

MULTIPULSE PHASES IN k -MIXTURES OF BOSE-EINSTEIN CONDENSATES *

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Abstract

For the system

$$-\Delta U_i + U_i = U_i^3 - \beta U_i \sum_{j \neq i} U_j^2, \quad i = 1, \dots, k,$$

(with $k \geq 3$) we prove the existence, for β large, of positive radial solutions on \mathbb{R}^N . We show that, as $\beta \rightarrow +\infty$, the profile of each component U_i separates, in many pulses, from the others. Moreover, we can prescribe the location of such pulses in terms of the oscillations of the changing-sign solutions of the scalar equation $-\Delta W + W = W^3$. Within an Hartree-Fock approximation, this provides a theoretical indication of phase separation into many nodal domains for the k -mixtures of Bose-Einstein condensates.

1 Introduction

In this paper we seek radial solutions to the system of elliptic equations

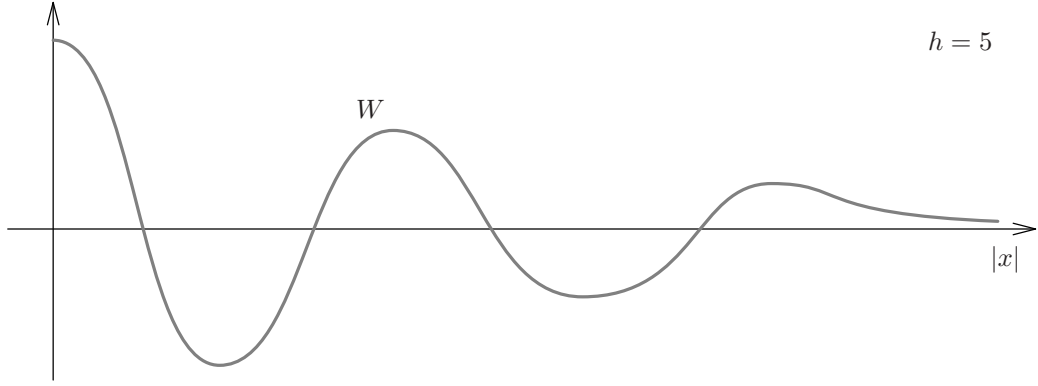
$$\begin{cases} -\Delta U_i + U_i = U_i^3 - \beta U_i \sum_{j \neq i} U_j^2, & i = 1, \dots, k \\ U_i \in H^1(\mathbb{R}^N), \quad U_i > 0, \end{cases} \quad (1)$$

with $N = 2, 3$, $k \geq 3$, and β (positive and) large, in connection with the changing-sign solutions of the scalar equation

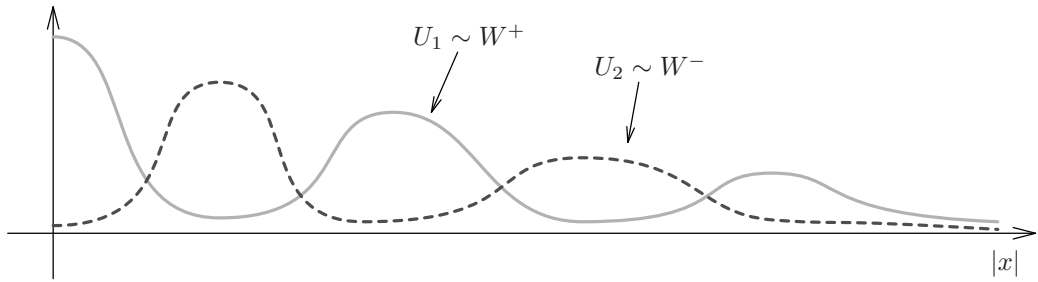
$$-\Delta W + W = W^3, \quad W \in H^1(\mathbb{R}^N). \quad (2)$$

It is well known (see, for instance, [15, 7]) that this equation admits infinitely many nodal solutions. More precisely, following Bartsch and Willem [2], for any $h \in \mathbb{N}$ equation (2) possesses radial solutions with exactly $h - 1$ changes of sign, that is h nodal components (“bumps”), with a variational characterization.

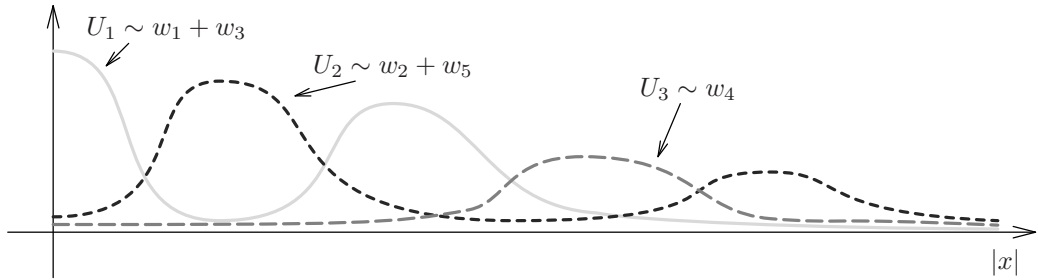
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In the recent paper [16], Wei and Weth have shown that, in the case of $k = 2$ components, there are solutions (U_1, U_2) such that the difference $U_1 - U_2$, for large values of β , approaches some sign-changing solution W of (2). Hence, one can prescribe the limit shape of U_1 and U_2 as W^+ and W^- : this means that each U_i can be seen as the sum of pulses, each converging to one of the bumps of $|W|$.



In the present paper we extend this result to the case of an arbitrary number of components $k \geq 3$, proving the existence of solutions to (1) with the property that, for β large, each component U_i is near the sum of some non-consecutive bumps of $|W|$ (see Theorems 2.7 and 2.8).



Furthermore, we can prescribe the correspondence between such bumps of $|W|$ and the index i of the component U_i (see Example 2.3). This, compared with the case $k = 2$, provides a much richer structure of the solution set for (1). This goal will be achieved by a suitable construction inspired by the extended Nehari method (see [12]) developed in [5].

System (1) arises in the study of solitary wave solutions of systems of $k \geq 3$ coupled nonlinear Schrödinger equations, known in the literature as Gross-Pitaevskii equations:

$$\begin{cases} -i\partial_t(\phi_i) = \Delta\phi_i - V_i(x)\phi_i + \mu_i|\phi_i|^2\phi_i - \sum_{j \neq i} \beta_{ij}|\phi_j|^2\phi_i, & i = 1, \dots, k \\ \phi_i \in H^1(\mathbb{R}^N; \mathbb{C}), & N = 1, 2, 3. \end{cases}$$

This system has been proposed as a mathematical model for multispecies Bose–Einstein condensation in k different hyperfine spin states (see [3] and references therein); such a condensation has been experimentally observed in the triplet states (see [13]). Here the complex valued functions ϕ_i 's are the wave functions of the i -th condensate, the functions V_i 's represent the trapping magnetic potentials, and the positive constants μ_i 's and β_{ij} 's are the intraspecies and the interspecies scattering lengths, respectively. With this choice the interactions between like particles are attractive, while the interactions between the unlike ones are repulsive; we shall assume that $\beta_{ij} = \beta_{ji}$, which gives the system a gradient structure. To obtain solitary wave solutions we set

$$\phi_i(t, x) = e^{-i\lambda_i t} U_i(x),$$

obtaining that the real functions U_i 's satisfy

$$\begin{cases} -\Delta U_i + [V_i(x) + \lambda_i] U_i = \mu_i U_i^3 - \sum_{j \neq i} \beta_{ij} U_j^2 U_i, & i = 1, \dots, k \\ U_i \in H^1(\mathbb{R}^N). \end{cases} \quad (3)$$

For the sake of simplicity we assume $V_i(x) \equiv 0$, $\lambda_i = \mu_i = 1$ and $\beta_{ij} = \beta$, for every i and j , and $N = 2, 3$, even though our method works also in more general cases, see Remark 5.2 at the end of the paper. With this choice, system (3) becomes system (1).

For a fixed k , as the interspecific competition goes to infinity, the wave amplitudes U_i 's segregate, that is, their supports tend to be disjoint. This phenomenon, called “phase separation”, has been studied, starting from [4, 5], in the case of $\mu_i > 0$ and in [3] in the case $\mu_i < 0$, for least energy solutions in non necessarily symmetric bounded domains. Of course, the number of connected domains of segregation is at least the number of different phases surviving in the limit. For the minimal solutions, the limiting states have *connected* supports¹. This is not necessarily the case for solutions which are not characterized as ground states. This is indeed what we show in the present paper, proving the existence of solutions converging to limiting states which supports have a large number of connected components. In this way we obtain a large number of connected domains of segregation with a few phases. Taking the limiting supports as unknown, this can be seen as a free boundary problem. The local analysis of the interfaces and the asymptotic analysis, as the interspecific scattering length grows to infinity has been carried in [5] for the minimal solutions.

In the recent literature, systems of type (1) have been the object of an intensive research also in different ranges of the interaction parameters, for their possible applications to a number of other physical models, such as the study of incoherent solutions in nonlinear optics. We refer the reader to the recent papers [1, 10, 6, 9, 17] mainly dealing with systems of two equations. For the general k -systems we refer to [8, 14] and the references therein.

2 Preliminaries and main results

In the absence of a magnetic trapping potential we shall work in the Sobolev space of radial functions $H_r^1(\mathbb{R}^N)$, endowed with the standard norm $\|U\|^2 = \int_{\mathbb{R}^N} |\nabla U_i|^2 + U_i^2 dx$; it is well known that such functions are continuous everywhere but the origin, thus we are allowed to evaluate them pointwise. As $N = 2, 3$ implies that $p = 4$ is a subcritical exponent, the (compact) embedding of $H_r^1(\mathbb{R}^N)$ in $L^4(\mathbb{R}^N)$ (see [15]) will be available:

(2.1) Lemma (Sobolev–Strauss). *If $U \in H_r^1(\mathbb{R}^N)$ then $\int_{\mathbb{R}^N} U^4 dx \leq C_S^4 \|U\|^4$, and the immersion $H_r^1(\mathbb{R}^N) \hookrightarrow L^4(\mathbb{R}^N)$ is compact for $N = 2, 3$.*

¹This is rigorously proven in [5], while it results from numerical evidence in [3].

We search for solutions of (1) as critical points of the related energy functional

$$J_\beta(U_1, \dots, U_k) = \sum_{i=1}^k \left[\frac{1}{2} \|U_i\|^2 - \frac{1}{4} \int_{\mathbb{R}^N} U_i^4 dx \right] + \frac{\beta}{4} \sum_{\substack{i,j=1 \\ i \neq j}}^k \int_{\mathbb{R}^N} U_i^2 U_j^2 dx$$

(we will always omit the dependence on β when no confusion arises). In the same way we associate with equation (2) the corresponding functional

$$J^*(W) = \frac{1}{2} \|W\|^2 - \frac{1}{4} \int_{\mathbb{R}^N} W^4 dx.$$

Let $h \in \mathbb{N}$ be fixed. We introduce the set of the nodal components of radial functions having (at least) $h - 1$ ordered zeroes as

$$\mathcal{X}^* = \left\{ (w_1, \dots, w_h) \in (H_r^1(\mathbb{R}^N))^h : \begin{array}{l} \text{for every } l = 1, \dots, h \text{ it holds } w_l \geq 0, w_l \not\equiv 0 \text{ and} \\ w_l(x_0) > 0 \implies w_p(x) = 0 \begin{cases} \forall |x| \geq |x_0| \text{ if } p < l \\ \forall |x| \leq |x_0| \text{ if } p > l. \end{cases} \end{array} \right\}.$$

We will often write $W = (w_1, \dots, w_h)$. By definition, if $W \in \mathcal{X}^*$, then $w_l \cdot w_p = 0$ a.e. when $l \neq p$. More precisely, the sets $\{w_l > 0\}$ are contained in disjoint annuli² and, for $l < p$, the annulus containing $\{w_l > 0\}$ is closer to the origin than the one containing $\{w_p > 0\}$. As a consequence, we have $J^*(\sum_l w_l) = J^*(\sum_l (-1)^l w_l) = \sum_l J^*(w_l)$.

We are interested in solutions of (2) with h nodal regions. The *Nehari manifold* related to this problem is defined as

$$\mathcal{N}^* = \left\{ W \in \mathcal{X}^* : J^*(w_l) = \sup_{\lambda > 0} J^*(\lambda w_l) \right\} = \left\{ W \in \mathcal{X}^* : \|w_l\|^2 = \int_{\mathbb{R}^N} w_l^4 dx \right\}.$$

As a matter of fact one has

(2.2) Proposition. *Let*

$$c_\infty = \inf_{W \in \mathcal{N}^*} J^*(W) = \inf_{W \in \mathcal{X}^*} \sup_{\lambda_l > 0} J^* \left(\sum_{l=1}^h \lambda_l w_l \right).$$

Then the set

$$\mathcal{K} = \{W \in \mathcal{N}^* : J^*(W) = c_\infty\}$$

is non empty and compact, and, for every $W \in \mathcal{K}$, the functions

$$\pm \sum_{l=1}^h (-1)^l w_l \text{ solve (2).}^3$$

Moreover there exist two constants $0 < C_1 < C_2$ such that, for every $W \in \mathcal{K}$ and for every l it holds

$$C_1^2 \leq \|w_l\|^2 = \int_{\mathbb{R}^N} w_l^4 dx \leq C_2^2.$$

For the proof of this result, very well known in the literature, we refer to [2].

Now, let us consider system (1). Roughly speaking, we want to construct solutions of (1) in the following way: each $U_i > 0$ is the sum of pulses u_{im} , where each u_{im} is near some w_l for an appropriate $W \in \mathcal{K}$. Maybe an example will make the situation more clear.

²Here and in the following by annuli we mean also balls or exteriors of balls.

³As a consequence $\text{supp } w_l$ is an annulus for every l , and $\text{supp } W = \mathbb{R}^N$.

(2.3) Example. Let $h = 5$ and $k = 3$. A possible setting is to search for solutions $U_1 = u_{11} + u_{12}$, $U_2 = u_{21} + u_{22}$, $U_3 = u_{31}$, in such a way that, for some $W \in \mathcal{K}$, (for instance) u_{11} is near w_1 , u_{21} is near w_2 , u_{12} is near w_3 , u_{31} is near w_4 , and u_{22} is near w_5 . The only rule we want to respect is that two consecutive pulses w_l and w_{l+1} must belong to different components U_i and U_j (see the last figure in the introduction).

The general situation can be treated as follows. Let $h \geq k$ and consider any surjective map

$$\sigma : \{1, \dots, h\} \rightarrow \{1, \dots, k\} \quad \text{such that} \quad \sigma(l+1) \neq \sigma(l) \text{ for } l = 1, \dots, h-1$$

(a map that associates each pulse of an element of \mathcal{K} to a component U_i). The numbers $h_i = \#\sigma^{-1}(i)$ (the number of pulses associated to the i -th component) are such that $h_i \geq 1$ and $\sum_1^k h_i = h$. This means that we can (uniquely) define a bijective map onto the set of double indexes

$$\tilde{\sigma} : \{1, \dots, h\} \rightarrow \bigcup_{i=1}^k \{(i, m) : m = 1, \dots, h_i\}$$

where the first index of $\tilde{\sigma}$ is given by σ , and the second is increasing (when the first is fixed). In this setting, Example 2.3 can be read as $\tilde{\sigma}(1) = (1, 1)$, $\tilde{\sigma}(2) = (2, 1)$, $\tilde{\sigma}(3) = (1, 2)$, $\tilde{\sigma}(4) = (3, 1)$, $\tilde{\sigma}(5) = (2, 2)$.⁴

According to the previous notation we define, for $\varepsilon \leq 1$,

$$\mathcal{X}_\varepsilon = \left\{ (u_{11}, \dots, u_{kh_k}) \in (H_r^1(\mathbb{R}^N))^h : \begin{array}{l} u_{im} \geq 0 \text{ and there exists } W \in \mathcal{K} \text{ such that} \\ \sum_{i=1}^k \sum_{m=1}^{h_i} \|u_{im} - w_{\tilde{\sigma}^{-1}(im)}\|^2 < \varepsilon^2 \end{array} \right\},$$

and $U_i = \sum_{m=1}^{h_i} u_{im}$. Sometimes we will use the distance

$$d_{\tilde{\sigma}}^2((u_{11}, \dots, u_{kh_k}), W) = \sum_{i=1}^k \sum_{m=1}^{h_i} \|u_{im} - w_{\tilde{\sigma}^{-1}(im)}\|^2$$

(2.4) Remark. Using Proposition 2.2 it is easy to see that

1. \mathcal{X}_ε is contained in an ε -neighborhood of \mathcal{K} , in the sense of $d_{\tilde{\sigma}}$; $\mathcal{K} \subset \mathcal{X}_\varepsilon$ (understanding the identification $w_l = u_{\tilde{\sigma}(im)}$);
2. there exist constants C_1, C_2 not depending on β and $\varepsilon < 1$, such that $0 < C_1 \leq \|u_{im}\| \leq C_2$, $0 < C_1^2 \leq \int_{\mathbb{R}^N} u_{im}^4 dx \leq C_2^2$
3. $m \neq n$ implies $\int_{\mathbb{R}^N} \nabla u_{im} \cdot \nabla u_{in} < C\varepsilon$, $\int_{\mathbb{R}^N} u_{im} u_{in} < C\varepsilon$, $\int_{\mathbb{R}^N} U_i^2 u_{im} u_{in} < C\varepsilon$.

A first important result we want to give, that is underlying the spirit of this whole paper, is the following: in the classical Nehari's method described above, it is not necessary to perform the min-max procedure on pulses with disjoint support, but we can "mix up", even though not too much, the pulses with non adjacent supports.

(2.5) Proposition. *There exist $\varepsilon_0 \leq 1$ such that for every $0 < \varepsilon < \varepsilon_0$ the following hold. If $(v_{11}, \dots, v_{kh_k}) \in \mathcal{X}_\varepsilon$ is such that*

$$V_i \cdot V_j = 0 \quad \text{almost everywhere, for every } i, j,$$

⁴For easier notation we will write u_{im} instead of $u_{(i,m)}$; from now on we will use the letters i, j for the first index and the letters m, n for the second, while l is reserved for the components of W .

(but $v_{im} \cdot v_{in}$ is not necessarily null) then

$$\sup_{\lambda_{im} > 0} J^* \left(\sum_{i,m} \lambda_{im} v_{im} \right) \geq c_\infty.$$

Proof. To prove the result, we will construct a h -tuple $(\tilde{w}_1, \dots, \tilde{w}_h) \in \mathcal{N}^*$ such that, for a suitable choice of the positive numbers $\tilde{\lambda}_{im}$'s, it holds

$$J^* \left(\sum \tilde{\lambda}_{im} v_{im} \right) = J^* \left(\sum \tilde{w}_{\bar{\sigma}^{-1}(im)} \right).$$

As a first step, we have to select the (disjointed) supports of such \tilde{w}_l . To do that, using the definition of \mathcal{X}_ε , we choose a $W \in \mathcal{K}$ such that $\sum_{i,m} \|v_{im} - w_{\bar{\sigma}^{-1}(im)}\|^2 < \varepsilon^2$. When ε is small we can find h positive radii r_1, \dots, r_h such that, for $|x_l| = r_l$, $v_{\bar{\sigma}(l)}(x_l)w_l(x_l) > 0$.⁵ Using this fact we can construct the open connected annuli A_1, \dots, A_h in such a way that

- $|x_l| = r_l$ implies $x_l \in A_l$,
- $\bigcup_{l=1}^h \overline{A_l} = \mathbb{R}^N$ and $A_l \cap A_p = \emptyset$ when $p \neq l$,
- $\text{supp} V_i \subset \bigcup_{m=1}^{h_i} \overline{A_{\bar{\sigma}^{-1}(im)}}$ for every i

(recall that, by assumption, $\text{int}(\text{supp} V_i) \cap \text{int}(\text{supp} V_j) = \emptyset$). By construction, we obtain that obviously $\text{supp } w_l \cap I_l \neq \emptyset$, while, by connectedness,

$$\text{supp } w_{\bar{\sigma}^{-1}(im)} \cap A_{\bar{\sigma}^{-1}(in)} = \emptyset \text{ when } n \neq m.$$

In particular this last fact implies that, for $n \neq m$,⁶

$$\left\| v_{im}|_{A_{\bar{\sigma}^{-1}(in)}} \right\|^2 \leq \varepsilon^2 \text{ and hence } \left\| v_{im}|_{A_{\bar{\sigma}^{-1}(im)}} \right\|^2 \geq C_1^2 - (h_i - 1)\varepsilon^2 \quad (4)$$

(with C_1 as in Remark 2.4). Now, depending on the positive parameters λ_{im} 's, let us define the functions \tilde{v}_{im} 's as

$$\tilde{v}_{im} = \left(\sum_{n=1}^{h_i} \lambda_{in} v_{in} \right) \Big|_{A_{\bar{\sigma}^{-1}(im)}}.$$

By construction we have that $\tilde{v}_{im} \cdot \tilde{v}_{jn} \equiv 0$ for every choice of the λ_{im} 's. We claim the existence of $\tilde{\lambda}_{im}$'s such that the corresponding \tilde{v}_{im} 's satisfy

$$F_{im}(\tilde{\lambda}_{11}, \dots, \tilde{\lambda}_{kh_k}) = \|\tilde{v}_{im}\|^2 - \int_{\mathbb{R}^N} \tilde{v}_{im}^4 dx = 0 \quad \text{for every } (i, m).$$

This will imply that, writing $\tilde{w}_l = \tilde{v}_{\bar{\sigma}(l)}$, the h -tuple (w_1, \dots, w_h) belongs to \mathcal{N}^* . Since $J^* \left(\sum_{i,m} \tilde{v}_{im} \right) = J^* \left(\sum_{i,m} \tilde{\lambda}_{im} v_{im} \right)$, this will conclude the proof of the lemma. In order to prove the claim we will use (k times) a classic result by Miranda concerning the zeroes of maps from a rectangle of \mathbb{R}^{h_i} into \mathbb{R}^{h_i} (see [11]), proving the existence, when ε is sufficiently small, of constants $0 < t \leq T$ such that, for every (i, m) ,

$$\lambda_{im} = T, t \leq \lambda_{in} \leq T \implies F_{im} < 0, \quad \lambda_{im} = t, t \leq \lambda_{in} \leq T \implies F_{im} > 0; \quad (5)$$

⁵Otherwise we would obtain, for some l , $v_{\bar{\sigma}(l)}w_l \equiv 0$, and hence $\varepsilon^2 > \|v_{\bar{\sigma}(l)} - w_l\|^2 = \|v_{\bar{\sigma}(l)}\|^2 + \|w_l\|^2 \geq 2C_1^2$ by Remark 2.4.

⁶Since v_{im} vanishes on ∂I_l for every choice of the indexes, $v_{im}|_{I_l}$ belongs to $H_r^1(\mathbb{R}^N)$ and then, when $\bar{\sigma}(l) \neq (i, m)$, $\varepsilon^2 > \|(v_{im} - w_{\bar{\sigma}^{-1}(im)})|_{I_l}\|^2 = \|v_{im}|_{I_l}\|^2$.

from this and Miranda's Theorem the claim will follow. Let then (i, m) be fixed, $\lambda_{im} = T$ and, for $n \neq m$, $0 \leq \lambda_{in} \leq T$. Exploiting Remark 2.4, equation (4), and the Sobolev embedding of $H_r^1(\mathbb{R}^N)$ in $L^4(\mathbb{R}^N)$, we obtain⁷

$$\|\tilde{v}_{im}\| = \lambda_{im} \left\| v_{im}|_{A_l} + \sum_{n \neq m} \frac{\lambda_{in}}{\lambda_{im}} v_{in}|_{A_l} \right\| \leq T (C_2 + (h_i - 1)\varepsilon)$$

and

$$\int_{\mathbb{R}^N} \tilde{v}_{im}^4 dx \geq \lambda_{im}^4 \int_{A_l} v_{im}^4 dx = \lambda_{im}^4 \left(\int_{\mathbb{R}^N} v_{im}^4 dx - \sum_{p \neq l} \int_{A_p} v_{im}^4 dx \right) \geq T^4 (C_1 - (h_i - 1)C_S^4 \varepsilon^4).$$

Choosing ε_0 in such a way that, for $\varepsilon < \varepsilon_0$, the last term is positive, we obtain an inequality of the form

$$F_{im} \leq aT^2 - bT^4 < 0 \quad \text{if } T \text{ is fixed sufficiently large,}$$

and the first part of (5) is proved. On the other hand, let now T be fixed as above, $\lambda_{im} = t$, and, for $n \neq m$, $t \leq \lambda_{in} \leq T$. Using again the Sobolev embedding, we have

$$F_{im} = \|\tilde{v}_{im}\|^2 - \int_{\mathbb{R}^N} \tilde{v}_{im}^4 dx \geq \|\tilde{v}_{im}\|^2 - C_S^4 \|\tilde{v}_{im}\|^4 = \|\tilde{v}_{im}\|^2 (1 - C_S^4 \|\tilde{v}_{im}\|^2).$$

Then we simply have to prove that, when t is sufficiently small, $\|\tilde{v}_{im}\| < 1/C_S^2$. As before, by Remark 2.4 and equation (4), we obtain

$$\|\tilde{v}_{im}\| = \left\| \lambda_{im} v_{im}|_{A_l} + \sum_{n \neq m} \lambda_{in} v_{in}|_{A_l} \right\| \leq tC_2 + (h_i - 1)\varepsilon T.$$

Hence we can choose t and ε_0 sufficiently small in such a way that, for every $\varepsilon < \varepsilon_0$, $\|\tilde{v}_{im}\| < 1/C_S^2$. As we just observed, this implies the second part of (5), concluding the proof of the proposition. \square

(2.6) Corollary. *Since by Remark 2.4 we have $\mathcal{N}^* \subset \mathcal{X}_\varepsilon$ for every ε , using the previous proposition we obtain the following equivalent characterizations of c_∞ :*

$$\begin{aligned} c_\infty &= \inf_{\substack{w_l \neq 0 \\ w_l \cdot w_p \equiv 0}} \sup_{\lambda_l > 0} J^* \left(\sum_l \lambda_l w_l \right) \\ &= \inf_{\substack{(v_{im}) \in \mathcal{X}_\varepsilon \\ V_i \cdot V_j \equiv 0}} \sup_{\lambda_{im} > 0} J^* \left(\sum_{i,m} \lambda_{im} v_{im} \right) \\ &= \inf_{\substack{(v_{im}) \in \mathcal{X}_\varepsilon \\ V_i \cdot V_j \equiv 0}} \sup_{\lambda_{im} > 0} J_\beta \left(\sum_m \lambda_{1m} v_{1m}, \dots, \sum_m \lambda_{km} v_{km} \right). \end{aligned}$$

In the same spirit of the previous corollary, for $i = 1, \dots, k$ and $m = 1, \dots, h_i$, let $\lambda_{im} \geq 0$ and $u_{im} \in H_r^1(\mathbb{R}^N) \setminus \{0\}$. We write

$$\begin{aligned} \Lambda_i &= (\lambda_{i1}, \dots, \lambda_{ih_i}), & U_i &= \sum_m u_{im}, & \Lambda_i U_i &= \sum_m \lambda_{im} u_{im}, \\ \Phi_\beta(\Lambda_1, \dots, \Lambda_k) &= J_\beta(\Lambda_1 U_1, \dots, \Lambda_k U_k), \end{aligned}$$

⁷For easier notation we write $l = \bar{\sigma}^{-1}(im)$.

in such a way that Φ is a C^2 -function. Moreover, let

$$M_\beta(u_{11}, \dots, u_{kh_k}) = \sup_{\lambda_{im} > 0} \Phi_\beta(\Lambda_1, \dots, \Lambda_k)$$

and finally

$$c_{\varepsilon, \beta} = \inf_{\mathcal{X}_\varepsilon} M_\beta. \quad (6)$$

Our main results are the following.

(2.7) Theorem. *There exist $\bar{\varepsilon} > 0$ and $\bar{\beta} = \bar{\beta}(\bar{\varepsilon})$ such that, if $0 < \varepsilon < \bar{\varepsilon}$ and $\beta > \bar{\beta}$, then $c_{\varepsilon, \beta}$ is a critical value for J_β , corresponding to a solution of (1) belonging to \mathcal{X}_ε .*

(2.8) Theorem. *Let $0 < \varepsilon < \bar{\varepsilon}$ (as in the previous theorem) be fixed and $(\beta_s)_{s \in \mathbb{N}}$ be such that $\beta_s \rightarrow +\infty$. Finally, let (U_1^s, \dots, U_k^s) be any solution of (1) at level $c_{\varepsilon, \beta}$ and belonging to \mathcal{X}_ε . Then, up to subsequences,*

$$d_{\bar{\sigma}}((U_1^s, \dots, U_k^s), \mathcal{K}) \rightarrow 0$$

as $s \rightarrow +\infty$.

3 Estimates for any β and ε small

Let us start with some estimates on $c_{\varepsilon, \beta}$.

(3.1) Lemma. *When ε is fixed, $c_{\varepsilon, \beta}$ is non decreasing in β , and $c_{\varepsilon, \beta} \leq c_\infty$.*

Proof. First of all, if $\beta_1 < \beta_2$, then for any $(u_{11}, \dots, u_{kh_k}) \in \mathcal{X}_\varepsilon$ we have $J_{\beta_1}(\Lambda_1 U_1, \dots, \Lambda_k U_k) < J_{\beta_2}(\Lambda_1 U_1, \dots, \Lambda_k U_k)$, and we can pass to the inf-sup obtaining $c_{\varepsilon, \beta_1} \leq c_{\varepsilon, \beta_2}$. Now, let β be fixed. By definition we have

$$c_\infty = \inf_{\mathcal{X}_0} M_\beta = \inf \left\{ M_\beta(u_{11}, \dots, u_{kh_k}) : \text{there exists } W \in \mathcal{X}^* \text{ such that } u_{im} = w_{\bar{\sigma}^{-1}(im)} \right\}.$$

Indeed, in such situation $w_l w_p = 0$ for $l \neq p$ and thus, letting $\mu_l = \lambda_{\bar{\sigma}(l)}$, we obtain $J_\beta(\Lambda_1 U_1, \dots, \Lambda_k U_k) = J^*(\sum \mu_l w_l)$. To conclude we simply observe that

$$\mathcal{X}_0 \subset \mathcal{X}_\varepsilon \quad \text{for every } \varepsilon. \quad \square$$

(3.2) Corollary. *Let*

$$\tilde{\mathcal{X}}_\varepsilon = \left\{ (u_{11}, \dots, u_{kh_k}) \in \mathcal{X}_\varepsilon : M_\beta(u_{11}, \dots, u_{kh_k}) < c_\infty + \min \left(1, \frac{1}{\beta} \right) \right\}.$$

Then

$$c_{\varepsilon, \beta} = \inf_{\tilde{\mathcal{X}}_\varepsilon} M_\beta.$$

From now on, we will restrict our attention to the elements of $\tilde{\mathcal{X}}_\varepsilon$. We remark that $\tilde{\mathcal{X}}_\varepsilon$ depends on β : actually, if, for some $i \neq j$, $U_i U_j \neq 0$ on a set of positive measure, then the corresponding $(u_{11}, \dots, u_{kh_k})$ may not belong to $\tilde{\mathcal{X}}_\varepsilon$ if β is sufficiently large. Nevertheless, all the results we will prove in this section will depend only on ε , and not on β .

(3.3) Lemma. *Let $(u_{11}, \dots, u_{kh_k}) \in \tilde{\mathcal{X}}_\varepsilon$. Then $M_\beta(u_{11}, \dots, u_{kh_k})$ is positive and achieved:*

$$0 < M_\beta(u_{11}, \dots, u_{kh_k}) = \Phi_\beta(\bar{\Lambda}_1, \dots, \bar{\Lambda}_k),$$

with

$$\nabla \Phi_\beta(\bar{\Lambda}_1, \dots, \bar{\Lambda}_k) \cdot (\bar{\Lambda}_1, \dots, \bar{\Lambda}_k) = 0.$$

Proof. We drop the dependence on β . We observe that Φ is the sum of two polynomials, which are homogenous of degree two and four, respectively:

$$\Phi(\Lambda_1, \dots, \Lambda_k) = \frac{1}{2}P_2(\Lambda_1, \dots, \Lambda_k) + \frac{1}{4}P_4(\Lambda_1, \dots, \Lambda_k)$$

where $P_2 = \sum_{i,m} \|\lambda_{im}u_{im}\|^2$. Therefore, for $t \geq 0$,

$$\Phi(t\Lambda_1, \dots, t\Lambda_k) = \frac{1}{2}t^2P_2(\Lambda_1, \dots, \Lambda_k) + \frac{1}{4}t^4P_4(\Lambda_1, \dots, \Lambda_k),$$

Since $u_{im} \neq 0$ for every i, m , if some λ_{im} is different from 0 then $P_2 > 0$. Thus, for t small, $0 < \Phi(t\Lambda_1, \dots, t\Lambda_k) \leq M$.

As a consequence, we can write

$$M = \sup \left\{ \Phi(t\Lambda_1, \dots, t\Lambda_k) : t > 0, \sum_{i=1}^k |\Lambda_i|^2 = 1 \right\}.$$

On one hand, we have

$$\max_{\sum |\Lambda_i|=1} P_2(\Lambda_1, \dots, \Lambda_k) = a > 0.$$

On the other hand, since $M < c_\infty + 1/\beta$, then

$$\max_{\sum |\Lambda_i|=1} P_4(\Lambda_1, \dots, \Lambda_k) = -b < 0$$

(otherwise we would have $M = +\infty$). But then

$$\Phi(t\Lambda_1, \dots, t\Lambda_k) \leq \frac{a}{2}t^2 - \frac{b}{4}t^4 < 0 \quad \text{when } t^2 > \frac{2a}{b},$$

thus Φ is negative outside a compact set, and M is achieved by some $(\bar{\Lambda}_1, \dots, \bar{\Lambda}_k)$.

Finally, since this is a maximum for $\lambda_{im} \geq 0$, $\bar{\lambda}_{im} > 0$ implies $\partial_{\lambda_{im}} \Phi(\bar{\Lambda}_1, \dots, \bar{\Lambda}_k) = 0$, therefore

$$\partial_{\lambda_{im}} \Phi(\bar{\Lambda}_1, \dots, \bar{\Lambda}_k) \cdot \bar{\lambda}_{im} = 0 \quad \text{for every } (i, m). \quad \square$$

Thus, if $(u_{11}, \dots, u_{k h_k}) \in \tilde{\mathcal{X}}_\varepsilon$, then M_β is a maximum. We want to prove that, when ε is small (not depending on β), the maximum point is uniquely defined, and it smoothly depends on $(u_{11}, \dots, u_{k h_k})$. As a first step, we provide some uniform estimates for its coordinates.

(3.4) Lemma. *There exists $R > 0$, not depending on⁸ $\varepsilon \leq \varepsilon_0$ and β , such that, for every $(u_{11}, \dots, u_{k h_k}) \in \tilde{\mathcal{X}}_\varepsilon$,*

1. $\nabla \Phi(\bar{\Lambda}_1, \dots, \bar{\Lambda}_k) \cdot (\bar{\Lambda}_1, \dots, \bar{\Lambda}_k) = 0$ implies $\sum_i |\bar{\Lambda}_i|^2 < R^2$;
2. $\sum_i |\Lambda_i|^2 = R^2$ implies $\nabla \Phi(\Lambda_1, \dots, \Lambda_k) \cdot (\Lambda_1, \dots, \Lambda_k) < 0$.

Proof. Since $\Phi \leq M$, using the notations of the proof of the previous lemma we can write

$$\Phi(\bar{\Lambda}_1, \dots, \bar{\Lambda}_k) = \frac{1}{2}P_2(\bar{\Lambda}_1, \dots, \bar{\Lambda}_k) + \frac{1}{4}P_4(\bar{\Lambda}_1, \dots, \bar{\Lambda}_k) < c_\infty + 1$$

(recall Corollary 3.2) and

$$\nabla \Phi(\bar{\Lambda}_1, \dots, \bar{\Lambda}_k) \cdot (\bar{\Lambda}_1, \dots, \bar{\Lambda}_k) = P_2(\bar{\Lambda}_1, \dots, \bar{\Lambda}_k) + P_4(\bar{\Lambda}_1, \dots, \bar{\Lambda}_k) = 0,$$

⁸Here ε_0 is as in Proposition 2.5.

providing

$$P_2(\bar{\Lambda}_1, \dots, \bar{\Lambda}_k) = \sum_{i=1}^k \|\bar{\Lambda}_i U_i\|^2 < 4(c_\infty + 1). \quad (7)$$

Since every u_{im} is non negative, we have $\|\bar{\Lambda}_i U_i\|^2 \geq \sum_m \bar{\lambda}_{im}^2 \|u_{im}\|^2$. But we know (Remark 2.4) that each $\|u_{im}\|$ is bounded from below, providing

$$\sum_{i=1}^k |\bar{\Lambda}_i|^2 < \frac{4(c_\infty + 1)}{C_1^2} = R^2.$$

Now let $(\Lambda_1, \dots, \Lambda_k)$ be fixed with $\sum_i |\Lambda_i|^2 = R^2$. For $t > 0$ we write

$$f(t) = \nabla \Phi(t\Lambda_1, \dots, t\Lambda_k) \cdot (t\Lambda_1, \dots, t\Lambda_k) = t^2 P_2(\Lambda_1, \dots, \Lambda_k) + t^4 P_4(\Lambda_1, \dots, \Lambda_k),$$

and we know, from the discussion above, that $f(\bar{t}) = 0$ implies $\bar{t} < 1$. Recalling that P_4 must be negative (otherwise $M = +\infty$) we deduce that $f(1) < 0$, concluding the proof. \square

(3.5) Remark. As a consequence of the previous proof (and of Lemma 3.3) we have that, if $(u_{11}, \dots, u_{khk}) \in \tilde{\mathcal{X}}_\varepsilon$ and $M_\beta(u_{11}, \dots, u_{khk}) = \Phi_\beta(\bar{\Lambda}_1, \dots, \bar{\Lambda}_k)$, then the three quantities

$$\sum_{i=1}^k \|\bar{\Lambda}_i U_i\|^2, \quad \sum_{i=1}^k \int_{\mathbb{R}^N} (\bar{\Lambda}_i U_i)^4 dx, \quad \beta \sum_{\substack{i,j=1 \\ i \neq j}}^k \int_{\mathbb{R}^N} (\bar{\Lambda}_i U_i)^2 (\bar{\Lambda}_j U_j)^2 dx,$$

are bounded not depending on β : the first by equation (7); the second by the first bound and by the continuous immersion of $H_r^1(\mathbb{R}^N)$ in $L^4(\mathbb{R}^N)$; the third by the previous bounds and the fact that $M_\beta < c_\infty + 1$.

(3.6) Lemma. *There exists $0 < \varepsilon_1 \leq \varepsilon_0$ (not depending on β) such that if $\varepsilon < \varepsilon_1$, $(u_{11}, \dots, u_{khk}) \in \tilde{\mathcal{X}}_\varepsilon$, and $\Phi_\beta(\bar{\Lambda}_1, \dots, \bar{\Lambda}_k) = M_\beta(u_{11}, \dots, u_{khk})$ then*

$$\bar{\lambda}_{im} > \frac{1}{2} \quad \text{for every } (i, m).$$

In particular, since $\bar{\lambda}_{im} > 0$, $\nabla \Phi_\beta(\bar{\Lambda}_1, \dots, \bar{\Lambda}_k) = 0$.

Proof. By the previous lemma we know that $0 \leq \bar{\lambda}_{im} \leq R$. We choose (i, m) and, for any $(j, n) \neq (i, m)$, we fix $0 \leq \lambda_{jn} \leq R$. We will prove that (for ε sufficiently small)

$$\Phi|_{\lambda_{im}=0} < \Phi|_{\lambda_{im}=1/2}, \quad \text{and} \quad 0 \leq \lambda_{im} \leq 1/2 \implies \partial_{\lambda_{im}}^2 \Phi(\Lambda_1, \dots, \Lambda_k) > 0, \quad (8)$$

and the result will follow. Let $V = \sum_{n \neq m} \lambda_{in} u_{in}$. Since all the λ_{in} 's are bounded, by Remark 2.4 we know that

$$\langle V, u_{im} \rangle = o(1), \quad \int_{\mathbb{R}^N} V^p u_{im}^q dx = o(1), \quad \text{as } \varepsilon \rightarrow 0.$$

Moreover, according to the definition of $\tilde{\mathcal{X}}_\varepsilon$, let $l = \tilde{\sigma}^{-1}(im)$ and w_l be such that $\|u_{im} - w_l\| < \varepsilon$, with $\|w_l\|^2 = \int_{\mathbb{R}^N} w_l^4 dx$. On one hand, we have (as $\varepsilon \rightarrow 0$)

$$\begin{aligned} \Phi|_{\lambda_{im}=1/2} - \Phi|_{\lambda_{im}=0} &\geq \frac{1}{2} \left[\left\| V + \frac{1}{2} u_{im} \right\|^2 - \|V\|^2 \right] - \frac{1}{4} \int_{\mathbb{R}^N} \left[\left(V + \frac{1}{2} u_{im} \right)^4 - (V)^4 \right] dx \\ &= \frac{1}{2} \left\| \frac{1}{2} u_{im} \right\|^2 - \frac{1}{4} \int_{\mathbb{R}^N} \left(\frac{1}{2} u_{im} \right)^4 dx + o(1) \\ &= \frac{1}{2} \left\| \frac{1}{2} w_l \right\|^2 - \frac{1}{4} \int_{\mathbb{R}^N} \left(\frac{1}{2} w_l \right)^4 dx + o(1) \\ &= \frac{7}{64} \|w_l\|^2 + o(1) > 0. \end{aligned}$$

On the other hand, with similar calculations, we obtain

$$\partial_{\lambda_{im}} \Phi = \langle \Lambda_i U_i, u_{im} \rangle - \int_{\mathbb{R}^N} (\Lambda_i U_i)^3 u_{im} dx + \beta \int_{\mathbb{R}^N} (\Lambda_i U_i) u_{im} \sum_{j \neq i} (\Lambda_j U_j)^2 dx \quad (9)$$

and

$$\begin{aligned} \partial_{\lambda_{im}}^2 \Phi(\Lambda_1, \dots, \Lambda_k) &= \|u_{im}\|^2 - 3 \int_{\mathbb{R}^N} (\Lambda_i U_i)^2 u_{im}^2 dx + \beta \int_{\mathbb{R}^N} u_{im}^2 \sum_{j \neq i} (\Lambda_j U_j)^2 dx \\ &\geq \|w_l\|^2 - 3 \int_{\mathbb{R}^N} \lambda_{im}^2 w_l^2 dx + o(1) \\ &= (1 - 3\lambda_{im}^2) \|w_l\|^2 + o(1) > 0 \end{aligned} \quad (10)$$

since $\lambda_{im} \leq 1/2$. □

(3.7) Remark. As a byproduct of the previous proof (equation (8)), we have that, if $\varepsilon < \varepsilon_1$, $(u_{11}, \dots, u_{kh_k}) \in \tilde{\mathcal{X}}_\varepsilon$, and $0 \leq \lambda_{jn} \leq R$, then

$$\partial_{\lambda_{im}} \Phi(\Lambda_1, \dots, \Lambda_k)|_{\lambda_{im}=1/2} > 0.$$

(3.8) Lemma. *There exists $0 < \varepsilon_2 \leq \varepsilon_1$ (not depending on β) such that if $\varepsilon < \varepsilon_2$, $(u_{11}, \dots, u_{kh_k}) \in \tilde{\mathcal{X}}_\varepsilon$ and $\nabla \Phi(\bar{\Lambda}_1, \dots, \bar{\Lambda}_k) = 0$ with $\bar{\lambda}_{im} > 1/2$ then the Hessian matrix*

$$D^2 \Phi(\bar{\Lambda}_1, \dots, \bar{\Lambda}_k) \text{ is negative definite.}$$

Proof. From (9) we obtain

$$\begin{aligned} \partial_{\lambda_{im} \lambda_{jn}}^2 \Phi &= \beta \int_{\mathbb{R}^N} 2(\Lambda_i U_i)(\Lambda_j U_j) u_{im} u_{jn} dx \quad \text{if } j \neq i \\ \partial_{\lambda_{im} \lambda_{in}}^2 \Phi &= \langle u_{im}, u_{in} \rangle - 3 \int_{\mathbb{R}^N} (\Lambda_i U_i)^2 u_{im} u_{in} dx + \beta \int_{\mathbb{R}^N} u_{im} u_{in} \sum_{j \neq i} (\Lambda_j U_j)^2 dx \end{aligned}$$

(we computed $\partial_{\lambda_{im}}^2 \Phi$ in (10)). For $m \neq n$ we write

$$M_{mn}^i = \langle u_{im}, u_{in} \rangle - \int_{\mathbb{R}^N} (\bar{\Lambda}_i U_i)^2 u_{im} u_{in} dx$$

in such a way that

$$\langle \bar{\Lambda}_i U_i, u_{im} \rangle - \int_{\mathbb{R}^N} (\bar{\Lambda}_i U_i)^3 u_{im} dx = \bar{\lambda}_{im} \left[\|u_{im}\|^2 - \int_{\mathbb{R}^N} (\bar{\Lambda}_i U_i)^2 u_{im}^2 dx + \sum_{n \neq m} \frac{\bar{\lambda}_{in}}{\bar{\lambda}_{im}} M_{mn}^i \right].$$

Since $\nabla \Phi(\bar{\Lambda}_1, \dots, \bar{\Lambda}_k) = 0$ we have, for every i, m ,

$$\int_{\mathbb{R}^N} (\bar{\Lambda}_i U_i)^2 u_{im}^2 dx = \|u_{im}\|^2 + \sum_{n \neq m} \frac{\bar{\lambda}_{in}}{\bar{\lambda}_{im}} M_{mn}^i + \frac{\beta}{\bar{\lambda}_{im}} \int_{\mathbb{R}^N} (\bar{\Lambda}_i U_i) u_{im} \sum_{j \neq i} (\bar{\Lambda}_j U_j)^2 dx.$$

Substituting we obtain

$$\partial_{\lambda_{im} \lambda_{im}}^2 \Phi(\bar{\Lambda}_1, \dots, \bar{\Lambda}_k) = -2 \underbrace{\|u_{im}\|^2}_{(A)} - 3 \underbrace{\sum_{n \neq m} \frac{\bar{\lambda}_{in}}{\bar{\lambda}_{im}} M_{mn}^i}_{(B)} + \beta \underbrace{\int_{\mathbb{R}^N} u_{im} \left(u_{im} - \frac{3}{\bar{\lambda}_{im}} \bar{\Lambda}_i U_i \right) \sum_{j \neq i} (\bar{\Lambda}_j U_j)^2 dx}_{(C)}, \quad (11)$$

and, for $n \neq m$,

$$\partial_{\lambda_{im}\lambda_{in}}^2 \Phi(\bar{\Lambda}_1, \dots, \bar{\Lambda}_k) = \underbrace{M_{mn}^i - 2 \int_{\mathbb{R}^N} (\bar{\Lambda}_i U_i)^2 u_{im} u_{in} dx}_{(B)} + \beta \underbrace{\int_{\mathbb{R}^N} u_{im} u_{in} \sum_{j \neq i} (\bar{\Lambda}_j U_j)^2 dx}_{(C)}. \quad (12)$$

As a consequence we can split $D^2\Phi$ as

$$D^2\Phi(\bar{\Lambda}_1, \dots, \bar{\Lambda}_k) = -2A + B + \beta \int_{\mathbb{R}^N} C(x) dx,$$

where each of the matrices A , B , and C contains the corresponding terms in (11) and (12), and C also contains the terms appearing in $\partial_{\lambda_{im}\lambda_{jn}}^2 \Phi$, $i \neq j$. First of all, using Remark 2.4 and the boundedness of the $\bar{\lambda}_{im}$'s, we observe that A is diagonal and strictly positive definite, independent of ε , while B is arbitrary small as ε goes to zero (not depending on β). We will show that $C(x)$ is negative semidefinite for every x : this will conclude the proof.

To do that, we will only use that $u_{im} \geq 0$ and $\sum_m \bar{\lambda}_{im} u_{im} = \bar{\Lambda}_i U_i$, therefore, without loss of generality, we can put $\bar{\lambda}_{im} = 1$ for every (i, m) ⁹. The matrix $C(x)$ can be written as the sum of matrices $C_{ij}(x)$ where only two components, say U_i and U_j , interact. Such matrices, for x fixed, contain many null blocks, corresponding both to the interaction with the other components U_p , $p \neq i, j$, and to the pulses of U_i and U_j vanishing in x . All those null blocks do not incide on the semidefiniteness of C_{ij} ; up to the null terms, C_{ij} writes like

$$\left(\begin{array}{ccc|ccc} U_j^2(u_{i1} - 3U_i)u_{i1} & \cdots & U_j^2 u_{i1} u_{ih_i} & 2U_i U_j u_{i1} u_{j1} & \cdots & 2U_i U_j u_{i1} u_{jh_j} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ U_j^2 u_{ih_i} u_{i1} & \cdots & U_j^2 (u_{ih_i} - 3U_i) u_{ih_i} & 2U_i U_j u_{ih_i} u_{j1} & \cdots & 2U_i U_j u_{ih_i} u_{jh_j} \\ \hline 2U_i U_j u_{i1} u_{j1} & \cdots & 2U_i U_j u_{ih_i} u_{j1} & U_i^2 (u_{j1} - 3U_j) u_{j1} & \cdots & U_i^2 u_{j1} u_{jh_j} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 2U_i U_j u_{jh_j} u_{i1} & \cdots & 2U_i U_j u_{ih_i} u_{jh_j} & U_i^2 u_{jh_j} u_{j1} & \cdots & U_i^2 (u_{jh_j} - 3U_j) u_{jh_j} \end{array} \right)$$

(where every term is strictly positive), which has the same signature than

$$\left(\begin{array}{ccc|ccc} 1 - 3U_i/u_{i1} & \cdots & 1 & & & \\ \vdots & \ddots & \vdots & & & 2 \\ 1 & \cdots & 1 - 3U_i/u_{ih_i} & & & \\ \hline & & 2 & 1 - 3U_j/u_{j1} & \cdots & 1 \\ & & & \vdots & \ddots & \vdots \\ & & & 1 & \cdots & 1 - 3U_j/u_{jh_j} \end{array} \right)$$

(we mean that in the two blocks every term is equal to 2). The last matrix can be seen as the sum

$$2 \left(\begin{array}{c|c} -1 & 1 \\ \hline 1 & -1 \end{array} \right) + 3 \left(\begin{array}{ccc|ccc} 1 - 1/\alpha_1 & \cdots & 1 & & & 0 \\ \vdots & \ddots & \vdots & & & \\ 1 & \cdots & 1 - 1/\alpha_{h_i} & & & \\ \hline & & 0 & 1 - 1/\beta_1 & \cdots & 1 \\ & & & \vdots & \ddots & \vdots \\ & & & 1 & \cdots & 1 - 1/\beta_{h_j} \end{array} \right),$$

⁹replacing $\bar{\lambda}_{im} u_{im}$ with u_{im} and $\bar{\Lambda}_i U_i$ with U_i .

where $\alpha_m = u_{im}/U_i$, $\beta_m = u_{jm}/U_j$, in such a way that $\sum \alpha_m = \sum \beta_m = 1$. It is easy to see that the first addend is negative semidefinite, so the last thing we have to prove is that $\sum_m \alpha_m = 1$, $\alpha_m > 0$, implies that

$$D = \begin{pmatrix} 1 - 1/\alpha_1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 1 - 1/\alpha_h \end{pmatrix}$$

is negative semidefinite. Let $\xi = (\xi_1, \dots, \xi_h) \in \mathbb{R}^h$. Then it is easy to prove that

$$\sum_{m=1}^h \frac{\xi_m^2}{\alpha_m^2} \geq h|\xi|^2, \quad \text{thus} \quad D\xi \cdot \xi \leq \left(\sum_{m=1}^h \xi_m \right)^2 - h|\xi|^2,$$

that is trivially non positive for every h and ξ . \square

(3.9) Proposition. *Let $\varepsilon < \varepsilon_2$ in such a way that all the previous results hold. Then, for every $(u_{11}, \dots, u_{kh_k}) \in \tilde{\mathcal{X}}_\varepsilon$ there exists one and only one choice*

$$\bar{\Lambda}_i = \bar{\Lambda}_i(u_{11}, \dots, u_{kh_k})$$

such that

$$J_\beta(\bar{\Lambda}_1(u_{11}, \dots, u_{kh_k})U_1, \dots, \bar{\Lambda}_k(u_{11}, \dots, u_{kh_k})U_k) = M_\beta(u_{11}, \dots, u_{kh_k}).$$

Moreover, each $\bar{\Lambda}_i$ is well defined and of class C^1 on a neighborhood $N(\tilde{\mathcal{X}}_\varepsilon)$ of $\tilde{\mathcal{X}}_\varepsilon$.

Proof. To start with we will show, via a topological degree argument, that $(\bar{\Lambda}_1, \dots, \bar{\Lambda}_k)$ is uniquely defined on $\tilde{\mathcal{X}}_\varepsilon$. Indeed, consider the set

$$D = \left\{ (\Lambda_1, \dots, \Lambda_k) \in \mathbb{R}^h : \sum_i |\Lambda_i|^2 < R^2, \lambda_{im} > 1/2 \right\}.$$

By Lemma 3.4 and Remark 3.7 we know that $\nabla\Phi$ points inward on ∂D . As a consequence $-\nabla\Phi$ is homotopically equivalent to a translation of the identity map, and

$$\deg(\nabla\Phi, 0, D) = (-1)^h.$$

On the other hand, such degree must be equal to the sum of the local degrees of all the critical points of Φ in D : since these points are all non degenerate maxima (by Lemma 3.8), and they have local degree $(-1)^h$, we conclude that there is only one critical point of Φ in D , and it must be the global maximum point. Therefore the maps $\bar{\Lambda}_i(u_{11}, \dots, u_{kh_k})$ are well defined in $\tilde{\mathcal{X}}_\varepsilon$. Moreover, they are implicitly defined by

$$\nabla\Phi(\bar{\Lambda}_1, \dots, \bar{\Lambda}_k) = 0,$$

thus to conclude we can apply the Implicit Function Theorem in a neighborhood of any point of $\tilde{\mathcal{X}}_\varepsilon$: indeed, $\nabla\Phi$ is a C^1 map (both in the λ -variables and in the u -variables); moreover, its differential with respect to the λ -variables is invertible by Lemma 3.8 (it is simply $D^2\Phi$). \square

We observe that, even if $(u_{11}, \dots, u_{kh_k})$ belongs to $\tilde{\mathcal{X}}_\varepsilon$, nevertheless this might not be true for the corresponding $(\bar{\lambda}_{11}u_{11}, \dots, \bar{\lambda}_{kh_k}u_{kh_k})$. At this point we can only state a weaker property for those elements of $\tilde{\mathcal{X}}_\varepsilon$ with the corresponding U_i 's having disjoint supports.

(3.10) Lemma. *Let $\varepsilon < \varepsilon_2$ in such a way that all the previous results hold, and $(v_{11}, \dots, v_{kh_k}) \in \tilde{\mathcal{X}}_\varepsilon$ be such that*

$$V_i \cdot V_j = 0 \quad \text{almost everywhere, for every } i, j.$$

Then there exists $\delta = \delta(\varepsilon)$ (not depending on β) such that $(\bar{\lambda}_{11}v_{11}, \dots, \bar{\lambda}_{kh_k}v_{kh_k}) \in \tilde{\mathcal{X}}_\delta$.¹⁰ Moreover, δ goes to 0 as ε does.

Proof. To start with, we observe that $v_{im} \geq 0$ implies $\bar{\lambda}_{im}v_{im} \geq 0$, and that $M_\beta(u_{11}, \dots, u_{kh_k}) = M_\beta(\bar{\lambda}_{11}u_{11}, \dots, \bar{\lambda}_{kh_k}u_{kh_k})$. As a consequence, if we prove that

$$\sum_{i,m} \|v_{im} - w_{\bar{\sigma}^{-1}(im)}\|^2 < \varepsilon^2 \quad \implies \quad \sum_{i,m} \|\bar{\lambda}_{im}v_{im} - w_{\bar{\sigma}^{-1}(im)}\|^2 < \delta^2,$$

with δ vanishing when ε does, we have finished. By assumption we have that $\beta \int (\bar{\Lambda}_i V_i) v_{im} (\bar{\Lambda}_j V_j)^2 = 0$ for every choice of the indexes, so that the functions $\bar{\lambda}$ are implicitly defined by

$$\left\langle \sum_n \bar{\lambda}_{in} v_{in}, v_{im} \right\rangle - \int_{\mathbb{R}^N} \left(\sum_n \bar{\lambda}_{in} v_{in} \right)^3 v_{im} dx = 0.$$

On the other hand, by the definition of \mathcal{K} , we know that

$$\left\langle \sum_n \bar{\lambda}_{in} w_{\bar{\sigma}^{-1}(in)}, w_{\bar{\sigma}^{-1}(im)} \right\rangle - \int_{\mathbb{R}^N} \left(\sum_n \bar{\lambda}_{in} w_{\bar{\sigma}^{-1}(in)} \right)^3 w_{\bar{\sigma}^{-1}(im)} dx = 0 \iff \bar{\lambda}_{in} = 1 \text{ for every } n.$$

But then our claim directly follows from the Implicit Function Theorem. \square

4 Estimates for ε fixed and β large

From now on we choose $\bar{\varepsilon} > 0$ in such a way that, for every $\varepsilon < \bar{\varepsilon}$, it holds

$$\varepsilon < \varepsilon_2 \quad \text{and} \quad \delta < \varepsilon_2$$

with ε_2 as in Proposition 3.9 and $\delta = \delta(\varepsilon)$ as in Lemma 3.10. As we said, $\bar{\varepsilon}$ do not depend on β . In the following ε and δ are considered fixed as above.

Since in the following we will let β move, we observe that, as we already remarked, the set $\tilde{\mathcal{X}}_\varepsilon = \tilde{\mathcal{X}}_{\varepsilon,\beta}$ also depends on β , since the functions inside satisfy $M_\beta < c_\infty + 1/\beta$.

(4.1) Remark. If $\beta_1 \leq \beta_2$, then for any $(u_{11}, \dots, u_{kh_k}) \in \mathcal{X}_\varepsilon$ and any choice of the Λ_i 's, it holds $J_{\beta_1}(\Lambda_1 U_1, \dots, \Lambda_k U_k) \leq J_{\beta_2}(\Lambda_1 U_1, \dots, \Lambda_k U_k)$. Passing to the supremum we obtain

$$\beta_1 \leq \beta_2 \implies \tilde{\mathcal{X}}_{\varepsilon,\beta_2} \subset \tilde{\mathcal{X}}_{\varepsilon,\beta_1}.$$

In the following we will deal with sequence of h -tuples in $\tilde{\mathcal{X}}_\varepsilon$, with increasing β . For this reason, we start this section with a general result about some convergence property for such sequences.

(4.2) Lemma. *Let the sequence $\beta_s \rightarrow +\infty$, $s \in \mathbb{N}$, and let us consider a sequence of h -tuples*

$$(u_{11}^s, \dots, u_{kh_k}^s) \in \tilde{\mathcal{X}}_{\varepsilon,\beta_s}.$$

Then, up to a subsequence, $u_{im}^s \rightarrow u_{im}^$, strongly in $H_r^1(\mathbb{R}^N)$, for every (i, m) . Moreover*

$$U_i^* \cdot U_j^* \equiv 0 \text{ for } i \neq j, \quad \text{and } (u_{11}^*, \dots, u_{kh_k}^*) \in \tilde{\mathcal{X}}_{\varepsilon,\beta} \text{ for every } \beta.$$

Finally, writing $\bar{\lambda}_{im}^ = \bar{\lambda}_{im}(u_{11}^*, \dots, u_{kh_k}^*)$, we have that $\bar{\lambda}_{im}^s \rightarrow \bar{\lambda}_{im}^*$ and*

$$J_\beta(\Lambda_1^* U_1^*, \dots, \Lambda_k^* U_k^*) = c_\infty \quad \text{for every } \beta.$$

¹⁰Here and in the following $\bar{\lambda}_{im} = \bar{\lambda}_{im}(v_{11}, \dots, v_{kh_k})$, according to Proposition 3.9.

Proof. By assumption we can find a sequence $(w_1^s, \dots, w_l^s) \in \mathcal{K}$ such that $\sum_{i,m} \|u_{im}^s - w_l^s\|^2 < \varepsilon^2$, where $l = \tilde{\sigma}^{-1}(im)$. Since \mathcal{K} is compact, we have that, up to a subsequence, $w_l^s \rightarrow w_l^*$ strongly, and $u_{im}^s \rightharpoonup u_{im}^*$ weakly in $H_r^1(\mathbb{R}^N)$ (since they are bounded, independently on β (see Lemma 3.4), also each $\bar{\lambda}_{im}^\beta$ converges to some number). By the compact immersion of $H_r^1(\mathbb{R}^N)$ in $L^4(\mathbb{R}^N)$, we deduce that $u_{im}^s \rightarrow u_{im}^*$ strongly in $L^4(\mathbb{R}^N)$. We know by Remark 3.5 that

$$\beta_s \sum_{\substack{i,j=1 \\ i \neq j}}^k \int_{\mathbb{R}^N} (\bar{\Lambda}_i^s U_i^s)^2 (\bar{\Lambda}_j^s U_j^s)^2 dx \leq C$$

not depending on β . Using the strong L^4 -convergence and Lemma 3.6 we conclude that $U_i^* \cdot U_j^* \equiv 0$ for $i \neq j$. To prove that $(u_{11}^*, \dots, u_{khk}^*) \in \tilde{\mathcal{X}}_\varepsilon$ we observe that:

- $u_{im}^* \geq 0$ by the strong L^4 -convergence;
- $\sum_{i,m} \|u_{im}^* - w_l^*\|^2 < \varepsilon^2$ by weak lower semicontinuity of $\|\cdot\|$;
- finally, for every choice of the λ_{im} 's, we have that $\liminf \|\Lambda_i U_i^s\| \geq \|\Lambda_i U_i^*\|$, $\lim \int (\Lambda_i U_i^s)^4 = \int (\Lambda_i U_i^*)^4$ and

$$\liminf \beta_s \int_{\mathbb{R}^N} (\bar{\Lambda}_i U_i^s)^2 (\bar{\Lambda}_j U_j^s)^2 dx \geq 0 = \beta \int_{\mathbb{R}^N} (\bar{\Lambda}_i U_i^*)^2 (\bar{\Lambda}_j U_j^*)^2 dx \quad \text{for every } \beta,$$

providing, for every β , $J_\beta(\Lambda_i U_i^*) \leq \liminf J_{\beta_s}(\Lambda_i U_i^s) < c_\infty + 1/\beta_s$, that implies

$$M_\beta(u_{11}^*, \dots, u_{khk}^*) \leq c_\infty < c_\infty + \min(1, 1/\beta).$$

Thus $(u_{11}^*, \dots, u_{khk}^*) \in \tilde{\mathcal{X}}_{\varepsilon, \beta}$ and we can write $\bar{\lambda}_{im}^* = \bar{\lambda}_{im}(u_{11}^*, \dots, u_{khk}^*)$. Now, since $U_i^* \cdot U_j^* \equiv 0$, by Proposition 2.5 we know that

$$J_\beta(\bar{\Lambda}_1^* U_1^*, \dots, \bar{\Lambda}_k^* U_k^*) = \sup_{\lambda_{im} > 0} J^* \left(\sum_{i,m} \lambda_{im} v_{im} \right) \geq c_\infty.$$

On the other hand, for what we said,

$$c_\infty + 1/\beta_s \geq J_{\beta_s}(\bar{\Lambda}_1^s U_1^s, \dots, \bar{\Lambda}_k^s U_k^s) \geq J_{\beta_s}(\bar{\Lambda}_1^* U_1^s, \dots, \bar{\Lambda}_k^* U_k^s) \geq J_\beta(\bar{\Lambda}_1^* U_1^*, \dots, \bar{\Lambda}_k^* U_k^*) + o(1),$$

where the second inequality is strict if and only if $\bar{\lambda}_{im}^\beta \not\rightarrow \bar{\lambda}_{im}^*$, and the third is strict if and only if $u_{im}^\beta \not\rightarrow u_{im}^*$. Comparing the last two equations, we obtain that

$$J(\bar{\Lambda}_1^* U_1^*, \dots, \bar{\Lambda}_k^* U_k^*) = c_\infty, \quad \bar{\lambda}_{im}^\beta \rightarrow \bar{\lambda}_{im}^*, \quad \text{and } u_{im}^\beta \rightarrow u_{im}^* \text{ strongly,}$$

concluding the proof. \square

Now we want to show that, if β is sufficiently large, then the result of Lemma 3.10 holds on the whole $\tilde{\mathcal{X}}_\varepsilon$, without restrictions.

(4.3) Lemma. *There exists β_1 such that if $\beta > \beta_1$ then*

$$(u_{11}, \dots, u_{khk}) \in \tilde{\mathcal{X}}_\varepsilon \implies (\bar{\lambda}_{11} u_{11}, \dots, \bar{\lambda}_{khk} u_{khk}) \in \tilde{\mathcal{X}}_\delta.$$

Proof. As in Lemma 3.10, since $M_\beta(u_{11}, \dots, u_{kh_k}) = M_\beta(\bar{\lambda}_{11}u_{11}, \dots, \bar{\lambda}_{kh_k}u_{kh_k})$, and $u_{im} \geq 0$ implies $\bar{\lambda}_{im}u_{im} \geq 0$, we only have to prove that, when β is sufficiently large,

$$\sum_{i,m} \|u_{im} - w_l\|^2 < \varepsilon^2 \quad \implies \quad \sum_{i,m} \|\bar{\lambda}_{im}u_{im} - w_l\|^2 < \delta^2,$$

where $l = \tilde{\sigma}^{-1}(im)$. By contradiction, let $\beta_s \rightarrow +\infty$ and $(u_{11}^s, \dots, u_{kh_k}^s) \in \tilde{\mathcal{X}}_\varepsilon$ be such that $\sum_{i,m} \|u_{im}^s - w_l^s\|^2 < \varepsilon^2$, and $\sum_{i,m} \|\bar{\lambda}_{im}^s u_{im}^s - w_l^s\|^2 \geq \delta^2$ for any $(w_1, \dots, w_h) \in \mathcal{K}$. Using Lemma 4.2 we have that $u_{im}^s \rightarrow u_{im}^*$, $\bar{\lambda}_{im}^s \rightarrow \bar{\lambda}_{im}^*$, in such a way that

$$(u_{11}^*, \dots, u_{kh_k}^*) \in \tilde{\mathcal{X}}_\varepsilon, \quad U_i^* \cdot U_j^* \equiv 0, \quad \text{and} \quad \sum_{i,m} \|\bar{\lambda}_{im}^* u_{im}^* - w_l\|^2 \geq \delta,$$

for every $(w_1, \dots, w_h) \in \mathcal{K}$. But this is in contradiction with Lemma 3.10. \square

By the previous lemma we have that, for every $(u_{11}, \dots, u_{kh_k}) \in \tilde{\mathcal{X}}_\varepsilon$, the corresponding maximum point $(\bar{\lambda}_{11}u_{11}, \dots, \bar{\lambda}_{kh_k}u_{kh_k})$ belongs to $\tilde{\mathcal{X}}_\delta$ and $\bar{\lambda}_{im}(\bar{\lambda}_{11}u_{11}, \dots, \bar{\lambda}_{kh_k}u_{kh_k}) = 1$ for every (i, m) . Motivated by this fact we define

$$\mathcal{N}_\beta = \left\{ (u_{11}, \dots, u_{kh_k}) \in \tilde{\mathcal{X}}_{\delta, \beta} : \bar{\lambda}_{im}(u_{11}, \dots, u_{kh_k}) = 1 \text{ for every } (i, m) \right\},$$

immediately obtaining that, on \mathcal{N}_β , $M_\beta \equiv J_\beta$ and

$$c_{\varepsilon, \beta} \geq \inf_{\mathcal{N}_\beta} J_\beta(U_1, \dots, U_k). \quad (13)$$

As a matter of fact, if β is sufficiently large, also the opposite inequality holds.

(4.4) Lemma. *There exists $\beta_2 \geq \beta_1$ such that, if $\beta > \beta_2$ then*

$$\mathcal{N}_\beta \subset \tilde{\mathcal{X}}_{\varepsilon/2}.$$

Proof. We argue again by contradiction. Let (up to a subsequence) $\beta_s \rightarrow +\infty$, $(u_{11}^s, \dots, u_{kh_k}^s) \in \mathcal{N}_\beta$ (and hence $\bar{\lambda}_{im}^s = 1$) be such that

$$\sum_{i,m} \|u_{im}^s - w_l\|^2 \geq \frac{\varepsilon^2}{4} \quad \text{for every } W \in \mathcal{K}. \quad (14)$$

Using Lemma 4.2, we have that $u_{im}^s \rightarrow u_{im}^*$ strongly in $H_r^1(\mathbb{R}^N)$ and $\bar{\lambda}_{im}^* = 1$, for every (i, m) . As a consequence, defining $w_l^* = u_{\tilde{\sigma}(l)}^*$, we obtain that $J^*(w_l^*) = \sup_\lambda J^*(\lambda w_l^*)$ and $J(\sum_l w_l^*) = c_\infty$. Therefore $(w_1^*, \dots, w_h^*) \in \mathcal{K}$, and, obviously $\sum_{i,m} \|u_{im}^* - w_l^*\|^2 = 0$. But this, using strong convergence in (14), provides a contradiction. \square

(4.5) Remark. Taking into account (13), and the previous lemma (beside the inclusion $\tilde{\mathcal{X}}_{\varepsilon/2} \subset \tilde{\mathcal{X}}_\varepsilon$) we obtain

$$c_{\varepsilon, \beta} = \inf_{\tilde{\mathcal{X}}_\varepsilon} M_\beta \geq \inf_{\mathcal{N}_\beta} J_\beta(U_1, \dots, U_k) \geq \inf_{\tilde{\mathcal{X}}_{\varepsilon/2}} M_\beta \geq \inf_{\tilde{\mathcal{X}}_\varepsilon} M_\beta = c_{\varepsilon, \beta},$$

obtaining three equivalent characterizations of $c_{\varepsilon, \beta}$.

5 Proof of the main results

In order to prove our main results, we present an useful abstract lemma.

(5.1) Lemma. *Let H be an Hilbert space, d an integer, $I \in C^2(H^d; \mathbb{R})$, $\mathcal{X} \subset H^d$, and $N(\mathcal{X})$ an open neighborhood of \mathcal{X} . Let us assume that:*

1. *d functionals $\bar{\lambda}_i \in C^1(N(\mathcal{X}); \mathbb{R})$, $i = 1, \dots, d$, are uniquely defined, in such a way that $\bar{\lambda}_i > 0$ for every i and*

$$\sup_{\lambda_i > 0} I(\lambda_1 x_1, \dots, \lambda_d x_d) = I(\bar{\lambda}_1(x_1, \dots, x_d)x_1, \dots, \bar{\lambda}_d(x_1, \dots, x_d)x_d);$$

2. *$(\bar{x}_1, \dots, \bar{x}_d) \in \mathcal{X}$ is such that $\bar{\lambda}_i(\bar{x}_1, \dots, \bar{x}_d) = 1$ for every i ,*

$$\inf_{\mathcal{X}} \sup_{\lambda_i > 0} I(\lambda_1 x_1, \dots, \lambda_d x_d) = I(\bar{x}_1, \dots, \bar{x}_d),$$

and the $d \times d$ matrix $H = \left(\partial_{x_i x_j}^2 I(\bar{x}_1, \dots, \bar{x}_d)[\bar{x}_i, \bar{x}_j] \right)_{i,j=1, \dots, d}$ is invertible;

3. *$(y_1, \dots, y_d) \in H^d$ is such that, for some $\bar{t} > 0$ and $0 < \delta < 1$,*

$$(s_1 \bar{x}_1 + ty_1, \dots, s_d \bar{x}_d + ty_d) \in \mathcal{X} \quad \text{as} \quad 0 \leq t \leq \bar{t}, 1 - \delta \leq s_i \leq 1 + \delta.$$

Then

$$\nabla I(\bar{x}_1, \dots, \bar{x}_d) \cdot (y_1, \dots, y_d) \geq 0.$$

Proof. For easier notation we set $\bar{x} = (\bar{x}_1, \dots, \bar{x}_d)$. Since $(s_1 \bar{x}_1 + ty_1, \dots, s_d \bar{x}_d + ty_d) \in \mathcal{X}$ we can substitute it in each $\bar{\lambda}_i$. To start with, we want to apply the Implicit Function Theorem to

$$F(s_1, \dots, s_d, t) = \begin{pmatrix} \bar{\lambda}_1(s_1 \bar{x}_1 + ty_1, \dots, s_d \bar{x}_d + ty_d) \\ \vdots \\ \bar{\lambda}_d(s_1 \bar{x}_1 + ty_1, \dots, s_d \bar{x}_d + ty_d) \end{pmatrix} = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix},$$

in order to write $s_i = s_i(t)$, where s_i is C^1 , for every i . By assumption F is C^1 and $F(1, \dots, 1, 0) = (1, \dots, 1)$, thus we only have to prove that the $d \times d$ jacobian matrix

$$A = \partial_{(s_1, \dots, s_d)} F(1, \dots, 1, 0) = \left(\partial_{x_i} \bar{\lambda}_j(\bar{x})[\bar{x}_i] \right)_{i,j=1, \dots, d} \quad \text{is invertible.}$$

Therefore let us assume, by contradiction, the existence of a vector

$$v = (v_1, \dots, v_d) \in \mathbb{R}^d \setminus \{0\} \text{ such that } Av = 0. \quad (15)$$

Let us now consider the function $\Phi(\lambda_1, \dots, \lambda_d) = I(\lambda_1 x_1, \dots, \lambda_d x_d)$; by definition, the point $(\bar{\lambda}_1(x_1, \dots, x_d), \dots, \bar{\lambda}_d(x_1, \dots, x_d))$ is a free maximum of Φ , and hence $\nabla \Phi(\bar{\lambda}_1, \dots, \bar{\lambda}_d) = (0, \dots, 0)$, that is

$$\partial_{x_i} I(\lambda_1(x_1, \dots, x_d)x_1, \dots, \lambda_d(x_1, \dots, x_d)x_d)[x_i] = 0 \quad \text{for every } i$$

(in particular, $\partial_{x_i} I(\bar{x})[\bar{x}_i] = 0$ for every i). We can differentiate the previous equation with respect to x_j , obtaining, for every $(z_1, \dots, z_d) \in H^d$,

$$\partial_{x_i x_j}^2 I(\lambda_1 x_1, \dots, \lambda_d x_d)[x_i, \lambda_j z_j] + \sum_{n=1}^d \partial_{x_i x_n}^2 I(\lambda_1 x_1, \dots, \lambda_d x_d)[x_i, x_n] \cdot \partial_{x_j} \bar{\lambda}_n[z_j] = 0 \quad \text{for } j \neq i$$

and

$$\begin{aligned} \partial_{x_i x_i}^2 I(\lambda_1 x_1, \dots, \lambda_d x_d)[x_i, \lambda_i z_i] + \sum_{n=1}^d \partial_{x_i x_n}^2 I(\lambda_1 x_1, \dots, \lambda_d x_d)[x_i, x_n] \cdot \partial_{x_i} \bar{\lambda}_n[z_i] + \\ + \partial_{x_i} I(\lambda_1 x_1, \dots, \lambda_d x_d)[z_i] = 0. \end{aligned}$$

We can substitute $x_i = \bar{x}_i$, $z_i = v_i \bar{x}_i$, and $\bar{\lambda}_i = 1$ in the previous equations. Recalling that $\partial_{x_i} I(\bar{x})[\bar{x}_i] = 0$ for every i we obtain, for every i and j (not necessarily different),

$$\sum_{n=1}^d \partial_{x_i x_n}^2 I(\bar{x})[\bar{x}_i, \bar{x}_n] \cdot \partial_{x_j} \bar{\lambda}_n(\bar{x})[\bar{x}_j] \cdot v_j = -v_j \partial_{x_i x_j}^2 I(\bar{x})[\bar{x}_i, \bar{x}_j].$$

Summing up on j , and recalling the definitions of A , v (equation (15)) and H , (second assumption of the lemma) we have

$$HAv = -Hv,$$

providing a contradiction with the invertibility of H .

Hence we obtain the existence of the C^1 -functions $s_i = s_i(t)$ (for t sufficiently small) such that $\lambda_i(s_1 \bar{x}_1 + ty_1, \dots, s_d \bar{x}_d + ty_d) = 1$. Let us consider the function

$$\varphi(t) = I(s_1(t) \bar{x}_1 + ty_1, \dots, s_d(t) \bar{x}_d + ty_d).$$

By construction φ is C^1 and $\varphi(t) \geq 0$ for $t \geq 0$. We obtain that

$$0 \leq \varphi'(0) = \sum_{i=1}^d \partial_{x_i} I(\bar{x})[s'_i(t) \bar{x}_i + y_i] = \nabla I(\bar{x}) \cdot (y_1, \dots, y_d) + \sum_{i=1}^d s'_i(t) \partial_{x_i} I(\bar{x})[\bar{x}_i],$$

and the result follows recalling again that $\partial_{x_i} I(\bar{x})[\bar{x}_i] = 0$ for every i . \square

Now we are finally ready to prove our main results.

Proof of Theorem 2.7. Let $\bar{\varepsilon}$ as above and $\bar{\beta} = \beta_2$, in such a way that, for any fixed $\varepsilon < \bar{\varepsilon}$ and $\beta > \bar{\beta}$, all the previous results hold. By Remark 4.5, for every integer s we have an element $(u_{11}^s, \dots, u_{khk}^s) \in \mathcal{N}_\beta$ such that

$$c_{\varepsilon, \beta} \leq J_\beta(U_1^s, \dots, U_k^s) \leq c_{\varepsilon, \beta} + \frac{1}{s}.$$

We are in a situation very similar to that in Lemma 4.2 (much easier, in fact, since now β is fixed). Following the same scheme, one can easily prove that $u_{im}^s \rightarrow u_{im}^*$ strongly in $H_r^1(\mathbb{R}^N)$, with

$$(u_{11}^*, \dots, u_{khk}^*) \in \mathcal{N}_\beta, \quad J_\beta(U_1^*, \dots, U_k^*) = c_{\varepsilon, \beta}.$$

Moreover, by Lemma 4.4, the minimum point is $\varepsilon/2$ -near an element of \mathcal{K} .

It remains to prove that each U_i^* is strictly positive and that (U_1^*, \dots, U_k^*) solves (1). To do this we will apply Lemma 5.1, letting $H = H_r^1(\mathbb{R}^N)$, $d = h$, $\mathcal{X} = \tilde{\mathcal{X}}_\varepsilon$, $I(u_{11}, \dots, u_{khk}) = J_\beta(U_1, \dots, U_k)$, and $(\bar{x}_1, \dots, \bar{x}_h) = (u_{11}^*, \dots, u_{khk}^*)$. Assumptions 1. and 2. in Lemma 5.1 are satisfied by construction, therefore we have only to choose a variation (y_1, \dots, y_h) and to check assumption 3.:

$${}^*P = (s_{11} u_{11}^* + ty_1, \dots, s_{khk} u_{khk}^* + ty_h) \in \tilde{\mathcal{X}}_\varepsilon \text{ when } t > 0 \text{ is small and each } s_{im} \text{ is near } 1.$$

Under these assumptions on t and s_{im} , it is immediate to see that P is ε -near to the same element of \mathcal{K} to which $(u_{11}^*, \dots, u_{kh_k}^*)$ is $\varepsilon/2$ -near; moreover, by continuity, $M_\beta(P) = J_\beta(\bar{\Lambda}(P)P) < c_{\varepsilon,\beta} + 1/\beta$. Recalling the definition of $\tilde{\mathcal{X}}_\varepsilon$ (Corollary 3.2) we have that assumption 3. is fulfilled whenever each component of P is non negative.

First let us prove that each U_i is strictly positive. Assume not, there exists $x_0 \in \mathbb{R}^N$ with, say, $U_1(x_0) = 0$. Since $U_1 \not\equiv 0$ we can easily construct an open, relatively compact annulus $A \ni x_0$ such that $U_1 \leq 1/2$ on A and $U_1 \not\equiv 0$ on ∂A . For any radial $\varphi \in C_0^\infty(A)$, $\varphi \geq 0$, we choose the variation $y_1 = \varphi$, $y_l = 0$ for $l > 1$. Clearly each component of P is non negative, thus Lemma 5.1 implies

$$\begin{aligned} 0 &\leq \nabla I(u_{11}^*, \dots, u_{kh_k}^*) \cdot (\varphi, 0, \dots, 0) = \partial_{u_{11}} I(u_{11}^*, \dots, u_{kh_k}^*)[\varphi] = \\ &= \int_A \left[\nabla U_1 \cdot \nabla \varphi + U_1 \left(1 - U_1^2 + \beta \sum_{j \neq 1} U_j^2 \right) \varphi \right] dx \\ &= \int_A [\nabla U_1 \cdot \nabla \varphi + a(x)U_1 \varphi] dx \quad \text{for every radial } \varphi \in C_0^\infty(A). \end{aligned}$$

But then, since $a(x) \geq 3/4 > 0$ on A , and $U_1 \not\equiv 0$ on ∂A , the strong maximum principle implies $U_1 > 0$ on A , a contradiction.

Now let us prove that (U_1^*, \dots, U_k^*) solves (1). Again, assume by contradiction that, for instance, U_1 does not satisfy the corresponding equation. Then there exists one radial $\varphi \in C_0^\infty(\mathbb{R}^N)$, not necessarily positive, such that (up to a change of sign)

$$\int_{\mathbb{R}^N} \left(\nabla U_1 \cdot \nabla \varphi + U_1 \varphi - U_1^3 \varphi + \beta U_1 \sum_{j \neq 1} U_j^2 \varphi \right) dx < 0.$$

Moreover we can choose φ with support arbitrarily small. Since U_1 is strictly positive, we can then assume that one of its pulse, say u_{11} , is strictly positive on the support of φ . But then, for t small, also $s_{11}u_{11}^* + t\varphi$ is positive, therefore Lemma 5.1 (with $y_1 = \varphi$, $y_l = 0$ for $l > 1$) implies

$$0 \leq \nabla I(u_{11}^*, \dots, u_{kh_k}^*) \cdot (\varphi, 0, \dots, 0) = \int_{\mathbb{R}^N} \left(\nabla U_1 \cdot \nabla \varphi + U_1 \varphi - U_1^3 \varphi + \beta U_1 \sum_{j \neq 1} U_j^2 \varphi \right) dx,$$

a contradiction. □

Proof of Theorem 2.8. The proof readily follows by proving that

$$\text{for every } 0 < \nu < 1, \text{ if } \beta \text{ is sufficiently large, then } \mathcal{N}_\beta \subset \tilde{\mathcal{X}}_{\nu\varepsilon}.$$

But this can be done following the line of the proof of Lemma 4.4. □

(5.2) Remark. We proved the main result in the simplest case of system (1). Now we suggest how to modify this scheme in order to treat the general case of system (3). The main difference is that the role of the associated limiting equation is now played by the minimization problem

$$\min_{\mathcal{X}^*} \sum_{l=1}^h \frac{\int_{\mathbb{R}^N} |\nabla w_l|^2 + (V_{\sigma(l)}(x) + \lambda_{\sigma(l)}) w_l^2 dx}{\left(\int_{\mathbb{R}^N} \mu_{\sigma(l)} w_l^4 dx \right)^{1/2}},$$

where now the constants μ_i 's and λ_i 's are allowed to take different values, and also the potentials V_i 's, with the only constraint that each Schrödinger operator

$$-\Delta + V_i(x) + \lambda_i$$

must be positive. In such a situation, the above minimization problem is always solvable and we call \mathcal{K} its solution set. With these changes, in dimensions two and three, Theorem 2.7 and all its proof remain the same.

When $\lim_{|x| \rightarrow +\infty} V_i(x) = +\infty$ we can lower the dimension to cover also the dimension $N = 1$. In such a case we must change the definition of the Hilbert space we work in choosing for each i the different norm

$$\|U_i\|^2 = \int_{\mathbb{R}^N} |\nabla U_i|^2 + (V_i(x) + \lambda_i) U_i^2 dx,$$

with the advantage that the embedding in L^4 is now compact also in dimension $N = 1$.

Finally, let us mention that we can also allow bounded radially symmetric domains instead of \mathbb{R}^N , and some non-cubic nonlinearities, provided they are subcritical.

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