the monotonicity of  $\beta$  as a function of  $k$ :

**Proposition 5.1.** *The function  $\beta(\cdot, N) : \mathbb{N} \rightarrow \mathbb{R}^+$  is non decreasing. Moreover,  $\beta(k, N) > \beta(2, N)$  for  $k \geq 3$ .*

*Proof.* Let  $N \geq 2$  be fixed and let  $(\Omega_1, \dots, \Omega_{k+1}) \in \mathcal{P}(k+1, N)$  be a partition of  $S^{N-1}$  which achieves  $\beta(k+1, N)$ . Let us assume, to fix the ideas, that  $\lambda_1(\Omega_{k+1}) \geq \lambda_1(\Omega_i)$ ,  $i = 1, \dots, k$ . If we consider  $(\Omega_1, \dots, \Omega_k)$  as an element of  $\mathcal{P}(k, N)$  we easily obtain

$$\beta(k+1, N) = \frac{1}{k+1} \sum_{i=1}^{k+1} \sqrt{\lambda_1(\Omega_i)} \geq \frac{1}{k} \sum_{i=1}^k \sqrt{\lambda_1(\Omega_i)} \geq \beta(k, N),$$

and the equality holds iff  $\lambda_1(\Omega_i) = \lambda_1(\Omega_{k+1})$  for every  $i$  and  $(\Omega_1, \dots, \Omega_k)$  achieves  $\beta(k, N)$ . As a first consequence, we obtain the weak monotonicity of  $\beta(\cdot, N)$ .

To conclude the proof of the lemma, we will show that  $\beta(3, N) > \beta(2, N)$ . Assume by contradiction that  $\beta(3, N) = \beta(2, N)$ . By the above considerations (in the case  $k = 2$ ), we obtain that  $\lambda_1(\Omega_1) = \lambda_1(\Omega_2) = \lambda_1(\Omega_3) =: \lambda_2$  and that  $(\Omega_1, \Omega_2)$  achieves  $\beta(2, N)$ . Let  $(u_1, u_2) \in \mathcal{S}$  denote the associated eigenfunctions. Then, by definition of  $\mathcal{S}$ , we obtain both  $-\Delta \hat{u}_1 \geq \lambda_1(\Omega_1)u_1 - \lambda_1(\Omega_2)u_2$  and  $-\Delta \hat{u}_2 \geq \lambda_1(\Omega_2)u_2 - \lambda_1(\Omega_1)u_1$ , that is,  $-\Delta(u_1 - u_2) = \lambda_2(u_1 - u_2)$  on  $S^{N-1}$ ; but  $u_1 - u_2 \equiv 0$  on  $\Omega_3$ , in contradiction with the well known properties of the eigenfunctions of the Laplace operator (unique continuation property).  $\square$

As we said, the function  $\beta$  naturally appears when trying to extend a variant of the monotonicity formula to the case of many subharmonic densities. In this perspective we are going to prove Lemma 1.2.

*Proof of Lemma 1.2.* To start with, we observe that the function  $\Phi(r)$  is absolutely continuous. The idea of the proof consists then in showing that  $\Phi'(r) \geq 0$  for almost every  $r$ . Let us start with some estimates. First, since each  $w_i$  is positive and  $-\Delta w_i \leq 0$ , testing with  $w_i$  on the sphere  $B(x_0, r) =: B_r$  (with  $r \leq \bar{r}$ ) we obtain, for every  $i$ :

$$\int_{B_r} |\nabla w_i|^2 \leq \int_{\partial B_r} w_i \frac{\partial}{\partial n} w_i.$$

Let  $\nabla_T w_i := \nabla w_i - n \partial_n w_i$  be the tangential component of the gradient of  $w_i$ . We apply the Hölder inequality to the previous equation, then we multiply and divide by the  $L^2$ -norm of  $\nabla_T w_i$ , and finally we use the Young inequality. We have

$$\begin{aligned} \int_{B_r} |\nabla w_i|^2 &\leq \left( \int_{\partial B_r} w_i^2 \right)^{1/2} \left( \int_{\partial B_r} (\partial_n w_i)^2 \right)^{1/2} \leq \\ &\leq \left( \int_{\partial B_r} |\nabla_T w_i|^2 \right)^{1/2} \left( \int_{\partial B_r} (\partial_n w_i)^2 \right)^{1/2} \cdot \frac{\left( \int_{\partial B_r} w_i^2 \right)^{1/2}}{\left( \int_{\partial B_r} |\nabla_T w_i|^2 \right)^{1/2}} \leq \\ (28) \quad &\leq \frac{1}{2} \left[ \int_{\partial B_r} |\nabla_T w_i|^2 + \int_{\partial B_r} (\partial_n w_i)^2 \right] \cdot \left( \frac{\int_{\partial B_r} w_i^2}{\int_{\partial B_r} |\nabla_T w_i|^2} \right)^{1/2} \leq \\ &\leq \frac{1}{2} \int_{\partial B_r} |\nabla w_i|^2 \cdot \left( \frac{\int_{\partial B_r} w_i^2}{\int_{\partial B_r} |\nabla_T w_i|^2} \right)^{1/2}. \end{aligned}$$

Now let  $v_i^{(r)} : S^{N-1} \rightarrow \mathbb{R}$  be defined as  $v_i^{(r)}(\xi) := w_i(x_0 + r\xi)$ . Then  $\nabla v_i^{(r)}(\xi) = r^2 \nabla_T w_i(x_0 + r\xi)$ . By the previous inequality we obtain

$$\begin{aligned} \frac{\int_{\partial B_r} |\nabla w_i|^2}{\int_{B_r} |\nabla w_i|^2} &\geq 2 \left( \frac{\int_{\partial B_r} |\nabla_T w_i|^2}{\int_{\partial B_r} w_i^2} \right)^{1/2} \geq 2 \left( \frac{r^{-2} \int_{S^{N-1}} |\nabla v_i^{(r)}|^2}{\int_{S^{N-1}} (v_i^{(r)})^2} \right)^{1/2} \\ &\geq \frac{2}{r} \sqrt{\lambda_1(\{v_i^{(r)} > 0\})}. \end{aligned}$$

Since  $w_i w_j = 0$  a.e., the supports of the  $v_i^{(r)}$ 's constitute, for a.e.  $r$ , a partition of  $S^{N-1}$ . Therefore, summing up on  $i$  the previous inequality and recalling the definition of  $\beta$  (and also (18)), we finally have

$$(29) \quad \sum_{i=1}^h \frac{\int_{\partial B_r} |\nabla w_i|^2}{\int_{B_r} |\nabla w_i|^2} \geq \frac{h}{r} \beta(h, N) \quad \text{for almost every } r.$$

Now we are ready to prove the lemma: by computing  $\Phi'(r)$  we obtain

$$\begin{aligned}\Phi'(r) &= -\frac{h\beta(h, N)}{r^{h\beta(h, N)+1}} \prod_{i=1}^h \int_{B_r} |\nabla w_i|^2 \\ &\quad + \frac{1}{r^{h\beta(h, N)}} \sum_{i=1}^h \left[ \left( \prod_{j \neq i} \int_{B_r} |\nabla w_j|^2 \right) \int_{\partial B_r} |\nabla w_i|^2 \right] = \\ &= \Phi(r) \left( -\frac{h\beta(h, N)}{r} + \sum_{i=1}^h \frac{\int_{\partial B_r} |\nabla w_i|^2}{\int_{B_r} |\nabla w_i|^2} \right).\end{aligned}$$

Replacing (29) in the previous equation, the lemma follows.  $\square$

*Remark 5.1.* The argument above shows that the function  $\Phi$  is in fact strictly increasing, except in the case when  $w_i(r, \theta) = r^\alpha \phi_i$ , where  $\alpha = \beta(k, N) - N + 1$  and the  $\phi_i$ 's are the first eigenfunctions of the Laplace–Beltrami operator on the unit sphere, associated to the optimal partition (27).

With similar ideas we can prove also Lemma 1.3.

*Proof of Lemma 1.3.* We follow the outline of the proof of Lemma 1.2: we test the equation with  $u_i$  and, after some calculations, we obtain the counterpart of (28), that is

$$\begin{aligned}&\int_{B_r} \left[ |\nabla u_i|^2 + u_i^2 \sum_{j \neq i} a_{ij} u_j \right] \\ &\leq \frac{1}{2} \int_{\partial B_r} \left[ |\nabla u_i|^2 + u_i^2 \sum_{j \neq i} a_{ij} u_j \right] \cdot \left( \frac{\int_{\partial B_r} u_i^2}{\int_{\partial B_r} \left[ |\nabla u_i|^2 + u_i^2 \sum_{j \neq i} a_{ij} u_j \right]} \right)^{1/2}.\end{aligned}$$

As in the proof of that lemma, we let  $v_i^{(r)} : S^{N-1} \rightarrow \mathbb{R}$  be defined as  $v_i^{(r)}(\xi) := u_i(r\xi)$ , and again  $\nabla v_i^{(r)}(\xi) = r^2 \nabla_T u_i(r\xi)$ , providing

$$\begin{aligned}\frac{\int_{\partial B_r} |\nabla u_i|^2 + u_i^2 \sum_{j \neq i} a_{ij} u_j}{\int_{B_r} |\nabla u_i|^2 + u_i^2 \sum_{j \neq i} a_{ij} u_j} &\geq 2 \left( \frac{\int_{\partial B_r} |\nabla u_i|^2 + u_i^2 \sum_{j \neq i} a_{ij} u_j}{\int_{\partial B_r} u_i^2} \right)^{1/2} \\ &\geq \frac{2}{r} \sqrt{\Lambda_i(r)}\end{aligned}$$

where

$$\Lambda_i(r) = \frac{\int_{S^{N-1}} |\nabla v_i^{(r)}|^2 + r^2 (v_i^{(r)})^2 \sum_{j \neq i} a_{ij} v_j^{(r)}}{\int_{S^{N-1}} (v_i^{(r)})^2}.$$

By computing  $\Phi'$  as in the proof of Lemma 1.2, and taking into account the previous calculations, we have

$$\Phi'(r) \geq \Phi(r) \left( -\frac{hh'}{r} + \frac{2}{r} \sum_{i=1}^h \sqrt{\Lambda_i(r)} \right).$$

Hence we are lead to prove that there exists  $r_0$  sufficiently large, such that

$$(30) \quad \Phi'(r) \geq 0 \quad \forall r \geq r_0.$$

Observe that here we can not conclude as in the proof of Lemma 1.2: indeed, the supports of the functions  $v_i^{(r)}$ 's are not mutually disjoint, and thus we can not compare the value of  $\sum \sqrt{\Lambda_i(r)}$  with  $\beta(h, N)$ . To overcome this problem, we will let  $r \rightarrow \infty$ , proving the convergence of (suitable multiples of) the  $v_i^{(r)}$ 's to a  $k$ -tuple of functions on  $S^{N-1}$  having disjoint supports.

To start with, observe that we can assume w.l.o.g. that each  $\Lambda_i(r)$  is bounded in  $r$ , otherwise (30) would be already proved. By this boundedness, we derive that, for  $r$  large,

$$(31) \quad \int_{S^{N-1}} (v_i^{(r)})^2 \geq C > 0.$$

Indeed, assume not. This means that as  $r \rightarrow \infty$  we have (up to a subsequence)  $\int_{S^{N-1}} (v_i^{(r)})^2 \rightarrow 0$ . By the Hölder inequality and since  $r$  is large, we infer that

$$\frac{1}{|\partial B_r|} \int_{\partial B_r} u_i \rightarrow 0.$$

Now we recall that  $u_i$  is subharmonic, since it solves equation (4): hence, by the Mean Value Theorem and the previous inequality, we have  $u_i(0) = 0$ , a contradiction since  $u_i$  is strictly positive.

Let us now prove (30) by showing that

$$\frac{2}{h} \sum_{i=1}^h \sqrt{\Lambda_i(r)} > h' \quad \forall r \geq r_0.$$

To this aim we argue by contradiction, and we assume the existence of  $r_n \rightarrow \infty$  such that

$$(32) \quad \frac{2}{h} \sum_{i=1}^h \sqrt{\Lambda_i(r_n)} \leq h' < \beta(h, N).$$

Let us define

$$\tilde{v}_{i,n} = \frac{v_i^{(r_n)}}{\left(\int_{S^{N-1}} (v_i^{(r_n)})^2\right)^{1/2}}.$$

We have

$$(33) \quad C \geq \Lambda_i(r_n) = \int_{S^{N-1}} \left[ |\nabla \tilde{v}_{i,n}|^2 + r_n^2 \left( \prod_{j \neq i} \|v_j^{(r_n)}\|^2 \right) (\tilde{v}_{i,n})^2 \sum_{j \neq i} a_{ij} \tilde{v}_{j,n} \right].$$

Since by definition  $\int_{S^{N-1}} \bar{v}_{i,n}^2 = 1$ , equation (33) implies that  $\int_{S^{N-1}} \nabla(\bar{v}_{i,n})^2$  is bounded. We infer the existence of a subsequence  $n_k \rightarrow \infty$  and  $\bar{v}_i \neq 0$  such that  $\bar{v}_{i,n_k} \rightharpoonup \bar{v}_i$ , weakly in  $H^1(S^{N-1})$ , as  $k \rightarrow \infty$ . This immediately gives

$$\lim_{k \rightarrow \infty} \Lambda_i(r_{n_k}) \geq \frac{\int_{S^{N-1}} |\nabla \bar{v}_i|^2}{\int_{S^{N-1}} \bar{v}_i^2} = \lambda_1(\{\bar{v}_i > 0\}).$$

Taking into account (31), we infer by (33) that  $\bar{v}_{i,n_k} \bar{v}_{j,n_k} \rightarrow 0$  a.e., and therefore, by weak convergence,  $\bar{v}_i \cdot \bar{v}_j \equiv 0$  if  $i \neq j$ . Hence the supports of the  $\bar{v}_i$ 's constitute a partition of  $S^{N-1}$ . Using this information and summing up on  $i$  the last inequality, we have

$$\lim_{k \rightarrow \infty} \sum_{i=1}^h \sqrt{\Lambda_i(r_{n_k})} \geq \frac{h}{2} \beta(h, N).$$

This provides a contradiction with (32), concluding the proof.  $\square$

Let us conclude this section by pointing out a straightforward consequence of the above monotonicity formula. Indeed, it induces some growth restriction to the solutions of (4) with positive components, as the following argument proves. First, notice that Lemma 1.3 with the choice  $h = k$  and any  $k' < \beta(k, N)$  gives  $\Phi(r) \geq \Phi(r')$  for all  $r \geq r'$ . We can assume w.l.o.g. that  $\Phi(r') = 1$  so that

$$(34) \quad \prod_{i=1}^k \int_{B(0,r)} \left( |\nabla u_i(x)|^2 + u_i^2(x) \sum_{j \neq i} a_{ij} u_j(x) \right) dx > r^{kk'} \quad r > r'.$$

Let us now go back to the differential equation for  $u_i$ : multiplying by  $u_i$  and integrating we have

$$\int_{B(0,r)} \left( |\nabla u_i|^2 + u_i^2 \sum_{j \neq i} a_{ij} u_j \right) = \int_{\partial B(0,r)} u_i \partial_n u_i.$$

Let us now suppose that there exists  $\alpha > 0$  such that, for all  $i$

$$\partial_r u_i \leq Cr^\alpha.$$

Then the r.h.s. is bounded by  $Cr^{N-1} r^{\alpha+1} r^\alpha$ . Using this in (34) we have, for  $r$  large,

$$r^{kk'} < \prod_{i=1}^k \int_{B(0,r)} \left( |\nabla u_i|^2 + u_i^2 \sum_{j \neq i} a_{ij} u_j \right) \leq Cr^{k(2\alpha+N)}.$$

This provides  $\alpha \geq (k' - N)/2$  for every  $k' < \beta(k, N)$ . Hence we can state the following

**Proposition 5.2.** *Let  $U = (u_1, \dots, u_k)$  be a solution of (4) on  $\mathbb{R}^N$  with strictly positive components. Assume that there exists  $\alpha > 0$  such that  $|\nabla U| \leq Cr^\alpha$ . Then  $\alpha \geq (\beta(k, N) - N)/2$ .*

In particular in dimension  $N = 2$ , if  $k \geq 3$  all the positive solutions of (4) are unbounded at infinity, together with their gradients.

## 6. Liouville type Theorems

In the spirit of Proposition 5.2, let us focus our attention on some nonexistence results of Liouville–type which follow by application of the monotonicity formulae. We start by proving that the system (4) does not admit Hölder continuous solutions: this is a crucial step when analyzing the rate of convergence of a class of competition–diffusion systems, as the parameter of interspecific competition tends to infinity ([9]).

**Proposition 6.1.** *Let  $k \geq 2$  and let  $U$  be a solution of (4) on  $\mathbb{R}^2$ . Let  $\alpha \in (0, 1)$  such that*

$$(35) \quad \max_{i=1, \dots, k} \sup_{x \in \mathbb{R}^N} \frac{|u_i(x)|}{1 + |x|^\alpha} < \infty.$$

*Then,  $k - 1$  components annihilate and the last is a nonnegative constant.*

*Proof.* By the strong maximum principle, every  $u_i$  is either identically zero or strictly positive. Let  $h$  be the number of the components not identically zero. If  $h = 1$ , then the proposition follows, without any growth assumption, by the straight application of Liouville’s theorem. Hence let  $h \geq 2$  and  $u_1, \dots, u_h$  strictly positive. Let  $B_r = B(0, r)$ . By Lemma 1.3 we know that

$$(36) \quad \prod_{i=1}^h \frac{1}{r^{h'}} \int_{B_r} \left( |\nabla u_i(x)|^2 + u_i^2(x) \sum_{j \neq i} a_{ij} u_j(x) \right) dx \geq C > 0$$

when  $h' < \beta(h, N)$  and  $r$  is large enough.

On the other hand, let us consider a smooth, radial cut–off function which is equal 1 in  $B_r$  and vanishes outside  $B_{2r}$ . Let us multiply the  $i$ –th differential equation by  $\eta^2 u_i$  and then integrate:

$$\begin{aligned} & \int_{B_{2r}} \eta^2 |\nabla u_i|^2 + \eta^2 u_i^2 \sum_{j \neq i} a_{ij} u_j \leq \int_{B_{2r}} |2\eta u \nabla \eta \nabla u| \\ & \leq \frac{1}{2} \int_{B_{2r}} \eta^2 |\nabla u_i|^2 + 2 \int_{B_{2r}} u_i^2 |\nabla \eta|^2 \end{aligned}$$

and hence

$$\int_{B_r} |\nabla u_i|^2 + u_i^2 \sum_{j \neq i} a_{ij} u_j \leq 4 \int_{B_{2r} \setminus B_r} u_i^2 |\nabla \eta|^2 \leq \frac{C}{r^2} \int_{B_{2r} \setminus B_r} u_i^2.$$

By (35), when  $\rho$  is sufficiently large, we have that  $u(x) \leq C' \rho^\alpha$  for all  $x \in \partial B_\rho$ : using this fact in the above inequality and passing to polar coordinates we obtain

$$\int_{B_r} |\nabla u_i|^2 + u_i^2 \sum_{j \neq i} a_{ij} u_j \leq \frac{C}{r^2} \int_r^{2r} \rho^{N-1+2\alpha} d\rho = Cr^{N-2(1-\alpha)}$$

for all indices  $i$ . Let  $N = 2$ ; comparing with (36) we have  $r^{h'} \leq Cr^{2-2(1-\alpha)}$  for  $r$  large enough. But now, we can choose  $h' := 1 + \alpha < 2 \leq \beta(h, 2)$ , which provides a contradiction.  $\square$

Following the same line of the previous proof, but exploiting the monotonicity formula Lemma 1.2 (with  $\bar{r} = \infty$ ), the subsequent result follows at once:

**Proposition 6.2.** *Let  $k \geq 2$  and let  $U = (u_1, \dots, u_k)$  such that  $u_i \cdot u_j = 0$  if  $i \neq j$  and  $-\Delta u_i \leq 0$  on  $\mathbb{R}^2$  for all  $i$ . Let  $\alpha \in (0, 1)$  such that*

$$\sup_{x \in \mathbb{R}^N} \frac{|u_i(x)|}{1 + |x|^\alpha} < \infty$$

for all  $i = 1, \dots, k$ . Then each component  $u_i$  is constant.

It is worthwhile noticing that an analogous nonexistence result holds for harmonic functions on the entire space and it can be proved in a similar fashion by using the original monotonicity formula Lemma 1.1.

**Proposition 6.3.** *Let  $u$  be an harmonic function on  $\mathbb{R}^2$ . Let  $\alpha \in (0, 1)$  such that*

$$\sup_{x \in \mathbb{R}^N} \frac{|u(x)|}{1 + |x|^\alpha} < \infty.$$

Then  $u$  is constant.

In fact, the last assertion could be proved even easier, by a simple test of the equation  $-\Delta u = 0$  with  $u$ .

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