

# ASYMPTOTIC ESTIMATES FOR THE SPATIAL SEGREGATION OF COMPETITIVE SYSTEMS

MONICA CONTI, SUSANNA TERRACINI, GIANMARIA VERZINI

ABSTRACT. For a class of population models of competitive type, we study the asymptotic behavior of the positive solutions as the competition rate tends to infinity. We show that the limiting problem is a remarkable system of differential inequalities, which defines the functional class  $\mathcal{S}$  in (2). By exploiting the regularity theory recently developed in [10] for the elements of functional classes of the form  $\mathcal{S}$ , we provide some qualitative and regularity property of the limiting configurations. Besides, for the case of two competing species, we obtain a full description of the limiting states and we prove some quantitative estimates for the rate of convergence. Finally, we prove some new Liouville type results which allow to have uniform regularity estimates of the solutions.

## 1. INTRODUCTION

A central problem in population ecology is the understanding of the interactions between biological species. To this aim, different models based on reaction diffusion equations can be successfully employed, in particular to investigate phenomena of coexistence and exclusion of competing species. Several theoretical studies have been carried out mainly for competition models of Lotka–Volterra type, in the case of two competing species, see e.g. [4, 7, 22, 23, 20, 24, 27, 28, 29]. In these papers, the main point is to investigate the existence of spatially inhomogeneous solutions; in most cases, the pattern formation is shown to be driven by the presence of different diffusion rates when the coefficients of intra-specific and inter-specific competitions are suitable related. Recently, also the case of three competing densities, which is far more complex, has become object of an extensive investigation [14, 15, 25, 26]. A different approach to the occurrence of nontrivial solutions for models of Lotka–Volterra type, consists in studying the pattern formation driven by strong competition, see [12, 13, 18, 19, 21]. It turns out that the presence of large interactions of competitive type produces, to the limit, the spatial segregation of the densities. In other words, in the limiting configurations all the populations survive, but have disjoint habitats. The objective of this paper is to improve the understanding of the spatial segregation phenomenon governed by strong competition. To this aim, we consider the following model.

Let  $\Omega \subset \mathbb{R}^N$  be a connected, open bounded domain with smooth boundary  $\partial\Omega$  and let  $k \geq 2$  be a fixed integer. We consider the system of  $k$  differential equations

$$(1) \quad \begin{cases} -\Delta u_i(x) = -\kappa u_i(x) \sum_{j \neq i} a_{ij} u_j(x) + f_i(x, u_i) & x \in \Omega \\ u_i(x) = \varphi_i(x) & x \in \partial\Omega \end{cases}$$

---

*Date:* May 21, 2004.

*2000 Mathematics Subject Classification.* 35B40, 35R35, (35J60, 92D25).

*Key words and phrases.* Competition–diffusion systems, asymptotic behaviour of solutions, free boundary problems, Liouville–type theorems.

Research partially supported the Italian MIUR Research Projects *Metodi Variazionali ed Equazioni Differenziali Nonlineari*,

for  $i = 1, \dots, k$ . This system governs the steady-states of  $k$  competing species coexisting in the same area  $\Omega$ . Here  $u_i$  represents the population density of the  $i$ -th specie, whose internal dynamic is prescribed by  $f_i$ ; the positive constants  $\kappa \cdot a_{ij}$  determine the interaction between the population  $u_i$  and  $u_j$ , which is possibly asymmetric. Besides, the boundary data  $\varphi_i$  are positive  $W^{1,\infty}(\partial\Omega)$ -functions with disjoint supports, namely  $\varphi_i \cdot \varphi_j = 0$  for  $i \neq j$ , almost everywhere on  $\partial\Omega$ .

In our model, the interspecific competition rate  $\kappa > 0$  has to be regarded as a large parameter, and the major goal of the paper consists in analyzing the behavior of the *positive* steady-states of (1) (i.e. solutions where  $u_i > 0$  for all  $i$ ), as  $\kappa$  tends to infinity. For the sake of simplicity, in the course of this investigation we shall assume  $f_i \equiv 0$  for all  $i$ . The reader could easily extend all the results contained in the paper, with the exception of Sections 4 and 5, also in presence of large class of nontrivial dynamics  $f_i$ .

The segregation phenomenon has already been observed in literature, mainly in the case of two competing densities. Recently, suitable free boundary problems have been studied in connection with the asymptotic behavior of some models of population dynamics with diffusion, see in [16, 19, 32]. In the present paper, we suggest a different approach for the study of the limit configurations, which applies to systems with any number of densities. We associate to (1) the functional class

$$(2) \quad \mathcal{S} = \left\{ (u_1, \dots, u_k) \in (H^1(\Omega))^k : \begin{array}{l} u_i \geq 0, u_i|_{\partial\Omega} = \varphi_i, \\ u_i \cdot u_j = 0 \text{ if } i \neq j \\ -\Delta u_i \leq 0, -\Delta \hat{u}_i \geq 0 \end{array} \right\}.$$

where  $\hat{u}_i := u_i - \sum_{j \neq i} \frac{a_{ij}}{a_{ji}} u_j$ . At first we establish that  $\mathcal{S}$  contains all the asymptotic limits of (1):

**Theorem 1.** *Let  $(u_{i,\kappa})$  be a positive solution of (1). Then, there exists  $(u_i) \in \mathcal{S}$  such that, up to subsequences,  $\|u_{i,\kappa} - u_i\|_{H^1} \rightarrow 0$  as  $\kappa \rightarrow \infty$ .*

Therefore, the class  $\mathcal{S}$  is the natural framework where investigating the qualitative properties of the asymptotic limits of highly competing diffusion systems. In the recent paper [10], the authors have developed a theory concerning the regularity and the properties of the free boundary of the elements of analogous functional classes. This theory can be promptly used to give a description of the limiting configurations in presence of *symmetric* coefficients  $a_{ij} = a_{ji}$ . This is done in Section 3, where we also discuss those results that can be adapted to the case of non-symmetric interactions.

A main part of the paper is devoted to perform the asymptotic analysis as  $\kappa \rightarrow \infty$ . In the case of two species we are able to provide both a full description of the limits and precise quantitative estimates for the rate of convergence. Our main result reads as follows.

**Theorem 2.** *Let  $k = 2$  and let  $\Phi$  be the harmonic extension on  $\Omega$  of the boundary data  $\varphi_1 - \varphi_2$ . Then, for every  $\kappa > 0$ , problem (1) has a unique positive solution  $u_{1,\kappa} > \Phi^+$ ,  $u_{2,\kappa} > \Phi^-$ . Moreover, if we set  $\Phi_1 = \Phi^+$ ,  $\Phi_2 = \Phi^-$ , there exists  $C \geq 0$  such that*

$$\kappa^{1/6} \cdot \|u_{i,\kappa} - \Phi_i\|_{H^1(\Omega)} \leq C, \quad \text{as } \kappa \rightarrow \infty.$$

This analysis is carried out in Section 5, in light of the existence and uniqueness results obtained in Section 4 for the same system of 2 equations on the whole of  $\mathbb{R}^N$ . Besides, we prove that the convergence of  $u_{i,\kappa}$  to  $\Phi^\pm$  is actually the most regular possible, in the sense that the Lipschitz constants of  $u_{i,\kappa}$  are independent of  $\kappa$ .

**Theorem 3.** *Let  $k = 2$  and let  $(u_{i,\kappa})$  be a positive solution of (1) at fixed  $\kappa > 0$ . Then there exists  $L > 0$  such that*

$$\sup_{x,y \in \Omega} \frac{u_{i,\kappa}(x) - u_{i,\kappa}(y)}{|x - y|} < L$$

for all  $i = 1, 2$  and for all  $\kappa > 0$ .

The case of three or more competing species is far more complex. It seems to be hopeless to provide explicit formulae for the limiting densities, and the rate of convergence to the final configuration is no longer available. Nevertheless, besides some qualitative properties of the limiting state, that we already discussed, we can prove uniform regularity estimates of the solutions. Precisely, in Section 6, we show that the Hölder constants of any order  $\alpha \in (0, 1)$  of  $u_{i,\kappa}$ , are independent of  $\kappa$ .

**Theorem 4.** Let  $(u_{i,\kappa})$  be any solution of (1) at fixed  $\kappa > 0$ . Let  $\alpha \in (0, 1)$ . Then there exists  $L_\alpha > 0$  such that

$$\sup_{x,y \in \Omega} \frac{u_{i,\kappa}(x) - u_{i,\kappa}(y)}{|x - y|^\alpha} < L_\alpha$$

for all  $i = 1, \dots, k$  and for all  $\kappa > 0$ .

The proof relies on a blow up procedure and on some new Liouville type results proved in Section 7. Namely, we show that the system (1) on  $\mathbb{R}^N$  cannot have positive solutions which are Hölder continuous.

**Theorem 5.** Let  $k \geq 2$  and let  $U = (u_1, \dots, u_k)$  be a solution of

$$\begin{cases} -\Delta u_i(x) = -u_i(x) \sum_{j \neq i} a_{ij} u_j(x) & x \in \mathbb{R}^N \\ u_i(x) \geq 0 & x \in \mathbb{R}^N \end{cases}$$

for every  $i$ . Let  $\alpha \in (0, 1)$  such that

$$\max_{i=1, \dots, k} \sup_{x, y \in \mathbb{R}^N} \frac{|u_i(x) - u_i(y)|}{|x - y|^\alpha} < \infty.$$

Then,  $k - 1$  components annihilate and the last is a nonnegative constant.

An analogous nonexistence result holds true in the class  $\mathcal{S}$  as well, see Theorem 7.3. In order to establish our Liouville results, we need to provide suitable monotonicity-type formula, on the line of [1, 5], which applies to the solutions of (1) and to the element of  $\mathcal{S}$ , for any  $k$  and in any dimension. Similar results when  $N = 2$  have been recently proved in [11].

## 2. EXISTENCE AND ASYMPTOTIC BEHAVIOR

In this section we first prove that (1) has a positive solution at any fixed  $\kappa > 0$ ; then we perform the asymptotic analysis as  $\kappa \rightarrow \infty$ , in order to derive a suitable limiting problem for our population model.

Our approach to existence relies on the application of Leray–Schauder degree theory (see for instance [33]). As a first step, we need some apriori estimates for the solutions of (1). To this aim, let  $U = (u_1, \dots, u_k)$  be a solution of (1) at fixed  $\kappa$ . Since the r.h.s of the equation for  $u_i$  is non positive, we immediately obtain that

$$(3) \quad -\Delta u_i \leq 0 \quad \text{in } \Omega.$$

Let us now define, for all  $i$ :

$$\widehat{u}_i := u_i - \sum_{j \neq i} \frac{a_{ij}}{a_{ji}} u_j.$$

A straightforward calculation leads to

$$-\Delta \widehat{u}_i = \kappa \sum_{j \neq i} \frac{a_{ij}}{a_{ji}} \sum_{h \neq j} a_{jh} u_j u_h$$

and provides an opposite inequality for  $\widehat{u}_i$ :

$$(4) \quad -\Delta \widehat{u}_i \geq 0 \quad \text{in } \Omega.$$

Functions satisfying the above differential inequalities exhibit some remarkable boundedness properties that will be useful both in proving the existence and in studying the asymptotic behavior of the solutions to (1) as  $\kappa \rightarrow \infty$ . For this reason let

$$\mathcal{F} := \{U \in (H^1(\Omega))^k : u_i \geq 0 \text{ such that (3), (4) hold, } u_i = \varphi_i \text{ on } \partial\Omega, \forall i = 1, \dots, k\}.$$

The functions in  $\mathcal{F}$  share the following properties.

**Lemma 2.1.** *Let  $\Phi_i$  be the harmonic extension on  $\Omega$  of the boundary datum  $\varphi_i$  and let us denote by  $\Psi_i$  the harmonic extension on  $\Omega$  of  $\widehat{\varphi}_i := \varphi_i - \sum_{j \neq i} \frac{a_{ij}}{a_{ji}} \varphi_j$ . Then, for all  $U \in \mathcal{F}$  and for all  $i = 1, \dots, k$ , it holds*

- (i)  $\Psi_i \leq \widehat{u}_i \leq u_i \leq \Phi_i$  on  $\Omega$ ,
- (ii)  $\frac{\partial}{\partial \nu} \Phi_i \leq \frac{\partial}{\partial \nu} u_i \leq \frac{\partial}{\partial \nu} \Psi_i$  on  $\partial\Omega$ ,
- (iii)  $\mathcal{F}$  is bounded in  $(H^1(\Omega))^k$ .

**Proof:** since  $\widehat{u}_i$  satisfies

$$\begin{cases} -\Delta \widehat{u}_i \geq 0 & \text{in } \Omega \\ \widehat{u}_i = \widehat{\varphi}_i & \text{on } \partial\Omega, \end{cases}$$

then, by the maximum principle, it holds  $\widehat{u}_i \geq \Psi_i$  in  $\overline{\Omega}$  and

$$(5) \quad \frac{\partial}{\partial \nu} \widehat{u}_i \leq \frac{\partial}{\partial \nu} \Psi_i \quad \text{on } \partial\Omega.$$

Now note that if  $x \in \partial\Omega$  and  $u_i(x) = \varphi_i(x) > 0$ , then  $u_j(x) = 0$  for all  $j \neq i$ : this in particular ensures  $\frac{\partial}{\partial \nu} u_j(x) \leq 0$  (recall that  $u_j \geq 0$ ). Using this information in (5) we obtain the chain of inequalities:

$$\frac{\partial}{\partial \nu} u_i(x) \leq \sum_{j \neq i} \frac{a_{ij}}{a_{ji}} \frac{\partial}{\partial \nu} u_j(x) + \frac{\partial}{\partial \nu} \Psi_i(x) \leq \frac{\partial}{\partial \nu} \Psi_i.$$

In an analogous way, by the equation  $-\Delta u_i \leq 0$ , we immediately have that  $u_i \leq \Phi_i$  and

$$\frac{\partial}{\partial \nu} u_i \geq \frac{\partial}{\partial \nu} \Phi_i.$$

In order to prove the boundedness of  $\mathcal{F}$ , let  $U$  be fixed and consider the inequality  $-\Delta u_i \leq 0$ . Multiplying by  $u_i$  and then integrating we obtain

$$\int_{\Omega} |\nabla u_i|^2 - \int_{\partial\Omega} \varphi_i \frac{\partial}{\partial \nu} u_i \leq 0.$$

Since  $\frac{\partial}{\partial \nu} u_i \leq \frac{\partial}{\partial \nu} \Psi_i$ , then  $\|u_i\|_{H^1(\Omega)}$  is bounded by a constant depending only on the data.  $\blacksquare$

Now we are ready to prove the existence of solutions. We have

**Theorem 2.1.** *For every fixed  $\kappa > 0$  there exists at least one solution  $U_\kappa$  of (1). Moreover,  $U_\kappa$  is of class  $W^{1,\infty}(\overline{\Omega})$  and it belongs to the class  $\mathcal{F}$ .*

**Proof:** as already observed, every solution of (1) belongs to  $\mathcal{F}$ . Besides, the standard regularity theory for elliptic PDE ensures that every solution is  $W^{1,\infty}$  and hence of class  $C^{0,\alpha}$ , for every  $0 < \alpha < 1$ . To prove the existence, we perform the change of variables  $v_i := u_i - \Phi_i$  (with  $\Phi_i$  as in the previous theorem), so that  $\Delta v_i = \Delta u_i$  and  $v_i$  vanishes on  $\partial\Omega$ . It is easy to see that finding positive solutions to our problem is equivalent to solving the fixed point equation

$$(6) \quad V - \lambda LF(V) = 0$$

where  $V = (v_1, \dots, v_k) \in (C^{0,\alpha}(\Omega))^k$  (with  $\alpha$  fixed),  $\lambda = 1$  and

$$\begin{aligned} L : (C^{0,\alpha}(\Omega))^k &\longrightarrow (C_0^{2,\alpha}(\Omega))^k \subset (C^{0,\alpha}(\Omega))^k & (LV)_i &= (-\Delta)^{-1}v_i, \\ F : (C^{0,\alpha}(\Omega))^k &\longrightarrow (C^{0,\alpha}(\Omega))^k & (F(V))_i &= -\kappa(v_i + \Phi_i)^+ \sum_{j \neq i} a_{ij}(v_j + \Phi_j). \end{aligned}$$

Let  $W$  be any possible solution of (6) for a fixed  $\lambda \in [0, 1]$ . Then, apart from a change of variables, it solves (6) and it belongs to  $\mathcal{F}$ . Thus, by Lemma 2.1, we know that  $\lambda F(W)$  is bounded in  $L^\infty$ . Since  $L$  is compact, then  $\lambda LF(W)$  is bounded in  $(C^{0,\alpha})^k$ , not depending on  $\lambda$ . As a consequence, since  $W = \lambda LF(W)$  by assumption, there exists  $R > 0$  such that, if  $B_R$  denotes the ball of center 0 and radius  $R$  in  $(C^{0,\alpha})^k$ , then

$$V - \lambda LF(V) \neq 0 \text{ for every } V \in \partial B_R, \text{ for every } \lambda \in [0, 1].$$

By standard degree arguments (recall that degree theory trivially applies to the map  $I - \lambda LF$ ) we obtain

$$\deg(I - LF, B_R, 0) = \deg(I, B_R, 0) = 1.$$

Now the theorem follows in a standard way. ■

Now that we have a solution for every fixed  $\kappa > 0$ , we are interested in their behavior as  $\kappa \rightarrow \infty$ . We find that a limit always exists (up to subsequences), and it belongs itself to the class  $\mathcal{F}$ .

**Theorem 2.2.** *Let  $U_\kappa = (u_{1,\kappa}, \dots, u_{k,\kappa})$  be a solution of (1) at fixed  $\kappa$ . Let  $\kappa \rightarrow \infty$ : then, there exists  $U \in \mathcal{F}$  such that, for all  $i = 1, \dots, k$ :*

- (i) up to subsequences,  $u_{i,\kappa} \rightarrow u_i$  strongly in  $H^1$ ,
- (ii) if  $i \neq j$  then  $u_i \cdot u_j = 0$  a.e. in  $\Omega$ ,
- (iii)  $-\Delta u_i = 0$  in the set  $\{u_i > 0\}$ .

**Proof:** by applying Lemma 2.1, we immediately obtain the existence of a weak limit  $U$  such that, up to subsequences,  $u_{i,\kappa} \rightharpoonup u_i$  in  $H^1$ . The weak limit  $U$  belongs to  $\mathcal{F}$ , since the differential inequalities (3) and (4) for  $u_{i,\kappa}$  pass to the weak limit, and moreover the compact embedding  $H^1(\Omega) \subset H^{1/2}(\partial\Omega)$  provides  $u_i = \varphi_i$  on  $\partial\Omega$ .

Let us now multiply the differential equation for  $u_{i,\kappa}$  with  $u_{i,\kappa}$  itself and integrate on  $\Omega$ :

$$\int_\Omega |\nabla u_{i,\kappa}|^2 - \int_{\partial\Omega} \varphi_i \frac{\partial}{\partial \nu} u_{i,\kappa} = - \int_\Omega \kappa u_{i,\kappa}^2 \sum_{j \neq i} a_{ij} u_{j,\kappa}.$$

By Lemma 2.1 again, we know that the l.h.s is bounded independently of  $\kappa$ : then, letting  $\kappa \rightarrow \infty$  forces

$$\int_\Omega u_{i,\kappa}^2 \sum_{j \neq i} a_{ij} u_{j,\kappa} \rightarrow 0.$$

This implies  $u_{i,\kappa}(x) \cdot u_{j,\kappa}(x) \rightarrow 0$  a.e. in  $\Omega$  when  $i \neq j$ , and consequently  $u_i \cdot u_j = 0$  a.e. in  $\Omega$ . In particular this provides  $\widehat{u}_i = u_i$  on  $\{u_i > 0\}$  and consequently it holds  $-\Delta u_i = -\Delta \widehat{u}_i \geq 0$  on  $\{u_i > 0\}$ . Since the opposite inequality is satisfied for  $u_i$  on the whole domain, assertion (iii) is proven.

In order to prove that the convergence to  $U$  is in fact strong in  $H^1$ , let us test the inequality  $-\Delta \widehat{u}_i \geq 0$  with  $u_i$ : then

$$(7) \quad \int_\Omega |\nabla u_i|^2 - \int_{\partial\Omega} \varphi_i \frac{\partial}{\partial \nu} u_i \geq 0.$$

On the other side, by  $-\Delta u_{i,n} \leq 0$  tested with  $u_{i,n}$  it holds

$$(8) \quad \int_\Omega |\nabla u_{i,n}|^2 - \int_{\partial\Omega} \varphi_i \frac{\partial}{\partial \nu} u_{i,n} \leq 0.$$

Since  $\frac{\partial}{\partial \nu} u_{i,n} \rightarrow \frac{\partial}{\partial \nu} u_i$  in  $L^2(\partial\Omega)$ , passing to the limit in (8) and using (7) we obtain

$$\int_\Omega |\nabla u_i|^2 \geq \limsup \int_\Omega |\nabla u_{i,n}|^2.$$

This finally provides  $u_{i,n} \rightarrow u_i$  strongly in  $H^1$ . ■

### 3. QUALITATIVE PROPERTIES OF THE ASYMPTOTIC LIMITS

By Theorem 2.2, all the asymptotic limits of the solutions of problem (1) belong to the functional class  $\mathcal{S}$  defined in the introduction. A very similar class was introduced by the authors in [10] in connection with some optimal partition problems. In that paper, functions in  $\mathcal{S}$  were shown to enjoy regularity and qualitative properties of various nature. Compared with the definition of the class  $\mathcal{S}$  given in [10], in the present paper we consider a slight change in the “hat” operation there (namely,  $\hat{u}_i = u_i - \sum_{j \neq i} u_j$ ). This allows to take into account the presence of the coefficients  $a_{ij}$  in (1), where possibly  $a_{ij} \neq a_{ji}$ . Nevertheless, some of the results in [10] (cfr. Sections 6, 7, 8, 9.1, and the first part of Section 9.2) can be proven also for the class  $\mathcal{S}$  in (2), with trivial changes. We list all these results below, referring to [10] for the proofs. From now on,  $(u_1, \dots, u_k)$  will denote a limiting configuration of solutions to (1) as  $\kappa \rightarrow \infty$ , according to Theorem 2.2. Let us first recall a definition.

**Definition 3.1.** The multiplicity of a point  $x \in \Omega$  is

$$m(x) = \sharp \{i : \text{meas}(\{u_i > 0\} \cap B(x, r)) > 0, \forall r > 0\} .$$

**Theorem 3.1.** *The function  $U := \sum_{i=1}^k u_i$  verifies the following properties:*

- $U$  is Lipschitz continuous in the interior of  $\Omega$ ; if  $\partial\Omega$  is of class  $C^1$ , then  $U \in W^{1,\infty}(\bar{\Omega})$ ,
- let  $x \in \Omega$  such that  $m(x) = 2$ . Then, if

$$\lim_{\substack{y \rightarrow x \\ u_i(y) > 0}} \nabla u_i(y) = - \lim_{\substack{y \rightarrow x \\ u_j(y) > 0}} \nabla u_j(y) .$$

In dimension  $N = 2$ , we can prove some topological properties of the segregated configuration induced by  $U$ . Namely

**Theorem 3.2.** *Let  $N = 2$ . Then the following properties hold true.*

- Let  $x_0 \in \Omega$  such that  $m(x_0) = 2$ . Then  $\nabla U(x_0) \neq 0$  and the set  $\{x : m(x) = 2\}$  is locally a  $C^1$ -curve through  $x_0$ .
- Let  $x_0 \in \Omega$  such that  $m(x_0) \geq 3$ . Then  $|\nabla U(x)| \rightarrow 0$  as  $x \rightarrow x_0$ . Besides, there exists  $\{x_n\} \subset \Omega$  such that  $m(x_n) = 2$  and  $x_n \rightarrow x_0$

If furthermore, the interactions between the species are symmetric, then more is true.

**Theorem 3.3.** *Let  $N = 2$  and  $a_{ij} = a_{ji}$  for all  $i, j = 1, \dots, k$ . Assume furthermore that each  $\{u_i > 0\}$  is connected: then the following hold.*

- The set  $\{m(x) \geq 3\}$  consists of a finite numbers of points.
- Let  $x \in \Omega$  such that  $m(x) = h$ . Then,  $U$  admits a local expansion around  $x$  of the following form:

$$(9) \quad U(r, \theta) = r^{\frac{h}{2}} \left| \cos\left(\frac{h}{2}(\theta + \theta_0)\right) \right| + o(r^{\frac{h}{2}})$$

as  $r \rightarrow 0$ , where  $(r, \theta)$  denotes a system of polar coordinates around  $x$ .

- The set  $\{m(x) = 2\}$  consists in a finite number of  $C^1$ -arcs ending either at points with higher multiplicity, or at the boundary  $\partial\Omega$ .

The proof of Theorem 3.3 strongly depends on the validity of global differential equations for certain auxiliary functions related to the symmetric  $\hat{u}_i$ , as defined in [10]. In the case of nonsymmetric interactions, global equations are no longer available, and we cannot recover the validity of those qualitative properties of  $U$ . As a matter of fact, numerical computations seems to show that

when more than two populations meet, the segregated states induced by the strong competition in presence of nonsymmetric interactions, exhibit a truly different behavior when compared with the segregation in the symmetric case. In particular, the representation formula (9), asserting that the zero set departing from a multiple point is made up by straight lines which share the angle in equal parts, seems to be false. We intend to return on this in a subsequent paper.

#### 4. THE CASE OF TWO DENSITIES: THE PROBLEM ON $\mathbb{R}^N$ .

In this section we restrict our studies to (1) in the case of two competing species. At first, we study the model on the whole of  $\mathbb{R}^N$  and we establish some existence and uniqueness results for nontrivial solutions at fixed  $\kappa$ . This analysis will be crucial in the next section when performing the asymptotic study of the solutions of (1), as  $\kappa \rightarrow \infty$ . In order to simplify the notations, throughout the section we assume  $a_{12} = a_{21} = \kappa = 1$  (this is equivalent to choose as new unknowns the functions  $\kappa a_{21}u_1, \kappa a_{12}u_2$ ). In this setting, our problem becomes

$$(10) \quad \begin{cases} -\Delta u_1(x) = -u_1(x)u_2(x) & x \in \mathbb{R}^N \\ -\Delta u_2(x) = -u_1(x)u_2(x) & x \in \mathbb{R}^N \\ u_i(x) > 0 & x \in \mathbb{R}^N. \end{cases}$$

Subtracting the two equations, we obtain

$$u_1 - u_2 = \Phi,$$

where  $\Phi$  is an harmonic function on  $\mathbb{R}^N$ . With this notation, writing  $u = u_1$ , system (10) becomes equivalent to

$$(11) \quad -\Delta u = -u(u - \Phi), \quad u > \Phi^+ \text{ on } \mathbb{R}^N.$$

We start proving that  $\Phi$  cannot be constant.

**Lemma 4.1.** *Let  $C \geq 0$ . Then the equation*

$$-\Delta u = -u(u - C), \quad u > C$$

*has no solution defined on the whole  $\mathbb{R}^N$ .*

**Proof:** let  $u > C$  be a solution of the equation. We define

$$\varphi(r) := \int_{\partial B_r} u,$$

where  $B_r$  is the ball with center in the origin and radius  $r$ . Recalling that  $\int u^2 \geq (\int u)^2$ , we have

$$\varphi''(r) + \frac{N-1}{r}\varphi'(r) \geq \varphi^2(r) - C\varphi(r)$$

with  $\varphi(0) =: \varphi_0 > C$ ,  $\varphi'(0) = 0$ , and we want to prove that all the solutions of this class of initial value problems explode for finite  $r$ . In a standard way, the left hand side can be written as  $r^{1-N}(r^{N-1}\varphi'(r))'$ . Using this fact and trivial sign considerations, we have that  $\varphi'$  is non negative on the subinterval of  $[0, +\infty)$  where it is defined, and hence  $\varphi(r) \geq \varphi_0$ . We choose a positive constant  $k$  in such a way that  $s^2 - Cs \geq ks^2$  whenever  $s \geq \varphi_0$ . We finally obtain  $(r^{N-1}\varphi')' \geq k\varphi^2$ . We claim that

$$\begin{cases} (r^{N-1}\varphi')' \geq k\varphi^2 \\ (r^{N-1}\psi')' = k\psi^2 \\ \varphi(0) > \psi(0) \\ \varphi'(0) = \psi'(0) = 0 \end{cases}$$

implies  $\varphi(r) > \psi(r)$  where they are defined. From this the lemma will easily follow, since, as it is well known, all the regular positive radially symmetric solutions of  $\Delta u = ku^2$  blow up on bounded domains. To prove the claim, let  $w(t) = \varphi(t) - \psi(t)$ . We have

$$\begin{cases} (r^{N-1}w')' \geq k(\varphi + \psi)w \\ w(0) > 0 \\ w'(0) = 0. \end{cases}$$

Let by contradiction  $R$  be such that  $w(t) > 0$  on  $[0, R)$ ,  $w(R) = 0$ . As a consequence,  $r^{N-1}w'$  is strictly increasing on  $[0, R]$ , and thus  $w'(R) > 0$ . But this is incompatible with the definition of  $R$ .  $\blacksquare$

Let us now restrict our attention to the case  $N = 1$ . Since  $\Phi$  is harmonic and non constant, it is affine, and hence, after a suitable change of variables, we can take  $\Phi(x) = x$ .

We have

**Lemma 4.2.** *The equation*

$$(12) \quad -u'' = -u(u - x), \quad u > x^+ \text{ on } \mathbb{R}$$

has an unique solution  $\eta$ . Moreover  $\eta(x) = O(e^x)$  as  $x \rightarrow -\infty$ ,  $\eta(x) - x = O(e^{-x})$  as  $x \rightarrow +\infty$ ,  $\eta(x) = x + \eta(-x)$  and  $\eta$  is a strictly increasing convex function.

**Proof:** as a first remark, we notice that if  $u(x)$  is a solution (respectively a subsolution or a supersolution) of (12), then also  $x + u(-x)$  is a solution (respectively a subsolution or a supersolution) of (12); therefore, if we prove the uniqueness of the solution, then the symmetry property will follow. To prove the existence, we exhibit a global subsolution and a global supersolution for (12), that constitute an ordered pair. We have that  $\underline{u}(x) := x^+$  is a subsolution of (12) in the sense of distributions; indeed, 0 and  $x$  are solutions, and  $\underline{u}(x) = \max\{0, x\}$ . On the other hand, simple calculations show that  $e^x$  is a supersolution. As a consequence, we have that also  $x + e^{-x}$  is a supersolution, and thus  $\bar{u}(x) := \min\{e^x, x + e^{-x}\}$  is a supersolution of (12). Since obviously  $\bar{u}(x) > \underline{u}(x)$ , we obtain the existence of a solution  $\eta$  such that  $\underline{u}(x) < \eta(x) < \bar{u}(x)$ , and thus  $\eta$  has the required decay properties at  $\pm\infty$ . It remains to prove that  $\eta$  is unique and that  $\eta'$  is positive on  $\mathbb{R}$ .

To start with, we will prove that any  $\xi$  solution of (12) is strictly increasing and shares with  $\eta$  the same limits at  $\pm\infty$ . The uniqueness of  $\eta$  will then follow from the maximum principle.

Since  $\xi(x) > x^+$ , clearly  $\xi''(x) > 0$  on  $\mathbb{R}$ . We want to prove that also  $\xi'(x) > 0$  on  $\mathbb{R}$ . Let by contradiction  $x_0 < 0$ ,  $\xi(x_0) =: \xi_0 > 0$ ,  $\xi'(x_0) =: \xi'_0 < 0$ . We recall that, if  $v$  is such that  $v'' \leq v(v - x)$  for  $x \leq x_0$ ,  $v(x_0) \leq \xi_0$ ,  $v'(x_0) \geq \xi'_0$ , then  $\xi(x) \geq v(x)$  on the subinterval of  $(-\infty, x_0]$  where they are defined (this can be shown using elementary comparison argument; the proof, as a matter of fact, is very similar to the one of the (more complicated) analogous result in the proof of Lemma 4.1). But, if we set  $v(x) := 6/(x - A)^2$ , then  $v'' = v^2 \leq v(v - x)$  for  $x \leq x_0$  (recall that  $x_0 < 0$ ), and  $v(x_0) \rightarrow 0^+$ ,  $v'(x_0) \rightarrow 0^-$  as  $A \rightarrow -\infty$ . As a consequence, we can choose  $A$  such that  $\xi(x) \geq v(x)$  on the left of  $x_0$ , and thus  $\xi(x)$  blows up on the right of  $A$ , a contradiction. Hence we know that  $\xi(x)$ ,  $\xi'(x)$  and  $\xi''(x)$  are all strictly positive on  $\mathbb{R}$  (in particular, this is true also for  $\eta$ ). We deduce that  $\xi'(x) \rightarrow 0^+$  as  $x \rightarrow -\infty$ , and, recalling the equation, also  $\xi(-\infty) = 0^+$ . Hence we have shown that *every* solution of (12) annihilates at  $-\infty$ ; in particular, this is true also for  $\xi(-x) + x$ , and we deduce that  $\xi(x) - x$  annihilates at  $+\infty$ . Finally let  $w(t) := \xi(t) - \eta(t)$  and assume by contradiction that  $w(t) \not\equiv 0$ . By the previous considerations,  $w(-\infty) = w(+\infty) = 0$ , and thus  $w$  achieves either a positive maximum or a negative minimum in some point  $\bar{x}$ . But subtracting the equations we obtain  $w''(\bar{x}) = (\xi(\bar{x}) + \eta(\bar{x}) - \bar{x})w(\bar{x})$ , and this is clearly a contradiction to the fact that  $\xi + \eta - x$  is strictly positive on  $\mathbb{R}$ .  $\blacksquare$

Now we take into account the problem on  $\mathbb{R}^N$  in any dimension, that is, equation (11). In this situation we can not expect to extend straightforward the results we obtained in the case  $N = 1$ ; the reason, in fact, is that when  $N \geq 2$  there is a great variety of harmonic functions on  $\mathbb{R}^N$ . In the following Lemma, we will prove an existence result for a class of harmonic functions (containing, for example, the harmonic functions  $\Phi$  growing at infinity in a polynomial way). Next, we will prove an uniqueness result in the class of globally Lipschitz solutions.

**Lemma 4.3.** *Let  $\Phi \not\equiv C$  an harmonic function on  $\mathbb{R}^N$  such that*

$$\sup_{x \in \mathbb{R}^N} \frac{|\nabla\Phi(x)|^2 + \Phi(x)}{\cosh(\Phi(x))} \leq M < +\infty.$$

*Then problem (11) has at least a solution.*

**Proof:** as before, we look for ordered global subsolutions and supersolutions. First we observe that  $\underline{u} := \Phi^+$  is a subsolution, indeed 0 and  $\Phi$  are solutions and  $\underline{u}(x) = \max\{0, \Phi(x)\}$ . Moreover, since  $\Phi \not\equiv C$ ,  $\Phi$  must change sign on  $\mathbb{R}^N$  and thus  $\underline{u}$  is a proper subsolution and the possible solution  $u$  verify  $u(x) > \Phi^+(x)$  on  $\mathbb{R}^N$ . On the other hand, we assume without loss of generality that  $M \geq 1$  and we define  $\bar{u} := M \cosh(\Phi(x))$ . Through standard calculations one can easily verify that  $\bar{u}(x) > \underline{u}(x)$  for every  $x$ . If we show that  $\bar{u}$  is a supersolution, the lemma will follow. Since  $\bar{u}$  is smooth, we can verify the inequality pointwise using directly the equation. Recalling that, by assumption,  $|\nabla\Phi(x)|^2 \leq M \cosh(\Phi(x)) - \Phi(x)$ , we obtain

$$\begin{aligned} -\Delta\bar{u}(x) &= -M|\nabla\Phi(x)|^2 \cosh(\Phi(x)) \geq \\ &\geq -M \cosh(\Phi(x))(M \cosh(\Phi(x)) - \Phi(x)) = -\bar{u}(x)(\bar{u}(x) - \Phi(x)), \end{aligned}$$

and this concludes the proof of the lemma. ■

As we mentioned before, in general we are not able to prove uniqueness for the solutions of (11), and, by the previous lemma, we know that (10) has many families of solutions. Nevertheless, we can prove uniqueness of globally Lipschitz solutions. Let  $(u_1, u_2)$  a globally Lipschitz solution of (10); then  $\Phi = u_1 - u_2$  is globally Lipschitz. We recall that a globally Lipschitz harmonic function must be affine (indeed any of its directional derivatives is harmonic and bounded, and hence constant). Hence, after a suitable change of variables, we can assume  $\Phi(x) = x_1$  (where  $x = (x_1, \dots, x_N) = (x_1, x')$ ). Obviously in this setting the function  $u(x) = \eta(x_1)$ , where  $\eta$  is defined as in Lemma 4.2, is a globally Lipschitz solution of our problem. According to the following result, this will be the only Lipschitz solution of (11).

**Theorem 4.1.** *Let  $\Phi$  be harmonic and globally Lipschitz on  $\mathbb{R}^N$  (that is,  $\Phi$  is affine). Then there exists exactly one solution  $u$  of (11) such that  $u$  is globally Lipschitz on  $\mathbb{R}^N$ . In particular, the only globally Lipschitz solution when  $\Phi(x) = x_1$  is  $u(x) = \eta(x_1)$ .*

To prove this result we need to apply a technical lemma that will be useful also in the following sections.

**Lemma 4.4.** *Let  $B_{2R} \subset \mathbb{R}^N$  be any ball of radius  $2R$ , and  $B_R$  the ball of radius  $R$  sharing the center with  $B_{2R}$ . Let  $A, M$  be two positive constants and  $u \in H^1(B_{2R})$  be such that*

$$\begin{cases} -\Delta u \leq -Mu & \text{on } B_{2R} \\ u \geq 0 & \text{on } B_{2R} \\ u \leq A & \text{on } \partial B_{2R}. \end{cases}$$

*Then for every  $\alpha \in (0, 1)$  there exists  $C_\alpha > 0$  (not depending on  $M, A$  and  $R$ ) such that*

$$\|u\|_\infty \leq C_\alpha A e^{-\alpha R \sqrt{M}} \quad \text{on } B_R.$$

**Proof:** to start with, we claim that

$$(13) \quad \begin{cases} \varphi''(r) + \frac{N-1}{r}\varphi'(r) = M\varphi \\ \varphi(0) = \lambda > 0 \\ \varphi'(0) = 0 \end{cases} \implies \begin{cases} \varphi'(r) > 0 & r \in [0, +\infty) \\ \varphi(r) \leq \lambda e^{r\sqrt{M}} & r \in [0, +\infty) \\ \varphi(r) \geq \frac{\lambda r_0^{N-1}}{2e^{r_0\sqrt{M}}} \frac{e^{r\sqrt{M}}}{r^{N-1}} & r \in [r_0, +\infty), \end{cases}$$

where  $r_0 > 0$ .

To prove the claim, we observe that  $\varphi$  is defined on  $[0, +\infty)$  (the equation is linear) and that  $\varphi > 0$ ,  $\varphi' > 0$  on  $(0, +\infty)$  (indeed, as before, if by contradiction  $\varphi$  is positive on  $[0, R)$  and  $\varphi(R) = 0$  then  $\varphi'(R) \leq 0$ ; on the other hand, since  $(r^{N-1}\varphi)' = Mr^{N-1}\varphi$ , then  $r^{N-1}\varphi'$  is strictly increasing on  $[0, R]$  and hence  $\varphi'(R) > 0$ , a contradiction). Now, since  $\varphi'$  is positive, we have  $\varphi'' \leq M\varphi$  and hence, using the initial conditions and comparison arguments,  $\varphi(r) \leq \lambda e^{r\sqrt{M}}$ . Finally, we define  $\bar{\varphi}(t) := r^{N-1}\varphi(r)$ ; computing, we have  $\bar{\varphi}'' \geq M\bar{\varphi}$ ,  $\bar{\varphi}(r_0) \geq r_0^{N-1}\lambda$ ,  $\bar{\varphi}'(r_0) \geq 0$ . Using again comparison arguments, we obtain  $\bar{\varphi}(r) \geq r_0^{N-1}\lambda \cosh((r - r_0)\sqrt{M})$ , and the claim follows.

Now let  $\psi$  be the solution of

$$\begin{cases} \psi''(r) + \frac{N-1}{r}\psi'(r) = M\psi \\ \psi(2R) = A \\ \psi'(0) = 0. \end{cases}$$

Clearly  $\psi$  satisfies the assumptions in (13) for a suitable  $\lambda$ . If we let  $v(x) = \psi(|x - x_0|)$ , where  $x_0$  is the center of  $B_{2R}$ , then by construction we have that  $v$  is a radially symmetric function with  $-\Delta v = -Mv$  on  $B_{2R}$ ,  $v = A$  on  $\partial B_{2R}$ , and hence, by the maximum principle,  $0 \leq u(x) \leq v(x)$  on  $B_{2R}$ . Moreover, since  $\psi$  is an increasing function, if we prove that  $\psi(R) \leq C_{\alpha, R} A e^{-\alpha\sqrt{M}}$ , then we will obtain the required bound for  $\|u\|_\infty$  and the proof of the lemma will be concluded. Using the third inequality in (13) and choosing  $r_0 = \beta R$ ,  $\beta \in (0, 1)$ , we obtain

$$A = \psi(2R) \geq \frac{\lambda \beta^{N-1} R^{N-1}}{2e^{\beta R\sqrt{M}}} \frac{e^{2R\sqrt{M}}}{2^{N-1} R^{N-1}},$$

that gives

$$\lambda \leq A \frac{2^N}{\beta^{N-1}} e^{(-2+\beta)R\sqrt{M}}.$$

Substituting in the second inequality in (13), we finally have

$$\psi(R) \leq A \frac{2^N}{\beta^{N-1}} e^{(-1+\beta)R\sqrt{M}},$$

that, setting  $\alpha = 1 - \beta$ , provides the desired inequality.  $\blacksquare$

**Proof of Theorem 4.1:** let  $u$  a globally Lipschitz solution of (11) with  $\Phi(x) = x_1$ . We will prove that

$$\frac{\partial u}{\partial x_i} \equiv 0 \quad \forall i \neq 1;$$

from the uniqueness of  $\eta$  as solution on  $\mathbb{R}$  (see Lemma 4.2) the lemma will follow. Fix  $M \geq 1$ . We start proving that there exist positive constants  $A$ ,  $\alpha < 1$ , not depending on  $M$ , such that

$$(14) \quad x_1 \leq -M \implies 0 < u(x) \leq A e^{-\alpha\sqrt{M}}.$$

To this aim, let  $B_3 \subset \mathbb{R}^N$  be a ball of radius 3 such that for every  $x \in B_3$  we have  $x_1 \leq -M$ , and  $B_2$  and  $B_1$  the balls with the same center and radius respectively 2 and 1. We will prove the required inequality for every point such that  $x_1 \leq -M - 2$ ; since  $u$  is Lipschitz, the reader will easily see that, possibly changing the constant  $A$ , one can obtain the inequality up to  $x_1 \leq -M$ .

Let  $\gamma(x)$  a smooth cut-off function such that  $\gamma(x) \equiv 1$  on  $B_2$  and  $\gamma(x) \equiv 0$  outside  $B_3$ . Multiplying the equation with  $\gamma^2 u$  and integrating we obtain

$$0 = \int_{B_3} (\nabla u \cdot \nabla(\gamma^2 u) + \gamma^2 u^2(u - x_1)) = \int_{B_3} (\gamma^2 |\nabla u|^2 + 2\gamma u \nabla u \cdot \nabla \gamma + \gamma^2 u^2(u - x_1)).$$

Since  $|2\gamma u \nabla u \cdot \nabla \gamma| \leq (M/2)(\gamma u)^2 + (2/M)(\nabla u \cdot \nabla \gamma)^2$ , we obtain

$$\int_{B_3} \gamma^2 |\nabla u|^2 + \int_{B_3} \gamma^2 u^2(u - x_1 - \frac{M}{2}) \leq \frac{M}{2} \int_{B_3} (\nabla u \cdot \nabla \gamma)^2.$$

Using the fact that  $u$  is globally Lipschitz, we deduce that the right-hand side is bounded. Recalling that  $-x_1 \geq M$ , we can then write

$$\int_{B_3} \frac{M}{2} \gamma^2 u^2 \leq C,$$

that implies

$$\int_{B_2} u^2 \leq \frac{C}{M}.$$

Since  $M$  has been fixed larger than 1, we obtain the existence of a point  $x_0 \in B_2$  such that  $u(x_0) \leq C$  not depending on  $M$ . Using the fact that  $u$  is Lipschitz, we have that  $u$  is bounded on  $\partial B_2$ , not depending on  $M$  and on the position of  $B_2$ . But now, since  $u - x_1 \geq M$  on  $B_2$ , we are in a condition to apply Lemma 4.4 obtaining  $|u(x)| \leq Ae^{-\alpha\sqrt{M}}$ , and (14) is proved.

Arguing in the same way, an analogous inequality can be written for  $u - x_1$  when  $x_1 \geq M$ . Recalling that  $u$  is Lipschitz, we finally obtain

$$0 < u(x) - x_1^+ \leq C \quad \text{on } \mathbb{R}^N.$$

Now let  $j \neq 1$ . We want to prove that

$$(15) \quad |x_1| \geq M \quad \Rightarrow \quad \left| \frac{\partial u(x)}{\partial x_j} \right| \leq Ae^{-\alpha\sqrt{M}}.$$

Let  $w := \frac{\partial u(x)}{\partial x_j}$ ; we have that  $-\Delta w = (-2u + x_1)w$ . Since  $|x_1| \geq M$  and  $u > x_1^+$ , we obtain  $-2u + x_1 \leq -M$ . Taking the absolute values, we have

$$\begin{cases} -\Delta|w| \leq -M|w| & \text{on } B_2 \\ |w| \geq 0 & \text{on } B_2 \\ |w| \leq L & \text{on } \partial B_2, \end{cases}$$

where  $L$  is the Lipschitz constant of  $u$  and  $B_2$  any ball of radius 2 contained in  $\{|x_1| \geq M\}$ . Lemma 4.4 applies again and (15) follows.

Now we are ready to prove the lemma. We want to show that  $w(x) \equiv 0$ . Since  $\Delta w(x)$  and  $w(x)$  have always the same sign,  $w(x)$  does not admit positive maxima and negative minima. Assume by contradiction that  $w$ , for instance, is positive somewhere, and let  $\sup_{\mathbb{R}^N} w(x) =: \xi > 0$ .  $\xi$  can not be achieved. Let  $(x^{(n)})$  be a sequence such that  $w(x^{(n)}) \rightarrow \xi$ . By (15)  $x_1^{(n)} \not\rightarrow \infty$ , hence, up to a subsequence, we can assume  $x_1^{(n)} \rightarrow \bar{x}_1$ . Let  $y^{(n)}$  the projection of  $x^{(n)}$  on the hyperplane  $\{x_1 = 0\}$ , and define  $u_n(x) = u(x - y^{(n)})$ . Since  $y_1^{(n)} = 0$ , we have

$$-\Delta u_n(x) = -u_n(x)(u_n(x) - x_1).$$

Moreover we know that  $u - x_1^+$ ,  $\frac{\partial u(x)}{\partial x_i}$  are bounded on  $\mathbb{R}^N$ ; by Ascoli's Theorem we have that  $u_n$  converges uniformly on compact sets to a limit  $\bar{u}$  that solves the equation on  $\mathbb{R}^N$ , and its  $j$ -th partial derivative achieves its maximum in  $(\bar{x}_1, 0, \dots, 0)$ , a contradiction.  $\blacksquare$

We conclude the section with some asymptotic estimates. To this aim, we scale back in problem (10) and we study the behavior of the solutions in dependence of  $\kappa$ . We detail the procedure for  $N = 1$ , which is the case we need in the following section.

Let  $\eta$  be fixed as in Lemma 4.2 and define, for every  $\delta > 0$ ,

$$(16) \quad \begin{aligned} \eta_\delta(x) = \delta\eta\left(\frac{x}{\delta}\right) \quad \text{then} \quad \eta'_\delta(x) &= \eta'\left(\frac{x}{\delta}\right) \\ \eta''_\delta(x) &= \frac{1}{\delta}\eta''\left(\frac{x}{\delta}\right). \end{aligned}$$

It is clear that  $\eta_\delta$  solves

$$-\eta''_\delta = -\frac{1}{\delta^3}\eta_\delta(\eta_\delta - x).$$

Actually, when  $\delta = \kappa^{-1/3}$ , the evolution of  $\eta_\delta$  as  $\delta \rightarrow 0$  describes the behavior of problem (1), when  $\Omega = \mathbb{R}$  and  $k = 2$ .

We introduce the Dirac distribution  $D$ , and the Heavyside function  $H$ , and recall that, in distributional sense, we have  $H' = D$ . We want to prove the following lemma:

**Lemma 4.5.** *As  $\delta \rightarrow 0$  we have*

$$\begin{aligned} \eta''_\delta - D &= O(\delta^{1/2}) = O(\kappa^{-1/6}) \text{ in } \mathcal{D}' \\ \eta'_\delta - H &= O(\delta^{1/2}) = O(\kappa^{-1/6}) \text{ in } L^2(\mathbb{R}) \\ \eta_\delta - x^+ &= O(\delta^{3/2}) = O(\kappa^{-1/2}) \text{ in } L^2(\mathbb{R}). \end{aligned}$$

**Proof:** if we denote by  $\langle \cdot, \cdot \rangle_{\mathcal{D}'\mathcal{D}}$  the duality pairing between the distributions and the compact supported  $C^\infty$  functions on  $\mathbb{R}$ , for every  $\phi \in \mathcal{D}$  we have

$$\begin{aligned} \langle \eta''_\delta - D, \phi \rangle_{\mathcal{D}'\mathcal{D}} &= \int_{\mathbb{R}} \left( H\left(\frac{x}{\delta}\right) - \eta'\left(\frac{x}{\delta}\right) \right) \phi'(x) dx \\ &\leq \delta^{1/2} \|H - \eta'\|_{L^2} \|\phi'\|_{L^2} \\ \|\eta'_\delta - H\|_{L^2} &= \left( \int_{\mathbb{R}} |\eta'(y) - H(y)|^2 \delta dy \right)^{1/2} \\ \|\eta_\delta - x^+\|_{L^2} &= \delta \left( \int_{\mathbb{R}} |\eta(y) - y^+|^2 \delta dy \right)^{1/2}. \end{aligned}$$

■

## 5. TWO DENSITIES ON A BOUNDED DOMAIN

In this section we carry on the asymptotic analysis of the solutions of (1) in the case of two densities. As a result we shall give a complete proof of Theorem 2 stated in the Introduction.

As in the previous section, after a change of variables, we can assume  $a_{12} = a_{21} = 1$  (this is equivalent to choose as new unknowns the functions  $a_{21}u_1$ ,  $a_{12}u_2$  and as new data  $\varphi_1/a_{21}$ ,  $\varphi_2/a_{12}$ ).

In this setting, the problem writes

$$(17) \quad \begin{cases} -\Delta u_i(x) = -\kappa u_1(x)u_2(x) & x \in \Omega \\ u_i(x) = \varphi_i(x) & x \in \partial\Omega \\ u_i(x) > 0 & x \in \Omega \\ i = 1, 2. \end{cases}$$

We can write problem (17) in the equivalent form

$$(18) \quad \begin{cases} -\Delta u(x) = -\kappa u(x)(u(x) - \Phi(x)) & x \in \Omega \\ u(x) = \Phi(x) & x \in \partial\Omega \end{cases}$$

where  $u = u_1$  and  $\Phi$  is the function defined on the whole  $\Omega$  as

$$\begin{cases} \Delta\Phi(x) &= 0 & x \in \Omega \\ \Phi(x) &= \varphi_1(x) - \varphi_2(x) & x \in \partial\Omega. \end{cases}$$

### 5.1. Existence and asymptotic behavior.

**Theorem 5.1.** *For every positive  $\kappa$  Problem (18) has a unique solution  $u_\kappa > \Phi^+$  with*

$$\|u_\kappa - \Phi^+\|_{H^1(\Omega)} = O(\kappa^{-1/6}) \quad \text{as } \kappa \rightarrow \infty.$$

**Proof:** we first prove the existence of  $u_\kappa$ , and to do that we apply again the sub-supersolutions method. Clearly  $\underline{u}(x) := \Phi^+(x) = \sup\{0, \Phi(x)\}$  is a subsolution. Let  $\eta_\delta$  be defined as in (16). We define

$$\bar{u}_\delta(x) := \eta_\delta(\Phi(x)).$$

Since  $\eta_\delta(t) \geq t^+$  we have  $\bar{u}_\delta(x) \geq \underline{u}(x)$  for every positive  $\delta$ . Hence, to obtain the solution  $u_\kappa$ , we need to prove that through a suitable choice of  $\delta$   $\bar{u}_\delta$  is a supersolution of (18). Recalling the properties of  $\eta_\delta$  and the fact that  $\Delta\Phi = 0$  on  $\Omega$ , we obtain

$$-\Delta\bar{u}_\delta(x) = -\eta_\delta''(\Phi(x)) \cdot |\nabla\Phi(x)|^2 - \frac{|\nabla\Phi(x)|^2}{\delta^3} \bar{u}_\delta(x)(\bar{u}_\delta(x) - \Phi(x)).$$

Therefore, if we define  $L := \|\nabla\Phi\|_\infty$ , we obtain that, choosing  $\delta = (L^2/\kappa)^{1/3}$ ,  $\bar{u}_\delta$  is a supersolution.

Let us come to the uniqueness. Assume by contradiction the existence of two solutions,  $u$  and  $v$ , and consider the differential problem for  $w = u - v$ :  $-\Delta w = a(x)w$  on  $\Omega$  and  $w = 0$  on  $\partial\Omega$ , where  $a(x) = u(x) + v(x) - \Phi(x) > 0$ . Then  $w \equiv 0$  by the maximum principle.

Finally, the convergence properties: recalling Lemma 4.5 we have that  $\bar{u}_\delta(x) \rightarrow \Phi^+(x)$  almost everywhere on  $\Omega$ , and hence also  $u_\delta(x) \rightarrow \Phi^+(x)$  pointwise. Moreover  $u_\kappa$  and  $\Phi^+$  agree on the boundary of  $\Omega$ , therefore we can use the gradient norm for  $H^1(\Omega)$ . We have to estimate

$$\begin{aligned} \int_\Omega |\nabla(u_\kappa - \Phi^+)|^2 &= \int_\Omega -\Delta(u_\kappa - \Phi^+) \cdot (u_\kappa - \Phi^+) = \\ &= \int_\Omega -\Delta(\bar{u}_\delta - \Phi^+) \cdot (u_\kappa - \Phi^+) + \int_\Omega \Delta(\bar{u}_\delta - u_\kappa) \cdot (u_\kappa - \Phi^+). \end{aligned}$$

Using the equations verified by  $\bar{u}_\delta$  and  $u_\kappa$ , and recalling that  $\Phi(x) \leq \Phi^+(x) \leq u_\kappa(x) \leq \bar{u}_\delta(x)$  and that we chase  $\delta^3 \geq |\nabla\Phi|^2 \kappa$ , we find, after easy calculations,

$$\Delta(\bar{u}_\delta - u_\kappa) \leq \frac{|\nabla\Phi|^2}{\delta^3} (\bar{u}_\delta - \Phi)(2\bar{u}_\delta - \Phi).$$

Combining the last two equations, we obtain then

$$\int_\Omega |\nabla(u_\kappa - \Phi^+)|^2 \leq \int_\Omega -\Delta(\bar{u}_\delta - \Phi^+) \cdot (u_\kappa - \Phi^+) + \frac{1}{\delta^3} \int_\Omega |\nabla\Phi|^2 (\bar{u}_\delta - \Phi)(2\bar{u}_\delta - \Phi)(\bar{u}_\delta - \Phi^+).$$

To estimate the right hand side we use the well known coarea formula. The last term becomes

$$\begin{aligned} &\frac{1}{\delta^3} \int_\Omega |\nabla\Phi|^2 (\bar{u}_\delta - \Phi)(2\bar{u}_\delta - \Phi)(\bar{u}_\delta - \Phi^+) = \\ &= \frac{1}{\delta^3} \int_{-\infty}^{+\infty} dt \int_{\{\Phi(x)=t\}} |\nabla\Phi(x)|^2 (\eta_\delta(t) - t)(2\eta_\delta(t) - t)(\eta_\delta(t) - t^+) \cdot |\nabla\Phi(x)|^{-1} d\mathcal{H}^{N-1} = \\ &= \frac{1}{\delta^3} \int_{-\infty}^{+\infty} (\eta_\delta(t) - t)(2\eta_\delta(t) - t)(\eta_\delta(t) - t^+) \cdot \left( \int_{\{\Phi(x)=t\}} |\nabla\Phi(x)| d\mathcal{H}^{N-1} \right) dt \leq C\delta, \end{aligned}$$

where  $C$  only depends on the fixed functions  $\Phi$  and  $\eta$  (we used the fact that  $\eta_\delta(t) = \delta\eta(t/\delta)$ ).

$$\int_\Omega -\Delta(\bar{u}_\delta - \Phi^+) \cdot (u_\kappa - \Phi^+) = \int_\Omega \nabla(\bar{u}_\delta - \Phi^+) \cdot \nabla(u_\kappa - \Phi^+) =$$

$$\begin{aligned}
&= \int_{\Omega} [\eta'_{\delta}(\Phi) - H(\Phi)] \nabla \Phi \cdot \nabla (u_{\kappa} - \Phi^+) = \\
&\int_{-\infty}^{+\infty} (\eta'_{\delta}(t) - H(t)) \left( \int_{\{\Phi(x)=t\}} \frac{\nabla \Phi \cdot \nabla (u_{\kappa} - \Phi^+)}{|\nabla \Phi|} d\mathcal{H}^{N-1} \right) dt \leq \\
&\leq \|\eta'_{\delta} - H\|_2 \left\| \int_{\{\Phi(x)=t\}} \frac{\nabla \Phi \cdot \nabla (u_{\kappa} - \Phi^+)}{|\nabla \Phi|} d\mathcal{H}^{N-1} \right\|_2 \\
&\leq C\delta^{1/2} \|\nabla (u_{\kappa} - \Phi^+)\|_2
\end{aligned}$$

by Lemma 3.6 and again the coarea formula. This finally gives

$$\|\nabla (u_{\kappa} - \Phi^+)\|_2 = O(\delta^{1/2}) = O(\kappa^{-1/6})$$

as required.  $\blacksquare$

**5.2. Uniform Lipschitz Continuity.** Clearly, being our equations autonomous, the solution  $u_{\kappa}$  is smooth for every fixed  $\kappa > 0$ . Nevertheless, the limit  $\Phi^+$  is only Lipschitz continuous. In this section we prove that the convergence of  $u_{\kappa}$  to  $\Phi^+$  is actually the most regular possible.

**Theorem 5.2.** *Let  $u_{\kappa}$  be the unique solution of (18) such that  $u_{\kappa} \geq \Phi^+$  as in Theorem 5.1. Then the Lipschitz constants of  $\{u_{\kappa}\}$  are independent of  $\kappa$ .*

**Proof:** the proof is based on a blow-up technique; so we assume by contradiction the existence of a sequence  $(x_{\kappa}) \subset \bar{\Omega}$  such that it holds, as  $\kappa \rightarrow \infty$ ,

$$\mu_{\kappa} = |\nabla u(x_{\kappa})| \rightarrow \infty \text{ with } \|\nabla u_{\kappa}\|_{\infty} - \mu_{\kappa} < \kappa.$$

We consider the rescaled functions

$$v_{\kappa}(x) = \frac{1}{r_{\kappa}\mu_{\kappa}} u(r_{\kappa}x + x_{\kappa})$$

where  $r_{\kappa} = u_{\kappa}(x_{\kappa})/\mu_{\kappa}$  so that  $v_{\kappa}(0) = 1$ . Note that by construction  $|\nabla v_{\kappa}(0)| = 1$ , and  $\|\nabla v_{\kappa}\|_{\infty} \rightarrow 1$  too. We have that  $v_{\kappa}$  is solution of

$$(19) \quad -\Delta v_{\kappa} = -\frac{r_{\kappa}^3 \mu_{\kappa}}{\kappa} v_{\kappa}(v_{\kappa} - \Phi_{\kappa}) \text{ on } \Omega_{\kappa}$$

where  $\Phi_{\kappa}(x) = \Phi(r_{\kappa}x + x_{\kappa})(r_{\kappa}\mu_{\kappa})^{-1}$  and  $\Omega_{\kappa} = \{x \in \mathbb{R}^N : r_{\kappa}x + x_{\kappa} \in \Omega\}$ . We obtain the following convergence properties.

1. Since  $r_{\kappa} \rightarrow 0$  and  $\partial\Omega$  is regular, then  $\Omega_{\kappa} \rightarrow \Omega_{\infty}$ , where  $\Omega_{\infty}$  is the whole  $\mathbb{R}^N$  or an half space, depending on the behaviour of the distance between  $x_{\kappa}$  and  $\partial\Omega$ ; in any case,  $0 \in \bar{\Omega}_{\infty}$ .
2. Since  $\|\nabla \Phi_{\kappa}\|_{L^{\infty}(\Omega_{\kappa})} \leq \mu_{\kappa}^{-1} \|\nabla \Phi\|_{L^{\infty}(\Omega_{\infty})} \rightarrow 0$  and  $\Phi_{\kappa}(0) \leq (r_{\kappa}\mu_{\kappa})^{-1} u(x_{\kappa}) = 1$ , then  $\Phi_{\kappa}$  converges uniformly to some constant  $\Psi$ , with  $\Psi \leq 1$ .
3. Since  $v_{\kappa}$  is Lipschitz continuous, uniformly with respect to  $\kappa$ , and  $v_{\kappa}(0) = 1$  by construction, then there exists a limit function, say  $v$ , defined on  $\Omega_{\infty}$ , such that  $v_{\kappa} \rightarrow v$  uniformly on compact sets of  $\bar{\Omega}_{\infty}$ .

Let us now consider the sequence  $M_{\kappa} := r_{\kappa}^3 \mu_{\kappa} \kappa^{-1}$ ; we shall discuss separately the case when  $M_{\kappa}$  is bounded or not since we shall reach a contradiction using different arguments.

*Case 1:  $M_{\kappa}$  is unbounded.*

Let us multiply (19) with  $v_{\kappa}$  and integrate over a ball of fixed radius  $R$  (possibly intersected with  $\Omega_{\kappa}$ ). We get

$$M_{\kappa} \int_{B_R \cap \Omega_{\kappa}} v_{\kappa}^2 (v_{\kappa} - \Phi_{\kappa}) = - \int_{B_R \cap \Omega_{\kappa}} |\nabla v_{\kappa}|^2 + \int_{\partial(B_R \cap \Omega_{\kappa})} v_{\kappa} \frac{\partial}{\partial n} v_{\kappa}.$$

Note that the right-hand side is bounded uniformly in  $\kappa$ , since  $v_{\kappa}(0) = 1$  and  $\|\nabla v_{\kappa}\|_{\infty} \rightarrow 1$ .

Thus for  $R$  fixed we have

$$M_\kappa \int_{B_R \cap \Omega_\kappa} v_\kappa^2 (v_\kappa - \Phi_\kappa) \leq C$$

( $C$  depending only on  $R$ ) and since  $v_\kappa \geq \Phi_\kappa$  we deduce that  $v_\kappa^2(x)(v_\kappa(x) - \Phi_\kappa(x))$  vanishes almost everywhere as  $\kappa \rightarrow \infty$ . Taking into account that  $v_\kappa(0) = 1$  for all  $\kappa$ , jointly with 2. and 3. above, we conclude that, at least for  $R$  small,  $v \equiv \Psi \equiv 1$  on  $B_R \cap \Omega_\infty$ . In particular, by 3.,  $v_\kappa \rightarrow 1$  uniformly on compact sets. In fact, we shall prove that the convergence of  $v_\kappa$  to 1 is also of class  $C^1$ ; this will provide  $|\nabla v(0)| = 1$ , in contradiction with  $v \equiv 1$ .

In order to prove the  $C^1$  convergence of  $v_\kappa$  to 1, we shall apply Lemma 4.4 to  $w := v_\kappa - \Phi_\kappa$ . As a first remark, we can extend such a function to the whole  $B_R$ , setting  $w = 0$  where it is not defined. We denote again with  $w$  such extension, observing that it belongs to  $H^1(B_R)$  (indeed,  $v_\kappa$  and  $\Phi_\kappa$  agree on  $B_R \cap \partial\Omega_\kappa$ ). We observe that, with this extension,  $-\Delta w \leq -(v_\kappa M_\kappa)w$ . By the uniform convergence of  $v_\kappa \rightarrow 1$  on  $B_R$ , we can assume that  $v_\kappa > 1/2$ , and this provides

$$-\Delta w \leq -(M_\kappa/2)w \quad \text{on } B_R.$$

By applying Lemma 4.4 we then obtain  $v_\kappa - \Phi_\kappa \leq C_1 e^{-C_2 \sqrt{M_\kappa}}$  on  $B_{R/2}$ . Going back to the equation for  $v_\kappa$  we realize that

$$-\Delta v_\kappa = a_\kappa(x)v_\kappa \quad \text{on } B_{R/2}$$

where  $0 < a_\kappa(x) \leq C_1 M_\kappa e^{-C_2 \sqrt{M_\kappa}} \rightarrow 0$ . From this the required regularity is ensured.

*Case 2:  $M_\kappa$  is bounded.*

Let  $M = \lim_{\kappa \rightarrow \infty} M_\kappa$  (up to subsequences) so that the limit function  $v$  is solution of  $-\Delta v = -Mv(v - \Psi)$  on  $\Omega_\infty$  and  $v \geq \Psi$ . By the strong maximum principle, this means that either  $v \equiv \Psi$  or  $v > \Psi$ . In the first case, again  $-\Delta v_\kappa \rightarrow 0$  uniformly, that is,  $v_\kappa$  converges at least in the  $C^1$  topology and thus  $\nabla v(0) = 1$ , a contradiction. In the second case, the contradiction is easily provided by the nonexistence result Lemma 4.1.  $\blacksquare$

## 6. UNIFORM HÖLDER CONTINUITY

Let us now go back to the case of an arbitrary number of species governed by (1). By Section 2, we know that, at fixed  $\kappa > 0$ , the system admits a solution  $U_\kappa = (u_{1,\kappa}, \dots, u_{k,\kappa})$  with smooth components. Aim of this section is to establish some regularity of the  $U_\kappa$ 's, uniformly in  $\kappa$ . After Theorem 5.2, we know that when  $k = 2$ , the solutions of our model are Lipschitz continuous, uniformly with respect to  $\kappa$ . In the general case, we cannot prove uniform Lipschitz estimates, nonetheless there holds

**Theorem 6.1.** *Let  $U_\kappa$  be any solution of (1) at fixed  $\kappa > 0$ . Let  $\alpha \in (0, 1)$ . Then there exists  $L_\alpha > 0$  such that*

$$(20) \quad \sup_{x, y \in \Omega} \frac{u_{i,\kappa}(x) - u_{i,\kappa}(y)}{|x - y|^\alpha} < L_\alpha$$

for all  $i = 1, \dots, k$  and for all  $\kappa > 0$ .

The proof of this result requires several steps. It relies upon a blow up procedure which leads to apply the Liouville type results established in Section 7.

To start with, let us assume by contradiction that the supremum in (20) is unbounded in  $\kappa$ ; namely let

$$(21) \quad L_\kappa := \max_i \max_{x, y \in \bar{\Omega}} \frac{|u_{i,\kappa}(x) - u_{i,\kappa}(y)|}{|x - y|^\alpha}$$

be such that  $L_\kappa \rightarrow \infty$ . Note that this is a maximum, since  $\Omega$  is bounded and each  $u_i$  is Lipschitz continuous up to the boundary. We can assume w.l.o.g. that  $L_\kappa$  is achieved by  $u_{1,\kappa}$  at the pair

$(x_\kappa, y_\kappa)$ ; let us now blow up each  $u_{i,\kappa}$  with center at  $x_\kappa$  and parameters  $L_\kappa$  and  $r_\kappa$  (which will be chosen below, according to different cases):

$$v_{i,\kappa}(x) = \frac{1}{L_\kappa r_\kappa^\alpha} u_{i,\kappa}(x_\kappa + r_\kappa x)$$

with  $x \in \Omega_\kappa := \{x \in \mathbb{R}^N : x_\kappa + r_\kappa x \in \Omega\}$ ; by the regularity of  $\partial\Omega$  we have that, when  $r_\kappa \rightarrow 0$ , up to a subsequence  $\Omega_\kappa \rightarrow \Omega_\infty$ , where  $\Omega_\infty$  is the whole  $\mathbb{R}^N$  or an half-space (depending on the behaviour of the distance of  $x_\kappa$  from  $\partial\Omega$ ). We easily obtain that  $v_{i,\kappa}$  is solution of

$$\begin{cases} -\Delta v_{i,\kappa} &= -M_\kappa v_{i,\kappa} \sum_{j \neq i} a_{ij} v_{j,\kappa} & \text{on } \Omega_\kappa \\ v_{i,\kappa} &= \varphi_{i,\kappa} & \text{on } \partial\Omega_\kappa \end{cases}$$

where  $M_\kappa := L_\kappa r_\kappa^{2+\alpha} / \kappa$  and  $\varphi_{i,\kappa}$  is obviously defined. Note that

$$(22) \quad \max_i \max_{x,y \in \overline{\Omega}_\kappa} \frac{|v_{i,\kappa}(x) - v_{i,\kappa}(y)|}{|x - y|^\alpha} = \frac{|v_{1,\kappa}(0) - v_{1,\kappa}((y_\kappa - x_\kappa)/r_\kappa)|}{|(y_\kappa - x_\kappa)/r_\kappa|^\alpha} = 1$$

for all  $\kappa$ . We remark that  $|x_\kappa - y_\kappa| \rightarrow 0$  as  $\kappa \rightarrow \infty$ : indeed, let by contradiction  $|x_\kappa - y_\kappa| \geq l > 0$  and choose  $r_\kappa = |x_\kappa - y_\kappa|$ ; by definition we have that  $\|v_{i,\kappa}\|_{L^\infty(\Omega_\kappa)} \rightarrow 0$ , and hence  $|v_{1,\kappa}(0) - v_{1,\kappa}(1)| \rightarrow 0$ , in contradiction with (22). To proceed with the proof, we need a technical lemma.

**Lemma 6.1.** *Under the previous notations, let  $r_\kappa \rightarrow 0$  as  $\kappa \rightarrow \infty$ , such that:*

- (i) *there exists  $R' > 0$  such that  $|y_\kappa - x_\kappa| \leq R' r_\kappa$ ,*
- (ii)  *$M_\kappa \not\rightarrow 0$ .*

*Then  $v_{i,\kappa}(0)$  is uniformly bounded with respect to  $\kappa$ , for every  $i$ .*

**Proof:** assume to the contrary that  $v_{h,\kappa}(0)$  is unbounded for some  $h$ , and let  $R \geq R'$ , with  $R'$  as in (i). Since  $v_{h,\kappa}$  is Hölder continuous (uniformly in  $\kappa$ , see (22)) and  $M_\kappa \not\rightarrow 0$  we have that

$$I_\kappa := \inf_{B(0,2R) \cap \Omega_\kappa} M_\kappa v_{h,\kappa} \rightarrow \infty.$$

We claim that:

$$\|M_\kappa v_{h,\kappa} v_{i,\kappa}\|_{L^\infty(B(0,R) \cap \Omega_\kappa)} \rightarrow 0 \text{ for all } i \neq h.$$

To prove the claim, we will apply Lemma 4.4 to each  $v_{i,\kappa}$  with  $i \neq h$ . To this aim, we observe that, if  $\partial\Omega_\kappa$  intersect  $B(0,2R)$ , then, for  $\kappa$  sufficiently small,  $v_{h,\kappa}$  is strictly positive on  $\partial\Omega_\kappa \cap B(0,2R)$ . This means that  $v_{i,\kappa}$ ,  $i \neq h$ , is identically 0 on  $\partial\Omega_\kappa \cap B(0,2R)$ . Therefore we can extend  $v_{i,\kappa}$  on  $B(0,2R) \setminus \Omega_\kappa$  as the null function (we will write again  $v_{i,\kappa}$  for this extension). It is easy to prove that this extension belongs to  $H^1(B(0,2R))$ , inherits the same Hölder constant, and satisfies (in distributional sense)

$$-\Delta v_{i,\kappa} \leq -a_{ih} I_\kappa v_{i,\kappa} \text{ on } B(0,2R).$$

Thus, in order to apply the above mentioned lemma, it suffices to prove that  $v_{i,\kappa}$  is bounded on  $\partial B(0,2R)$ . To this aim we choose a smooth positive cut-off function  $\eta$ , vanishing out of  $B(0,2R)$ , we multiply the  $i$ -th differential equation by  $\eta^2 v_{i,\kappa}$  and then integrate:

$$\int_{B(0,2R)} \eta^2 |\nabla v_{i,\kappa}|^2 + \int_{B(0,2R)} 2\eta v_{i,\kappa} \nabla \eta \nabla v_{i,\kappa} + a_{ih} I_\kappa \int_{B(0,2R)} \eta^2 v_{i,\kappa}^2 \leq 0$$

that implies

$$\int_{B(0,2R)} |\nabla(\eta v_{i,\kappa})|^2 - \int_{B(0,2R)} |\nabla \eta|^2 v_{i,\kappa}^2 + a_{ih} I_\kappa \int_{B(0,2R)} \eta^2 v_{i,\kappa}^2 \leq 0$$

and finally

$$(23) \quad a_{ih} I_\kappa \int_{B(0,2R)} \eta^2 v_{i,\kappa}^2 \leq \int_{B(0,2R)} |\nabla \eta|^2 v_{i,\kappa}^2.$$

Clearly,

$$\int_{B(0,2R)} |\nabla \eta|^2 v_{i,\kappa}^2 \leq \left( \sup_{B(0,2R)} v_{i,\kappa} \right)^2 \int_{B(0,2R)} |\nabla \eta|^2.$$

On the other hand, recalling that  $v_{i,\kappa}$  is Hölder continuous with constant smaller or equal to 1, we have

$$\inf_{B(0,2R)} v_{i,\kappa} \geq \sup_{B(0,2R)} v_{i,\kappa} - (4R)^\alpha$$

and thus

$$\int_{B(0,2R)} \eta^2 v_{i,\kappa}^2 \geq \frac{1}{2} \left( \sup_{B(0,2R)} v_{i,\kappa} \right)^2 \int_{B(0,2R)} \eta^2 - (4R)^\alpha \int_{B(0,2R)} \eta^2;$$

Using these inequalities in (23) we immediately infer the boundedness of  $v_{i,\kappa}$  in  $B(0,2R)$  ( $\kappa \rightarrow \infty$ ), and hence on  $\partial B(0,2R)$ , as required. Now Lemma 4.4 applies providing

$$\sup_{B(0,R)} v_{i,\kappa} \leq C_1 e^{-C_2 \sqrt{I_\kappa}},$$

with  $C_1$  and  $C_2$  not depending on  $\kappa$ . Finally, using the Hölder continuity of  $v_{h,\kappa}$ , we have

$$\sup_{B(0,2R) \cap \Omega_\kappa} M_\kappa v_{h,\kappa} \leq \inf_{B(0,2R) \cap \Omega_\kappa} M_\kappa v_{h,\kappa} + (4R)^\alpha = I_\kappa + (4R)^\alpha,$$

and the claim easily follows. We observe that, as a consequence,

$$(24) \quad \| -\Delta v_{i,\kappa} \|_{L^\infty(B(0,R) \cap \Omega_\kappa)} \rightarrow 0 \quad \text{for every } i \text{ and every } R \geq R'.$$

Now consider  $\tilde{v}_{i,\kappa}(x) := v_{i,\kappa}(x) - v_{i,\kappa}(0)$ , for every  $i$ : by the uniform Hölder continuity,  $\tilde{v}_{i,\kappa} \rightarrow 0$  uniformly on compact sets. Moreover, by (24), the convergence is  $C^1$ , so that (22) holds for  $\tilde{v}_1$  on  $\overline{\Omega}_\infty$  (recall that, by assumption (i),  $(y_\kappa - x_\kappa)/r_\kappa$  belongs to  $B(0,R) \cap \Omega_\kappa$ ). Finally, again by (24), it holds

$$-\Delta \tilde{v}_1 = 0 \quad \text{in } \Omega_\infty.$$

Now, if  $\Omega_\infty = \mathbb{R}^N$ , we obtain that  $\tilde{v}_1$  is an entire harmonic function with strictly sublinear growth, and hence, by Proposition 7.3, it is constant, in contradiction with (22). On the other hand, if  $\Omega_\infty$  is the half space, we have that  $\tilde{v}_1$  is constant on the boundary of  $\Omega_\infty$  (this is because the boundary data are fixed, and thus uniform Hölder continuous with respect to  $\kappa$ ). Therefore we can extend  $\tilde{v}_1$  as an harmonic function on the whole  $\mathbb{R}^N$  by symmetry, obtaining again the same contradiction.  $\blacksquare$

**Proof of Theorem 6.1:** let us define

$$\bar{r}_\kappa := \left( \frac{\kappa}{L_\kappa} \right)^{\frac{1}{2+\alpha}}.$$

We distinguish the proof in two main cases, according to the behavior of the ratio

$$D_\kappa := \frac{|x_\kappa - y_\kappa|}{\bar{r}_\kappa}.$$

*Case 1:  $D_\kappa$  is bounded.*

In this situation we choose the parameter of the rescaling as  $r_\kappa = \bar{r}_\kappa$ . We observe that, with this choice,  $M_\kappa = 1$ . Hence Lemma 6.1 applies providing the boundedness of each  $v_{i,\kappa}(0)$ . Then, up to a subsequence, there exist  $v_1, \dots, v_n$  such that  $v_{i,\kappa} \rightarrow v_i$  uniformly on compact sets. Furthermore, since  $M_\kappa = 1$ , then  $\Delta v_{i,\kappa}$  is bounded; hence the convergence to  $v_i$  is in fact of class  $C^1$ . This ensures that each  $v_i$  is Hölder continuous and that

$$(25) \quad \max_{x,y \in \Omega_\infty} \frac{|v_1(x) - v_1(y)|}{|x - y|^\alpha} = 1.$$

Moreover the differential equations pass to the limit and we have

$$-\Delta v_i = -v_i \sum_{j \neq i} a_{ij} v_j \quad \text{in } \Omega_\infty$$

for all  $i$ . Let  $\Omega_\infty = \mathbb{R}^N$ : by Proposition 7.1 we obtain that  $v_i$  is constant for every  $i$ ; for  $i = 1$  this is in contradiction with (25).

On the other side, if  $\Omega_\infty = \mathbb{R}_+^N$  (any half space) we can argue as follows. Let us first observe that each  $v_{i,\kappa}$  satisfies a pinching properties as stated by Lemma 2.1:

$$\Psi_{i,\kappa} \leq v_{i,\kappa} \leq \Phi_{i,\kappa}.$$

Here  $\Phi_{i,\kappa}$  (resp.  $\Psi_{i,\kappa}$ ) is the harmonic extension on  $\Omega_\kappa$  of  $\varphi_{i,\kappa}$  (resp. of  $\widehat{\varphi}_{i,\kappa}$ ), the rescaled data. Since  $v_{i,\kappa}$  is bounded on compact subset of  $\overline{\mathbb{R}_+^N}$  for all  $i$ , then  $\varphi_{i,\kappa}$  is bounded on compact subset of  $\partial\mathbb{R}_+^N$  (observe that 0 does not necessarily belong to  $\partial\mathbb{R}_+^N$ ). We infer that  $\Phi_{i,\kappa}$  and  $\Psi_{i,\kappa}$  are bounded on compact subset of  $\overline{\mathbb{R}_+^N}$ , for every  $i$ . Then we obtain the existence of  $\overline{\Phi}_i$  (resp.  $\overline{\Psi}_i$ ) such that, up to subsequences,  $\Phi_{i,\kappa} \rightarrow \overline{\Phi}_i$  (resp.  $\Psi_{i,\kappa} \rightarrow \overline{\Psi}_i$ ) uniformly on compact sets of  $\mathbb{R}_+^N$ , and furthermore  $\overline{\Psi}_i \leq v_i \leq \overline{\Phi}_i$ . On the other hand, it is easy to see that

$$\max_i \max_{x,y \in \Omega} \frac{|\Phi_{i,\kappa}(x) - \Phi_{i,\kappa}(y)|}{|x-y|^\alpha} \rightarrow 0, \quad \max_i \max_{x,y \in \Omega} \frac{|\Psi_{i,\kappa}(x) - \Psi_{i,\kappa}(y)|}{|x-y|^\alpha} \rightarrow 0,$$

indeed the functions above are obtained rescaling functions that are Hölder continuous and do not depend on  $\kappa$ . This means that  $\overline{\Phi}_i$  (resp.  $\overline{\Psi}_i$ ) is constant for every  $i$ . Now observe that, since  $\varphi_{i,\kappa} \cdot \varphi_{j,\kappa} = 0$  a.e. if  $i \neq j$ , then possibly only one of the  $\overline{\Phi}_i$  is a positive constant while all the others are identically zero. We want to show that this implies that all the  $v_i$ 's are constant; this will be again in contradiction with (25). Let  $\overline{\Phi}_i \equiv 0$ : from  $0 \leq v_i \leq \overline{\Phi}_i$  we obtain  $v_i \equiv 0$ . On the other hand, let  $\overline{\Phi}_i > 0$ : by the properties of the boundary data we obtain  $\overline{\Phi}_i \equiv \overline{\Psi}_i$ , and again  $v_i$  is constant.

*Case 2:  $D_\kappa$  is unbounded.*

In this situation we choose a different rescaling, letting  $r_\kappa = |x_\kappa - y_\kappa|$ . Since  $D_\kappa = \frac{r_\kappa}{\overline{r}_\kappa} \rightarrow \infty$  by assumption, this choice provides  $M_\kappa = \left(\frac{r_\kappa}{\overline{r}_\kappa}\right)^{2+\alpha} \rightarrow \infty$ : hence no limit equation is available. Note that, with this choice, (22) becomes

$$(26) \quad \max_{x \in \partial B(0,1)} (v_{1,\kappa}(x) - v_{1,\kappa}(0)) = 1.$$

Again, Lemma 6.1 provides that  $v_{i,\kappa}(0)$  is bounded for every  $i$ . Then, up to a subsequence, there exist  $v_1, \dots, v_n$  such that  $v_{i,\kappa} \rightarrow v_i$  uniformly on compact sets of  $\mathbb{R}^N$ . This convergence is sufficient to guarantee that (26) holds for  $v_1$ . Let us recall at this point that the differential equation for  $v_{i,\kappa}$  does not pass to the limit; nevertheless we can apply to the  $k$ -uple  $V_\kappa$  Theorem 2.2. This gives the convergence of  $v_{i,\kappa}$  to  $v_i$  in  $H^1$ ; since  $-\Delta v_{i,\kappa} \leq 0$  by definition, we obtain that the same inequality holds for  $v_i$ . Furthermore, by that theorem we also know that  $v_i \cdot v_j = 0$  when  $i \neq j$ . If  $\Omega_\infty = \mathbb{R}^N$ , then we are in a position to apply Proposition 7.2, obtaining that  $v_i$  is constant, in contradiction with (26). If  $\Omega_\infty = \mathbb{R}_+^N$  we arrive to the same contradiction arguing as in the last part of the Case 1.  $\blacksquare$

## 7. NONEXISTENCE RESULTS ON $\mathbb{R}^N$

In this section we establish some Liouville type results that are exploited in Section 6, when performing the asymptotic analysis of the solutions of (1) as  $\kappa \rightarrow \infty$ . In particular we shall prove that the original problem on  $\mathbb{R}^N$  does not have (nontrivial) solutions which are globally Hölder continuous.

**Proposition 7.1.** *Let  $k \geq 2$  and let  $U = (u_1, \dots, u_k)$  be a solution of*

$$\begin{cases} -\Delta u_i(x) = -u_i(x) \sum_{j \neq i} a_{ij} u_j(x) & x \in \mathbb{R}^N \\ u_i(x) \geq 0 & x \in \mathbb{R}^N \end{cases}$$

for every  $i$ . Let  $\alpha \in (0, 1)$  such that

$$\max_{i=1, \dots, k} \sup_{x, y \in \mathbb{R}^N} \frac{|u_i(x) - u_i(y)|}{|x - y|^\alpha} < \infty.$$

Then,  $k - 1$  components annihilate and the last is a nonnegative constant.

A similar nonexistence result is true when studying  $k$ -uple of subharmonic functions on  $\mathbb{R}^N$  having disjoint supports.

**Proposition 7.2.** *Let  $k \geq 2$  and let  $U = (u_1, \dots, u_k)$  such that  $u_i \cdot u_j = 0$  if  $i \neq j$  and  $-\Delta u_i \leq 0$  on  $\mathbb{R}^N$  for all  $i$ . Let  $\alpha \in (0, 1)$  such that*

$$\max_{i=1, \dots, k} \sup_{x, y \in \mathbb{R}^N} \frac{|u_i(x) - u_i(y)|}{|x - y|^\alpha} < \infty.$$

Then,  $k - 1$  components annihilate and the last is a nonnegative constant.

Finally we observe that, if  $u$  is an harmonic function on  $\mathbb{R}^N$ , we can apply the above proposition with the choice  $k = 2$ ,  $u_1 = u^+$ ,  $u_2 = u^-$ . As a consequence the only harmonic functions on  $\mathbb{R}^N$  which are globally Hölder continuous are the constants.

**Proposition 7.3.** *Let  $u$  be an harmonic function on  $\mathbb{R}^N$ . Let  $\alpha \in (0, 1)$  such that*

$$\sup_{x, y \in \mathbb{R}^N} \frac{|u(x) - u(y)|}{|x - y|^\alpha} < \infty.$$

Then  $u$  is constant.

In dimension  $N = 2$ , the above results have been already proved by the authors in [11], based on suitable variants of the monotonicity formula [1]. In order to recover the above results in any dimension, we need to establish a full generalization of those formulae, as in the following section.

**7.1. Monotonicity Formulae.** The *monotonicity lemma* was originally stated by Alt, Caffarelli and Friedman in [1] (see also [3]) in the following way:

**Lemma 7.1** (The monotonicity formula). *Let  $(w_1, w_2) \in (H^1(\Omega))^2$  be non negative, continuous, subharmonic functions in a ball  $B(x_0, \bar{r}) \subset \Omega$  (i.e.  $-\Delta w_i \leq 0$  in distributional sense). Assume that  $w_1(x)w_2(x) = 0$ . Assume that  $x_0 \in \partial\{w_i > 0\} \cap \Omega$  for  $i = 1, 2$ . Define*

$$\Phi(r) = \prod_{i=1}^2 \frac{1}{r^2} \int_{B(x_0, r)} \frac{|\nabla w_i(x)|^2}{|x - x_0|^{N-2}} dx.$$

Then  $\Phi$  is a non decreasing function in  $[0, \bar{r}]$ .

In this section we first extend the result to the case of many subharmonic densities having mutually disjoint supports. To this aim, we consider the optimal partition value

$$\beta(k, N) := \inf_{\mathcal{P}(k, N)} \frac{2}{k} \sum_{i=1}^k \left( \sqrt{\left(\frac{N-2}{2}\right)^2 + \lambda_1(\omega_i)} - \frac{N-2}{2} \right),$$

where the minimization is taken over all possible partitions in  $k$  disjoint parts of the unit sphere  $S^{N-1}$ . The monotonicity formula then reads as

**Lemma 7.2.** *Let  $h \geq 2$ . Let  $(w_1, \dots, w_h) \in (H^1(\Omega))^h$  be non negative, continuous, subharmonic functions in a ball  $B(x_0, \bar{r}) \subset \Omega$  (i.e.  $-\Delta w_i \leq 0$  in distributional sense). Assume that  $w_i(x)w_j(x) = 0$  a.e. if  $i \neq j$  and that  $x_0 \in \partial\{w_j > 0\} \cap \Omega$  for all  $j = 1, \dots, h$ . Define*

$$\Phi(r) = \prod_{i=1}^h \frac{1}{r^{\beta(h,N)}} \int_{B(x_0,r)} \frac{|\nabla w_i(x)|^2}{|x-x_0|^{N-2}} dx.$$

Then  $\Phi$  is a non decreasing function in  $[0, \bar{r}]$ .

Note that the value  $\beta$  can be written as

$$\beta(k, N) = \inf_{(\omega_i) \in \mathcal{P}(k, N)} \frac{2}{k} \sum_{i=1}^k \gamma(\sqrt{\lambda_1(\omega_i)}),$$

where  $\gamma : \mathbb{R}^+ \rightarrow \mathbb{R}$  is defined as

$$\gamma(x) = \sqrt{\frac{(N-2)^2}{4} + x^2} - \frac{N-2}{2}.$$

Since in dimension  $N = 2$  the function  $\gamma$  is the identity, the above lemma reduces to the result proved in [11], Section 5. Besides, when there are only two parts, the optimal partition is achieved by the equator-cut sphere (see [31]), providing  $\lambda_1(\omega_i) = N - 1$ . In this case

$$\beta(2, N) = 2,$$

which precisely recovers the original formula.

**Proof of Lemma 7.2:** the proof follows the line of [1], Lemma 5.2. To start with, observe that the function  $\Phi(r)$  is absolutely continuous. Then, the idea of the proof consists in showing that  $\Phi'(r) \geq 0$  for almost every  $r$ . By computing  $\Phi'(r)$  we obtain

$$\begin{aligned} \Phi'(r) &= -\frac{h\beta(h, N)}{r^{h\beta(h, N)+1}} \prod_{i=1}^h \int_{B_r} \frac{|\nabla w_i|^2}{|x-x_0|^{N-2}} + \\ &\quad + \frac{1}{r^{h\beta(h, N)}} \sum_{i=1}^h \left[ \left( \prod_{j \neq i} \int_{B_r} \frac{|\nabla w_j|^2}{|x-x_0|^{N-2}} \right) \int_{\partial B_r} \frac{|\nabla w_i|^2}{|x-x_0|^{N-2}} \right] \\ &= \Phi(r) \left( -\frac{h\beta(h, N)}{r} + \sum_{i=1}^h \frac{\int_{\partial B_r} |\nabla w_i|^2 / |x-x_0|^{N-2}}{\int_{B_r} |\nabla w_i|^2 / |x-x_0|^{N-2}} \right). \end{aligned}$$

The goal now is to estimate the summation term in the last equality. Let us start with some calculations. First we essentially test the equation with  $w_i/|x-x_0|^{N-2}$  on the sphere  $B(x_0, r) =: B_r$  (with  $r \leq \bar{r}$ ). Due to the singularity of the test function, one should make use of regularizing cut-off functions. This can be done exactly as in [1], so we omit the calculations and we proceed formally with the singular test functions.

Since each  $w_i$  is positive and  $-\Delta w_i \leq 0$ , we obtain, for every  $i$ :

$$(27) \quad \int_{B_r} \frac{|\nabla w_i|^2}{|x-x_0|^{N-2}} \leq \int_{\partial B_r} \frac{w_i}{|x-x_0|^{N-2}} \frac{\partial}{\partial n} w_i - \int_{B_r} \nabla \left( \frac{w_i^2}{2} \right) \nabla \left( \frac{1}{|x-x_0|^{N-2}} \right).$$

Let  $\nabla_T w_i := \nabla w_i - n \partial_n w_i$  be the tangential component of the gradient of  $w_i$ . Let  $\eta$  be a positive constant to be chosen. We have

$$\begin{aligned} \int_{\partial B_r} \frac{w_i}{|x-x_0|^{N-2}} \frac{\partial}{\partial n} w_i &\leq \frac{1}{r^{N-2}} \left( \int_{\partial B_r} w_i^2 \right)^{1/2} \left( \int_{\partial B_r} (\partial_n w_i)^2 \right)^{1/2} \leq \\ &\leq \frac{1}{2r^{N-2}} \left[ \int_{\partial B_r} \frac{1}{\eta^2} |\nabla_T w_i|^2 + \int_{\partial B_r} \eta^2 (\partial_n w_i)^2 \right] \cdot \left( \frac{\int_{\partial B_r} w_i^2}{\int_{\partial B_r} |\nabla_T w_i|^2} \right)^{1/2}. \end{aligned}$$

Now let  $v_i^{(r)} : S^{N-1} \rightarrow \mathbb{R}$  be defined as  $v_i^{(r)}(\xi) := w_i(x_0 + r\xi)$  and set  $\omega_i := \{v_i^{(r)} > 0\}$ . Then  $\nabla v_i^{(r)}(\xi) = r^2 \nabla_T w_i(x_0 + r\xi)$ . This entails

$$\left( \frac{\int_{\partial B_r} |\nabla_T w_i|^2}{\int_{\partial B_r} w_i^2} \right)^{1/2} \geq \left( \frac{r^{-2} \int_{S^{N-1}} |\nabla v_i^{(r)}|^2}{\int_{S^{N-1}} (v_i^{(r)})^2} \right)^{1/2} \geq \frac{1}{r} \sqrt{\lambda_1(\omega_i)}.$$

Note that, since  $w_i \cdot w_j = 0$  a.e., then the  $\omega_i$ 's constitute, for a.e.  $r$ , a partition of  $S^{N-1}$ . Besides, since  $-\Delta \frac{1}{|x-x_0|^{N-2}} = c\delta_{x_0}$ ,  $w_i(x_0) = 0$ , and by the Poincaré inequality, we have

$$-\int_{B_r} \nabla \left( \frac{w_i^2}{2} \right) \nabla \left( \frac{1}{|x-x_0|^{N-2}} \right) = -\int_{\partial B_r} \frac{w_i^2}{2} \frac{\partial}{\partial n} \frac{1}{|x-x_0|^{N-2}} \leq \frac{N-2}{2\lambda_1(\omega_i)} \int_{\partial B_r} \frac{r^2}{r^{N-1}} |\nabla_T w_i|^2.$$

Collecting all the above information in (27), we obtain

$$(28) \quad \int_{B_r} \frac{|w_i|^2}{|x-x_0|^{N-2}} \leq \frac{1}{2r^{N-3}} \left[ \left( \frac{1}{\eta^2 \sqrt{\lambda_1(\omega_i)}} + \frac{N-2}{\lambda_1(\omega_i)} \right) \int_{\partial B_r} |\nabla_T w_i|^2 + \frac{\eta^2}{\sqrt{\lambda_1(\omega_i)}} \int_{\partial B_r} (\partial_n w_i)^2 \right]$$

Now we choose

$$(29) \quad \eta^2 = \frac{1}{\sqrt{\lambda_1(\omega_i)}} \left( \frac{N-2}{2} + \sqrt{\left( \frac{N-2}{2} \right)^2 + \lambda_1(\omega_i)} \right) = \frac{\sqrt{\lambda_1(\omega_i)}}{\gamma(\sqrt{\lambda_1(\omega_i)})}$$

in such a way that the coefficients of the tangential gradient and of the normal gradient coincide, namely

$$\frac{\eta^2}{\sqrt{\lambda_1(\omega_i)}} = \frac{1}{\eta^2 \sqrt{\lambda_1(\omega_i)}} + \frac{N-2}{\lambda_1(\omega_i)} = \frac{1}{\gamma(\sqrt{\lambda_1(\omega_i)})}.$$

In this way we obtain

$$\int_{B_r} \frac{|\nabla w_i|^2}{|x-x_0|^{N-2}} \leq \frac{r}{2} \frac{1}{\gamma(\sqrt{\lambda_1(\omega_i)})} \int_{\partial B_r} \frac{|\nabla w_i|^2}{|x-x_0|^{N-2}}.$$

Finally, summing up on  $i$  and recalling the definition of  $\beta$ , we have

$$\sum_{i=1}^h \frac{\int_{\partial B_r} |\nabla w_i|^2 / |x-x_0|^{N-2}}{\int_{B_r} |\nabla w_i|^2 / |x-x_0|^{N-2}} \geq \frac{h}{r} \beta(h, N) \quad \text{for almost every } r.$$

Using this information in the computation of  $\Phi'(r)$ , the lemma follows.  $\blacksquare$

In order to state a suitable form of the lemma which applies to the solutions of (1), we first introduce a smooth auxiliary function  $f(r) : [0, \infty) \rightarrow (0, \infty)$  such that  $f(r) = 1/r^{N-2}$  when  $r > 1$  and  $\Delta f(|x|) := 2m(|x|)$  is bounded and vanishes outside the ball of radius 1. Now we are ready to prove

**Lemma 7.3.** *Let  $N \geq 2$  and let  $(u_1, \dots, u_k)$  be a solution of (1) such that  $u_i > 0$  for all  $i$ . Let  $h \leq k$  be any integer, let  $h' < \beta(h, N)$  and define*

$$\Phi(r) = \prod_{i=1}^h \frac{1}{r^{h'}} \Phi_i(r)$$

where

$$\Phi_i(r) = \int_{B(0,r)} \left[ f(|x|) \left( |\nabla u_i(x)|^2 + u_i^2(x) \sum_{\substack{1 \leq j \leq k \\ j \neq i}} a_{ij} u_j(x) \right) - m(|x|) u_i^2(x) \right] dx.$$

Then there exists  $r' = r(h') > 1$  such that  $\Phi_i > 0$  and  $\Phi$  is an increasing function in  $[r', \infty)$ .

**Proof of Lemma 7.3:** let us fix  $h' < \beta(h, N)$ ,  $r > 1$  and set  $B_r = B(0, r)$ . We follow the outline of the proof of Lemma 7.2. By computing  $\Phi'(r)$  we have

$$\Phi'(r) = \Phi(r) \left[ -\frac{hh'}{r} + \sum_{i=1}^h \frac{\int_{\partial B_r} f(|x|)(|\nabla u_i|^2 + u_i^2 \sum_{j \neq i} a_{ij} u_j)}{\int_{B_r} [f(|x|)(|\nabla u_i|^2 + u_i^2 \sum_{j \neq i} a_{ij} u_j) - m(|x|)u_i^2(x)]} \right].$$

By testing the equation for  $u_i$  with  $f(|x|)u_i(x)$ , we have

$$\begin{aligned} \int_{B_r} f(|x|)(|\nabla u_i|^2 + u_i^2 \sum_{j \neq i} a_{ij} u_j) &= - \int_{B_r} \nabla \left( \frac{u_i^2}{2} \right) \nabla f(|x|) + \int_{\partial B_r} \frac{u_i}{|x|^{N-2}} \frac{\partial}{\partial n} u_i \\ &= \int_{B_r} m(|x|)u_i^2 - \int_{\partial B_r} \frac{u_i^2}{2} \frac{\partial}{\partial n} \frac{1}{|x|^{N-2}} + \int_{\partial B_r} \frac{u_i}{|x|^{N-2}} \frac{\partial}{\partial n} u_i, \end{aligned}$$

providing

$$\Phi_i(r) = \frac{N-2}{2r^{N-1}} \int_{\partial B_r} u_i^2 + \frac{1}{r^{N-2}} \int_{\partial B_r} u_i \frac{\partial}{\partial n} u_i.$$

Let us now set  $v_i^{(r)}(\xi) := u_i(r\xi)$  and define

$$\Lambda_i(r) = \frac{\int_{S^{N-1}} |\nabla v_i^{(r)}|^2 + r^2 (v_i^{(r)})^2 \sum_{j \neq i} a_{ij} v_j^{(r)}}{\int_{S^{N-1}} (v_i^{(r)})^2}.$$

Then we have

$$\begin{aligned} \left| \int_{\partial B_r} u_i \frac{\partial}{\partial n} u_i \right| &\leq \left( \int_{\partial B_r} u_i^2 \right)^{1/2} \left( \int_{\partial B_r} (\partial_n u_i)^2 \right)^{1/2} \leq \\ &\leq \left[ \int_{\partial B_r} \frac{1}{\eta^2} (|\nabla_T u_i|^2 + u_i^2 \sum_{j \neq i} a_{ij} u_j) + \int_{\partial B_r} \eta^2 (\partial_n u_i)^2 \right] \cdot \frac{r}{\sqrt{\Lambda_i(r)}}, \end{aligned}$$

for any positive constant  $\eta^2$ . Hence

$$\begin{aligned} \Phi_i(r) &\geq \frac{1}{2r^{N-3}} \left[ \left( -\frac{1}{\eta^2 \sqrt{\Lambda_i(r)}} + \frac{N-2}{\Lambda_i(r)} \right) \int_{\partial B_r} |\nabla_T u_i|^2 + \frac{\eta^2}{\sqrt{\Lambda_i(r)}} \int_{\partial B_r} (\partial_n u_i)^2 \right. \\ &\quad \left. + \int_{\partial B_r} \left( -\frac{1}{\eta^2 \sqrt{\Lambda_i(r)}} + \frac{N-2}{\Lambda_i(r)} \right) u_i^2 \sum_{j \neq i} a_{ij} u_j \right]. \end{aligned}$$

By selecting  $\eta^2$  large enough, we learn at once that  $\Phi_i$  is positive. Besides, we obtain the counterpart of (28), that is

$$\begin{aligned} \Phi_i(r) &\leq \frac{1}{2r^{N-3}} \left[ \left( \frac{1}{\eta^2 \sqrt{\Lambda_i(r)}} + \frac{N-2}{\Lambda_i(r)} \right) \int_{\partial B_r} |\nabla_T u_i|^2 + \frac{\eta^2}{\sqrt{\Lambda_i(r)}} \int_{\partial B_r} (\partial_n u_i)^2 \right. \\ &\quad \left. + \int_{\partial B_r} \left( \frac{1}{\eta^2 \sqrt{\Lambda_i(r)}} + \frac{N-2}{\Lambda_i(r)} \right) u_i^2 \sum_{j \neq i} a_{ij} u_j \right] \end{aligned}$$

Now choose

$$\eta^2 = \frac{\sqrt{\Lambda_i(r)}}{\gamma(\sqrt{\Lambda_i(r)})}.$$

With this choice, the three coefficients of the integrals on  $\partial B_r$  coincide, and the right end side of the previous inequality becomes

$$\Phi_i(r) \leq \frac{r}{2\gamma(\sqrt{\Lambda_i(r)})} \int_{\partial B_r} f(|x|)(|\nabla u_i|^2 + u_i^2 \sum_{j \neq i} a_{ij} u_j).$$

With this we can estimate the summation term which appears in the computation of  $\Phi'$  as follows

$$\Phi'(r) \geq \Phi(r) \left[ -\frac{hh'}{r} + \frac{2}{r} \sum_{i=1}^h \gamma(\sqrt{\Lambda_i(r)}) \right].$$

In order to prove that  $\Phi'(r) \geq 0$  when  $r$  is sufficiently large, we are thus left to prove that there exists  $r_0 > 1$  such that

$$(30) \quad \frac{2}{h} \sum_{i=1}^h \gamma(\sqrt{\Lambda_i(r)}) > h' \quad \forall r \geq r_0.$$

It is worthwhile noticing that the supports of the functions  $v_i^{(r)}$ 's are not mutually disjoint, and thus we can not directly compare the summation term involving  $\Lambda_i(r)$  with  $\beta(h, N)$  in order to conclude the validity of (30). Nevertheless, arguing as in the proof of Lemma 1.3 of [11], it is possible to prove that, when  $r \rightarrow \infty$ , (suitable multiples of) the  $v_i^{(r)}$ 's converge to a  $k$ -tuple of functions  $\bar{v}_i$  on  $S^{N-1}$  having disjoint supports  $\bar{\omega}_i$ . This process forces the convergence of  $\Lambda_i(r)$  to  $\lambda_1(\bar{\omega}_i)$ , and allows to recover (30) by a simple contradiction argument. The reader is referred to the aforementioned result for the details.  $\blacksquare$

Let us now observe that, if  $k \geq 3$ , then

$$(31) \quad \beta(k, N) > 2.$$

Indeed, reasoning as in Proposition 5.1 of [11], it turns out that  $\beta$  is a nondecreasing function of  $k$  and, moreover  $\beta(k, N) > \beta(2, N) = 2$  for  $k \geq 3$ . This information is crucial for the proof of our Liouville type results.

**Proof of Proposition 7.1:** let  $B_r = B(0, r)$ . By Lemma 7.3 and (31), we know that

$$(32) \quad \prod_{i=1}^h \frac{1}{r^{h'}} \int_{B_r} \left[ f(|x|)(|\nabla u_i(x)|^2 + u_i^2(x) \sum_{j \neq i} a_{ij} u_j(x)) - m(|x|) u_i^2(x) \right] dx \geq C > 0$$

for any  $h' < 2$ , if  $r > 1$  is large enough.

On the other hand, let us consider a smooth, radial cut-off function which is equal 1 in  $B_r$  and vanishes outside  $B_{2r}$ . Let us multiply the  $i$ -th differential equation by  $\eta^2 f(|x|) u_i$  and then integrate:

$$\begin{aligned} \int_{B_{2r}} f(|x|) (\eta^2 |\nabla u_i|^2 + \eta^2 u_i^2 \sum_{j \neq i} a_{ij} u_j) &\leq \frac{1}{2} \int_{B_{2r}} f(|x|) \eta^2 |\nabla u_i|^2 \\ &+ 2 \int_{B_{2r}} f(|x|) u_i^2 |\nabla \eta|^2 - \int_{B_{2r}} \nabla \frac{\eta^2 u^2}{2} \nabla f(|x|) + \int_{B_{2r}} u^2 \eta \nabla \eta \nabla f(|x|). \end{aligned}$$

Recalling that  $\Delta f = 2m$ , by testing with  $\eta^2 u^2/4$  on  $B_{2r}$ , we have

$$- \int_{B_{2r}} \nabla \frac{\eta^2 u^2}{2} \nabla f(|x|) = \frac{1}{2} \int_{B_{2r}} m(|x|) \eta^2 u_i^2.$$

To estimate the remaining terms on the r.h.s., observe first that  $\nabla \eta$  vanishes outside  $B_{2r} \setminus B_r$ , while on the annulus  $|\nabla \eta| \leq Cr^{-1}$ . Besides, since  $r > 1$ ,  $f(|x|) = |x|^{2-N}$  and  $|\nabla f(|x|)| = (N-2)|x|^{1-N}$ . Collecting all this information, we end up with

$$\int_{B_{2r}} \left[ f(|x|) (\eta^2 |\nabla u_i|^2 + \eta^2 u_i^2 \sum_{j \neq i} a_{ij} u_j) - m(|x|) \eta^2 u_i^2 \right] \leq \frac{C}{r^N} \int_{B_{2r} \setminus B_r} u_i^2.$$

By assumption, when  $\rho$  is sufficiently large, we have that  $u_i(x) \leq C'\rho^\alpha$  for all  $x \in \partial B_\rho$  and for all indices  $i$ . Hence

$$\int_{B_{2r} \setminus B_r} u_i^2 = \int_r^{2r} \rho^{N-1+2\alpha} d\rho = Cr^{N+2\alpha}.$$

Comparing with (32) we have  $r^{h'} \leq Cr^{2\alpha}$  for any  $r$  large enough. The choice of  $h' := 1 + \alpha$  in (32), which is admissible as  $\alpha < 1$ , immediately provides a contradiction when  $r \rightarrow \infty$ . ■

Arguing as in the previous proof, but exploiting the monotonicity formula Lemma 7.2, Proposition 7.2 follows at once.

#### REFERENCES

- [1] H.W. Alt, L.A. Caffarelli, A. Friedman *Variational problems with two phases and their free boundaries*, Trans. A.M.S. 282 (1984), 431–461
- [2] I. Athanassopoulos, L.A. Caffarelli, S. Salsa, *The free boundary in an inverse conductivity problem*, J. Reine Angew. Math. 534 (2001), 1–31
- [3] L.A. Caffarelli, L. Karp, H. Shahgholian, *Regularity of a free boundary with applications to the Pompeii problem*, Annals Math. 151 (2000), 269–292
- [4] J. Blat, K.J. Brown, *Bifurcation of steady-state solutions in predator-prey and competition systems*, Proc. Roy. Soc. Edinburgh Sect. A, 97 (1984) 21–34
- [5] L.A. Caffarelli, D. Jerison, C.E. Kenig, *Some new monotonicity theorems with applications to free boundary problems*, Ann. of Math. 155 (2002), no. 2, 369–404
- [6] L.A. Caffarelli, D. Jerison, C.E. Kenig, *Regularity for inhomogeneous two-phase free boundary problems*, in preparation
- [7] C. Cosner, A. Lazer, *Stable coexistence in the Volterra–Lotka competition model with diffusion*, SIAM J. Math. Anal. 44 (1984), 1112–1132
- [8] M. Conti, S. Terracini, G. Verzini, *Nehari’s Problem and Competing Species Systems*, Ann. Inst. H. Poincaré, AN 19, 6 (2002) 871–888
- [9] M. Conti, S. Terracini, G. Verzini, *An optimal partition problem related to nonlinear eigenvalues*, Journal of Funct. Anal. 198, 1 (2003) 160–196
- [10] M. Conti, S. Terracini, G. Verzini, *A variational problem for the spatial segregation of reaction–diffusion systems*, Indiana Univ. Math. J., to appear
- [11] M. Conti, S. Terracini, G. Verzini, *On a class of optimal partition problems related to the Fučík spectrum and to the monotonicity formulae*, Calculus of variation, to appear
- [12] E.N. Dancer, *Competing species systems with diffusion and large interaction*, Rend. Sem. Mat. Fis. Milano 65 (1995), 23–33
- [13] E.N. Dancer, Y.H. Du, *Competing species equations with diffusion, large interactions, and jumping nonlinearities*, J. Differential Equations 114 (1994), 434–475
- [14] E.N. Dancer, Y.H. Du, *Positive solutions for a three-species competition system with diffusion. II. The case of equal birth rates*, Nonlinear Anal., 24 (1995) 3, 359–373
- [15] E.N. Dancer, Y.H. Du, *Positive solutions for a three-species competition system with diffusion. I. General existence results*, Nonlinear Anal., 24 (1995) 3, 359–373
- [16] E.N. Dancer, Y.H. Du, *On a free boundary problem arising from population biology*, Indiana Univ. Math. J., 52 (2003) 1, 51–67
- [17] E.N. Dancer, Y.H. Du, L.Ma, *Asymptotic behavior of positive solutions of some elliptic problems*, Pacific J. Math., to appear
- [18] E.N. Dancer, Z.M. Guo, *Uniqueness and stability for solutions of competing species equations with large interactions*, Comm. Appl. Nonlinear Anal. 1 (1994), 19–45
- [19] E.N. Dancer, D. Hilhorst, M. Mimura, L.A. Peletier, *Spatial segregation limit of a competition–diffusion system*, European J. Appl. Math. 10 (1999), 97–115
- [20] P. Korman, A. Leung *On the existence and uniqueness of positive steady states in Lotka–Volterra ecological models with diffusion*, Appl. Anal. 26 (1987), 145–160
- [21] S-I. Ei, E. Yanagida, *Dynamics of interfaces in competition-diffusion systems*, SIAM J. Appl. Math., 54 (1994) 5, 1355–1373
- [22] J.C. Eilbeck, J.E. Furter, J. López-Gómez, *Coexistence in the competition model with diffusion*, J. Differential Equations, 107 (1994) 1, 96–139
- [23] C. Gui, Y. Lou, *Uniqueness and nonuniqueness of coexistence states in the Lotka–Volterra competition model*, Comm. Pure Appl. Math., 47 (1994) 12, 1571–1594
- [24] A.C. Lazer, P.J. McKenna, *On steady state solutions of a system of reaction–diffusion equations from biology*, Nonlinear Anal. TMA 6 (1982), 523–530

- [25] Y. Lou, S. Martínez, W-M. Ni, *On  $3 \times 3$  Lotka-Volterra competition systems with cross-diffusion*, Discrete Contin. Dynam. Systems, 6 (2000) 1, 175–190
- [26] Y. Lou, W-M. Ni, *Diffusion, self-diffusion and cross-diffusion*, J. Differential Equations, 131 (1996) 1, 79–131
- [27] H. Matano, M. Mimura, *Pattern formation in competition-diffusion systems in nonconvex domains*, Publ. Res. Inst. Math. Sci., 19 (1983) 3, 1049–1079
- [28] M. Mimura, Masayasu, *Spatial distribution of competing species*, Mathematical ecology (Trieste, 1982), Lecture Notes in Biomath. 54, 492–501
- [29] N. Shigesada, K. Kawasaki, E. Teramoto, *The effects of interference competition on stability, structure and invasion of a multispecies system*, J. Math. Biol., 21 (1984) 2, 97–113
- [30] M. Struwe, *Variational methods*, Springer, 1990
- [31] E. Sperner, *Zur symmetrisierung von funktionen auf sphären*, Math. Z, 134 (1973), 317–330
- [32] Y. Tonegawa, *On the regularity of a chemical reaction interface*, Comm. Partial Differential Equations, 23 (1998) 7-8, 1181–1207
- [33] E. Zeidler, *Nonlinear functional analysis and its applications. I. Fixed point theorems*, Springer, 1986

MONICA CONTI, GIANMARIA VERZINI  
DIPARTIMENTO DI MATEMATICA “F.BRIOSCHI”  
POLITECNICO DI MILANO  
VIA BONARDI 9, I-20133 MILANO, ITALY  
*monica.conti@mate.polimi.it, gianmaria.verzini@mate.polimi.it*

SUSANNA TERRACINI  
DIPARTIMENTO DI MATEMATICA E APPLICAZIONI  
UNIVERSITÀ DEGLI STUDI DI MILANO-BICOCCA  
VIA BICOCCA DEGLI ARCIMBOLDI 8, I-20126 MILANO, ITALY  
*suster@matapp.unimib.it*