HETEROCLINIC CONNECTIONS BETWEEN NONCONSECUTIVE EQUILIBRIA OF A FOURTH ORDER DIFFERENTIAL EQUATION

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ABSTRACT. Assuming that f is a potential having three minima at the same level of energy, we study for the conservative equation

$$u^{iv} - g(u)u'' - \frac{1}{2}g'(u)u'^2 + f'(u) = 0, \tag{1}$$

the existence of a heteroclinic connection between the extremal equilibria. Our method consists in minimizing the functional

$$\int_{-\infty}^{+\infty} [\frac{1}{2}[(u''^2) + g(u)u'^2] + f(u)]\, dx$$

whose Euler-Lagrange equation is given by (1), in a suitable space of functions.

1. INTRODUCTION

In the study of ternary mixtures containing oil, water and amphiphile, a modification of a Ginzburg-Landau model yields for the free energy a functional of the form (see[2])

$$\Phi(u) = \int_{\mathbb{R}^3} [c(\nabla^2 u)^2 + g(u)|\nabla u|^2 + f(u)]\, dx\, dy\, dz$$

where the scalar order parameter u is related to the local difference of concentrations of water and oil. The function $g(u)$ quantifies the amphiphilic properties and the "potential" $f(u)$ is the bulk free energy of the ternary mixture. In some relevant situations g may take negative values to an extent that is balanced by the positivity of c and f.

The admissible density profiles may therefore be identified with critical points of Φ in a suitable function space. In the simplest case where the order parameter depends only on one spatial direction, $u = u(x)$ is defined on the real line and (after scaling) the functional becomes

$$\mathcal{F}(u) = \int_{-\infty}^{+\infty} [\frac{1}{2}[(u''^2) + g(u)u'^2] + f(u)]\, dx. \tag{2}$$

The corresponding Euler-Lagrange equation is

$$u^{iv} - g(u)u'' - \frac{1}{2}g'(u)u'^2 + f'(u) = 0. \tag{3}$$

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When $g \equiv const = \beta$, we recognize here the well known extended Fisher-Kolmogorov equation. If the potential $f(u)$ has two nondegenerate minima, say, at ± 1, with $f(\pm 1) = 0$, one question of great interest is the existence of a heteroclinic that connects these two equilibria. Peletier and Troy (see [8] and related references therein) have extensively dealt with the case $g(u) = \beta > 0$ and the model potential $f(u) = \frac{1}{4}(u^2-1)^2$; they have shown that a heteroclinic connecting ± 1 exists for all values of $\beta > 0$. The cases $\beta^2 < 8$ ("saddle-focus" case) and $\beta^2 \geq 8$ ("saddle-node" case) need different treatments. Kalies and VanderVorst [7] have considered an even potential in the saddle-foci case. In [4] Kalies, Kwaspisz and VanderVorst have classified heteroclinic connections (with $\beta > 0$) between two consecutive saddle-focus equilibria according to their homotopy type. Jan Bouwe van den Berg [1] has proved that if $\beta^2 \geq 8$ the heteroclinic is asymptotically stable. More recently Smets and van den Berg [9] have considered the case $\beta < 0$ for which they prove (in a saddle-foci case), by a version of the mountain pass theorem, that at each equilibria there arise homoclinic solutions.

In this paper, we are interested in the case where f possesses three minima at the same level, since this is the framework where description of the three distinct phases becomes possible (see [2, 5, 6]). We prove that under some compatibility conditions on f and g, besides the heteroclinics that connect the consecutive equilibria, a heteroclinic connecting the extremal minima appears. That this is a meaningful problem, at least in the theory of ternary mixtures, can be checked in the considerations of [2], ch. 3.

Similar results have been previously presented in [3], where the two minima and the three minima case have been addressed separately, the letter under the extra assumption of symmetry for f and g.

We emphasize that our approach does not require the nondegeneracy of the extreme equilibria (see the assumption $(F1)$ below).

A variety of numerical results leading to solutions of equations of type (3), obtained by minimization of the free energy $\mathcal{F}$, can be found in the thesis of H. Leitão [5].

2. Settings and result

We consider a potential $f \in C^1(\mathbb{R})$ such that for some $0 < a < 1/2$ and $\alpha > 0$,

$$(F1) \qquad \frac{f(u)}{(u-1)^2} \leq \alpha, \quad \forall u \in (1-a, 1+a),$$
$$\frac{f(u)}{(u+1)^2} \leq \alpha, \quad \forall u \in (-1-a, -1+a),$$

$$(F2) \qquad f(u) = 0 \text{ if and only if } u = 0 \text{ or } u = \pm 1$$

and

$$(F3) \qquad f(u) \geq 0, \quad \forall u \in \mathbb{R}, \quad \text{and} \quad \liminf_{|u|\to\infty} f(u) > 0.$$

We assume that $g \in C^1(\mathbb{R})$ is such that for some function $\tilde{g} \in C^1(\mathbb{R})$ and some $k < 1$,

$$(G1) \qquad g(u) \geq \tilde{g}(u), \ |\tilde{G}(u)| \leq k\sqrt{8f(u)}, \quad \forall u \in \mathbb{R},$$

where $\tilde{G}(u) := \int_0^u \tilde{g}(s)\,ds$ and

$$\tilde{g}(u) \geq 0, \quad \forall u \in [-1-a, -1+a] \cup [1-a, 1+a]. \tag{$G2$}$$

Notice that assumptions $(F1)$ and $(G2)$ need only be satisfied in neighbourhoods of -1 and 1.

We also introduce the additional condition

$$f \text{ is } C^2 \text{ in a neighbourhood of } 0,\ g(0)^2 < 4f''(0). \tag{$F4$}$$

We emphasize that if $(G1)$ holds with $\tilde{g} = g$, we just need f to be C^2 in a neighbourhood of 0 and $f''(0) \neq 0$ as the inequality $g(0)^2 < 4f''(0)$ then holds. Indeed, by l'Hospital's rule, we obtain

$$\frac{g(0)^2}{4f''(0)} = \lim_{u \to 0} \frac{\tilde{G}^2(u)}{8f(u)} < 1.$$

In order to find a solution u of (3) that satisfies

$$\lim_{x \to \pm\infty} u(x) = \pm 1,$$

we minimize the functional $\mathcal{F}$ in a convenient functional space. As the assumptions on f are quite weak and g can vanish close to ± 1, a function u that satisfies $\mathcal{F}(u) < \infty$ does not necessarily belong to an affine translate of $H^2(\mathbb{R})$ as it is often the case for a heteroclinic. In fact it is sufficient to search a minimizer of $\mathcal{F}$ in the space

$$\mathcal{E} = \{u \in C^1(\mathbb{R}),\ u'' \in L^2(\mathbb{R}),\ u' \in L^\infty(\mathbb{R}) \text{ and } \lim_{x \to \pm\infty} u(x) = \pm 1\}.$$

We obtain in Section 3 an a priori bound in C^1 on functions u that satisfy $\mathcal{F}(u) \leq C$ for some $C > 0$, so that the condition $u' \in L^\infty$ is naturally satisfied by the limit of a minimizing sequence.

We prove in Section 5 the following theorem.

Theorem 1. *Suppose that f, $g \in C^1(\mathbb{R})$ satisfy $(F1)$, $(F2)$, $(F3)$, $(F4)$, $(G1)$ and $(G2)$. Then, there exists a minimizer u of $\mathcal{F}$ in $\mathcal{E}$ which is a solution of (3).*

The key argument in the minimizing process is the following proposition proved in Section 4.

Proposition 2. *Suppose that $f \in C^1(\mathbb{R})$ and $g \in C(\mathbb{R})$ satisfy $(F1)$, $(F2)$, $(F3)$, $(F4)$, $(G1)$ and $(G2)$. Let $C > 0$ be given. Then there exists $T > 0$ such that for any $u \in \mathcal{E}$ with $\mathcal{F}(u) \leq C$, there exists $z \in \mathcal{E}$ that satisfies*

$$\begin{aligned} |z(x) + 1| &\leq a \quad \forall x \leq -T, \\ |z(x) - 1| &\leq a \quad \forall x \geq T, \\ \mathcal{F}(z) &\leq \mathcal{F}(u). \end{aligned}$$

To prove this proposition, we need some sharp estimates on functions $u \in \mathcal{E}$, depending only on an upper bound on $\mathcal{F}$ as well as a good knowledge of the behaviour of the local minimizers close to -1, 0 and 1. We obtain such preliminary results in Section 3.

Most of the difficulties to prove the existence of a minimizer come from the fact that g can vanish and even be negative. The condition $(G1)$ can be interpreted as a "good balance" between f and g. Such a condition is crucial as it implies that u''^2 and $f(u)$ are the dominant terms in $\mathcal{F}$ and ensures therefore the positivity of the functional. Also, it should be pointed out that condition $(G1)$ is consistent *** with some of the considerations in [2].***

Another source of troubles in the minimization process is the middle equilibrium. Indeed, if a function stays on a long time-interval close to 0, the contribution to the functional on this interval can be very small so that it is not obvious that such a behaviour can be avoided as predicted by Proposition 2. To overcome this difficulty, we use the assumption $(F4)$. This condition implies that 0 is a "saddle-focus" equilibrium, and has the consequence that the small solutions of the differential equation (3) oscillate around 0. As far as we know, it is not known whether Theorem 1 still holds without $(F4)$. Numerical experiments seem to be in favor of a positive answer. On the other hand, as proved by van den Berg [1], the bounded solutions of the extended Fisher-Kolmogorov equation

$$u^{iv} - \beta u'' + u^3 - u = 0$$

for $\beta \geq \sqrt{8}$ behave like the bounded solutions of the Fisher equation

$$-u'' + u^3 - u = 0.$$

If the same is true for a fourth order equation with a triple-well potential such as $f(u) = (u^2-1)^2u^2$, this would mean that in general, a solution starting from -1 at $-\infty$ cannot pass throughout the middle equilibrium.

3. Preliminaries

3.1. **A priori bounds.**

The following lemma is useful to prove a priori bounds for functions $u \in \mathcal{E}$ that satisfy $\mathcal{F}(u) < \infty$.

Lemma 3. *Given an interval $[a,b] \subset \mathbb{R}$ and a function $u \in H^2(a,b)$ such that $u(a) = A$, $u(b) = B$, $u'(a) = A_1$, $u'(b) = B_1$ the following inequality holds:*

$$\int_a^b u''^2\,dx \geq \frac{4}{b-a}[(B_1-A_1)^2 + 3(\frac{B-A}{b-a} - A_1)(\frac{B-A}{b-a} - B_1)]$$

and equality holds if and only if u is a 3rd degree polynomial.

As the proof follows from an easy but rather tedious computation, we skip it. The following lemma justifies the minimization procedure. Indeed, the assumption $(G1)$ implies that for any function $u \in \mathcal{E}$, $\mathcal{F}(u) \geq 0$.

Lemma 4. *If f, $g \in C(\mathbb{R})$ satisfy $(G1)$, then there exists a constant $s > 0$ such that for all $u \in \mathcal{E}$*

$$\mathcal{F}(u) \geq s\int_{-\infty}^{+\infty} [\frac{u''^2}{2} + f(u)]\,dx.$$

Proof. Let k be given by the assumption $(G1)$. For $c \in]k,1[$, we compute

$$\begin{aligned}\mathcal{F}(u) &\geq \int_{-\infty}^{+\infty} [\frac{1}{2}(u''^2 + \tilde{g}(u)u'^2) + f(u)]\,dx \\ &\geq \int_{-\infty}^{+\infty} [\frac{1}{2}(1-c^2)u''^2 + \frac{1}{2}(cu'' - \frac{\tilde{G}(u)}{2c})^2 + (f(u) - \frac{\tilde{G}(u)^2}{8c^2})]\,dx\end{aligned}$$

where we have performed integration by parts and used the fact that u' is bounded and $\tilde{G}(u(\pm\infty)) = \tilde{G}(\pm 1) = 0$ to obtain

$$-\int_{-\infty}^{+\infty} \tilde{G}(u)u''\,dx = \int_{-\infty}^{+\infty} \tilde{g}(u)u'^2\,dx.$$

Hence by our assumption

$$\mathcal{F}(u) \geq \int_{-\infty}^{+\infty} [\tfrac{1}{2}(1-c^2)u''^2 + (1-(\tfrac{k}{c})^2)f(u)]\,dx. \qquad \square$$

Next, we observe that the first derivative of any function in $\mathcal{E}$ that satisfies $\mathcal{F}(u) < \infty$, vanishes at $\pm\infty$.

Lemma 5. *Let $u \in \mathcal{E}$ be such that $\mathcal{F}(u) < \infty$. Then*

$$\lim_{|x|\to\infty} u'(x) = 0.$$

Proof. Suppose by contradiction that $\lim_{x\to+\infty} u'(x) = 0$ is false and assume for example that there exists $\varepsilon > 0$ and a sequence $x_n \to +\infty$ with $u'(x_n) \geq \varepsilon$ for all $n \in \mathbb{N}$. Let $\delta > 0$ be such that

$$2\mathcal{F}(u) < \frac{s\varepsilon^3}{8\delta}$$

where s is given by Lemma 4. Then, as $\lim_{x\to+\infty} u(x) = 1$, there exist $R > 0$ such that for all $x \geq R$, $|u(x) - 1| \leq \delta/2$. Let $x_0 > R$ be such that $u'(x_0) = \varepsilon$. We claim that we can find $x_1 > 0$ such that $u'(x_1) = \frac{\varepsilon}{2}$, $u(x_1) \leq u(x_0)+\delta$ and $x_1 - x_0 \leq 2\delta/\varepsilon$. Indeed, if $x > x_0$, we have $|u(x) - u(x_0)| \leq \delta$ and as $\lim_{x\to+\infty} u(x) = 1$, there exists $x > x_0$ such that $u'(x) \leq \varepsilon/2$. We can therefore choose x_1 such that $u'(x) \geq \varepsilon/2$ for all $x \in [x_0, x_1]$. We then have

$$\frac{\varepsilon}{2}(x_1 - x_0) \leq \int_{x_0}^{x_1} u'(s)\,ds = u(x_1) - u(x_0) \leq \delta.$$

Now, letting $m = \frac{u(x_1)-u(x_0)}{x_1-x_0}$, we infer from Lemma 3 that

$$\int_{x_0}^{x_1} u''^2\,dx \geq \frac{2\varepsilon}{\delta}[(\frac{\varepsilon}{2})^2 + 3(m-\varepsilon)(m-\frac{\varepsilon}{2})] \geq \frac{\varepsilon^3}{8\delta}.$$

Hence, we obtain a contradiction since Lemma 4 implies that

$$2\mathcal{F}(u) \geq s\int_{x_0}^{x_1} u''^2\,dx \geq \frac{s\varepsilon^3}{8\delta}.$$

Similar arguments hold in other cases and in particular with respect to $-\infty$. $\square$

Assuming that $u \in \mathcal{E}$ is such that $\mathcal{F}(u)$ is bounded, we obtain in the next lemma an a priori bound on $\|u\|_\infty$ and $\|u'\|_\infty$ which only depend on the upper bound of the energy functional.

Lemma 6. *Let $f \in C(\mathbb{R})$ satisfy $(F2)$, $(F3)$, $g \in C(\mathbb{R})$ satisfy $(G1)$ and let C be a positive constant. Then, there are constants K and N such that, for any function $u \in \mathcal{E}$, $\mathcal{F}(u) \leq C$ implies*

$$\|u\|_\infty \leq K \quad \text{and} \quad \|u'\|_\infty \leq N.$$

Proof. Let s be given by Lemma 4. According to the assumptions on f there exist $K > 0$ and $b > 0$ such that

$$\frac{3K^2 s}{8b^3} > C \quad \text{and} \quad s\,f(u) > \frac{C}{b} \quad \text{for all } |u| \geq \frac{K}{2}.$$

We claim that $\|u\|_\infty \le K$. Otherwise, either the set $\{x : |u(x)| \ge K/2\}$ has measure greater than b and

$$\mathcal{F}(u) \ge s \int_{-\infty}^{+\infty} f(u)\,dx > s\,b\frac{C}{b\,s} = C,$$

a contradiction, or we can pick up an interval (c,d) such that $d-c<b$, $u(c)=|u|_\infty$, $u(d)=\frac{K}{2}$, $u(x)\ge\frac{K}{2}$, for all $x\in(c,d)$. It follows then using Lemma 3 that

$$\begin{aligned}\mathcal{F}(u) &\ge s\int_c^d \frac{u''^2}{2}\,dx\\ &\ge \frac{2s}{d-c}[u'(d)^2 + 3(\frac{u(d)-u(c)}{d-c})(\frac{u(d)-u(c)}{d-c} - u'(d))]\\ &\ge \frac{3s}{2(d-c)}[\frac{u(d)-u(c)}{d-c}]^2\\ &\ge \frac{3K^2 s}{8b^3} > C,\end{aligned}$$

leading again to a contradiction. Hence the first statement is proved.

Next, choosing N such that

$$N > 4K \quad \text{and} \quad N^2 > \frac{8C}{s},$$

we show that u' cannot attain the value N. Assume by contradiction that $u'(x_0) = N$. Then there exists $x_1 \in (x_0, x_0+1)$ such that $u'(x_1) = \frac{N}{2}$. Hence, letting $m = \frac{u(x_1)-u(x_0)}{x_1-x_0}$ it turns out that

$$\mathcal{F}(u) \ge s\int_{x_0}^{x_1} \frac{u''^2}{2}\,dx \ge \frac{2s}{x_1-x_0}[(\frac{N}{2})^2 + 3(m-N)(m-\frac{N}{2})] \ge \frac{N^2 s}{8} > C,$$

which is impossible. We show in a similar way that u' cannot attain the value $-N$. □

3.2. **Behaviour of the local minimizers close to the minima of f.** In this section, we show that solutions of the nonlinear equation that are small in the C^3-norm oscillate around zero. The following lemma, which is already contained in Section 4 of [4] (in a slightly different form), gives such a precise statement.

Lemma 7. *Suppose that f, $g \in C(\mathbb{R})$ satisfy $(F4)$ and $a<b$.*

Then there exist $S_0>0$, $\delta_0>0$, $\tau_0>0$ and $M>0$ such that if $\|y_0\|\le\delta_0$, $\|y_1\|\le\delta_0$ and u minimizes

$$\int_a^b [\frac{1}{2}[(u''^2) + g(u)u'^2] + f(u)]\,dx$$

on the set of functions $v\in H^2[a,b]$ satisfying $(u(a),u'(a)) = y_0$ and $(u(b),\, u'(b)) = y_1$, then

$$\|u\|_{C^3([a,b])} \le M\max(\|y_0\|,\|y_1\|),$$

and u changes sign in any subinterval of length τ_0 in $[a,b]$ for $b-a\ge S_0$.

Proof. The conclusion follows by a slight modification of the proofs of Theorems 4.1 and 4.2 in [4]. □

The next lemma is useful to show that the functions $u\in\mathcal{E}$ that satisfy $\mathcal{F}(u)<\infty$, stay close to -1 at $-\infty$ and close to 1 at $+\infty$.

Lemma 8. *Let* α, $\gamma \in \mathbb{R}$ *and let* u *be a solution of the linear equation*

$$u^{iv} - \gamma u'' + 2\alpha u = 0 \tag{4}$$

satisfying $u(-\infty) = 0$.

Then there exists a positive constant L *such that for any* $b \in \mathbb{R}$

$$\int_{-\infty}^{b} [\frac{1}{2}(u''^2 + \gamma u'^2) + \alpha u^2]\, dx \leq L(u^2(b) + u'^2(b))\,. \tag{5}$$

Proof. Integrating (5) by parts gives

$$\int_{-\infty}^{b} [\frac{1}{2}(u''^2 + \gamma u'^2) + \alpha u^2]\, dx = \frac{1}{2}\left(u''(b)u'(b) - u'''(b)u(b) + \gamma u'(b)u(b)\right),$$

where we have used the fact that u is bounded in the C^3-norm and

$$u(-\infty) = u'(-\infty) = 0.$$

The result follows now by computing $u''(b)$ and $u'''(b)$ and observing that they depend linearly on $u(b)$ and $u'(b)$. □

We can of course write a similar lemma concerning $\int_a^\infty [\frac{1}{2}(u''^2 - \gamma u'^2) + 2\alpha u^2]\, dx$ for the function that satisfies the differential equation (4) and vanishes at $+\infty$.

3.3. **Clipping.** In this section we recall the clipping procedure defined in [4], in a way which is convenient for us.

Lemma 9. *Suppose that* $f \in C(\mathbb{R})$ *is a nonnegative function and* $g \in C(\mathbb{R})$ *satisfies (G1). Let* $a \leq s_1 < s_2 \leq s_3 < s_4 \leq b$ *and let* $u \in H^2(a, b)$ *be such that* $u(x) \subset [u(a), u(b)]$ *for all* $x \in [a, b]$*,* u *is invertible on* $[s_1, s_2]$ *and* $[s_3, s_4]$*,*

$$u(s_1) = u(s_3),\ u(s_2) = u(s_4),\ (u'(s_1) - u'(s_3))(u'(s_2) - u'(s_4)) \leq 0.$$

Then there exist $s_1 \leq a_1 \leq s_2 \leq s_3 \leq b_1 \leq s_4$ *such that the function* $\hat{u} : [a, b - (b_1 - a_1)] \to \mathbb{R}$ *defined by*

$$\hat{u}(x) = \begin{cases} u(x) & \text{if } x \in [a, a_1] \\ u(x + b_1 - a_1) & \text{if } x \in (a_1, b - (b_1 - a_1)) \end{cases}$$

belongs to $H^2(a, b - (b_1 - a_1))$*. Moreover*

$$\int_a^{b-(b_1-a_1)} [\frac{1}{2}((\hat{u}''^2) + g(\hat{u})\hat{u}'^2) + f(\hat{u})]\, dx \leq \int_a^b [\frac{1}{2}((u''^2) + g(u)u'^2) + f(u)]\, dx. \tag{6}$$

Proof. From Lemma 3.1 of [4], we deduce the existence of $a_1 \in (s_1, s_2)$ and $b_1 \in (s_3, s_4)$ such that $u(a_1) = u(b_1),\ u'(a_1) = u'(b_1)$ so that the interval (a_1, b_1) can be discarded to construct the function $\hat{u}$ with the desired regularity. To complete the proof we just need to show (6). We thus compute

$$\begin{aligned}\int_a^{b-(b_1-a_1)} &[\tfrac{1}{2}((\hat{u}''^2) + g(\hat{u})\hat{u}'^2) + f(\hat{u})]\, dx \\ &= \int_a^{a_1} [\tfrac{1}{2}(u''^2 + g(u)u'^2) + f(u)]\, dx + \int_{b_1}^{b} [\tfrac{1}{2}(u''^2 + g(u)u'^2) + f(u)]\, dx.\end{aligned}$$

Using the arguments of Lemma 4 and the fact that $u(a_1) = u(b_1),\ u'(a_1) = u'(b_1)$, we obtain the inequality

$$\int_{a_1}^{b_1} [\frac{1}{2}[u''^2 + g(u)u'^2] + f(u)]\,dx \geq 0$$

so that the result follows. □

4. Proof of Proposition 2

The proof is divided in four main steps. In Steps 1 and 2, we approximate the minimum of $\mathcal{F}$ with functions that stay close to ± 1 at $\pm\infty$. Step 3 estimates the time for a function $u \in \mathcal{E}$ to travel in the (u, u')-plane from a neighbourhood of $(\pm 1, 0)$ to a neighbourhood of $(0, 0)$. Finally, in Step 4, we show that functions in $\mathcal{E}$ that stay close to 0 with small velocity can be replaced, using the clipping procedure, by functions that spend in such a neighbourhood a time which is a-priori bounded.

Proof. Let $C > 0$ be fixed and u be a function in $\mathcal{E}$ such that $\mathcal{F}(u) \leq C$. Define

$\gamma := \sup\{g(u) \mid u \in [-1-a, -1+a] \cup [1-a, 1+a]\}$,
$\eta := \min\{f(u) \mid u \in [-1-a, -1-\frac{a}{2}] \cup [-1+\frac{a}{2}, -1+a] \cup [1-a, 1-\frac{a}{2}] \cup [1+\frac{a}{2}, 1+a]\} > 0$,
$\delta := \max\{|\tilde{G}(u)| \mid |u+1| \leq a,\ |u-1| \leq a\} \geq 0$,
$s > 0$ from Lemma 4,
$N > 0$ from Lemma 6,
$\delta_0 > 0$ and $M > 1$ from Lemma 7,
$L > 0$ from Lemma 8 associated to γ and the value α given by $(F1)$.

Choose next $\varepsilon > 0$ so that

$$\varepsilon < \frac{a}{2}, \quad \delta\varepsilon \leq s\frac{\eta a}{2N}, \quad L\varepsilon^2 < s\frac{\eta a}{8N} \quad \text{and} \quad \varepsilon \leq \frac{\delta_0}{M}.$$

Define now

$$x_1 = \sup\{x \mid |u(x) + 1| \leq \varepsilon \text{ and } |u'(x)| \leq \varepsilon\},$$

and

$$x_4 = \inf\{x \mid |u(x) - 1| \leq \varepsilon \text{ and } |u'(x)| \leq \varepsilon\},$$

Observe that as $u \in \mathcal{E}$ and $\mathcal{F}(u) < \infty$, x_1 and x_4 are real numbers.

Step 1 - Modification of u in $]-\infty, x_1]$. Define

$$\mathcal{F}_1(v) = \int_{-\infty}^{x_1} [\frac{1}{2}(v''^2 + g(v)v'^2) + f(v)]\,dx$$

having as domain

$$\mathcal{E}_{\infty,1} = \{v \in \mathcal{E} \mid v = u \text{ on } [x_1, +\infty[\}.$$

Using integration by parts and arguing as in Lemma 4, we see that for any function $v \in \mathcal{E}_{\infty,1}$

$$\mathcal{F}_1(v) \geq \frac{1}{2}\tilde{G}(v(x_1))v'(x_1) + s\int_{-\infty}^{x_1} [\frac{v''^2}{2} + f(v)]\,dx. \tag{7}$$

We also consider the set

$$\mathcal{D}_{\infty,1} = \{v \in \mathcal{E}_{\infty,1} \mid \forall x \leq x_1, \quad v(x) \in [-1-a, -1+a]\}.$$

Claim – If $v \in \mathcal{E}_{\infty,1} \setminus \mathcal{D}_{\infty,1}$ then $\mathcal{F}_1(v) \geq s\frac{\eta a}{4N}$. Indeed, if there exists $x \leq x_1$ such that $v(x) \notin [-1-a, -1+a]$, either there exist $s_1 \leq s_2 \leq x_1$ so that

$$v(s_1) = -1 + \frac{a}{2}, \quad v(s_2) = -1 + a$$

and

$$\forall x \in [s_1, s_2],\ v(x) \in [-1 + \frac{a}{2}, -1 + a]$$

or there exist $s_3 \leq s_4 \leq x_1$ so that

$$v(s_3) = -1 - a, \quad v(s_4) = -1 - \frac{a}{2}$$

and

$$\forall x \in [s_3, s_4],\ v(x) \in [-1 - a, -1 - \frac{a}{2}].$$

Let us for instance consider the first possibility, the second being similar. We then have

$$N(s_2 - s_1) \geq \int_{s_1}^{s_2} v'(x)\, dx = v(s_2) - v(s_1) = \frac{a}{2}$$

and

$$\int_{-\infty}^{x_1} [\frac{v''^2}{2} + f(v)]\, dx \geq \int_{s_1}^{s_2} f(v)\, dx \geq \frac{\eta a}{2N}.$$

On the other hand, we have

$$|\tilde{G}(v(x_1))v'(x_1)| \leq \delta\varepsilon,$$

so that we deduce from (7)

$$\mathcal{F}_1(v) \geq \frac{1}{2}\tilde{G}(v(x_1))v'(x_1) + s\frac{\eta a}{2N} \geq s\frac{\eta a}{4N}.$$

Claim – There exists $v \in \mathcal{D}_{\infty,1}$ so that $\mathcal{F}_1(v) < s\frac{\eta a}{4N}$. For a function $v \in \mathcal{D}_{\infty,1}$, we have by virtue of $(F1)$

$$\mathcal{F}_1(v) \leq \int_{-\infty}^{x_1} [\frac{1}{2}(v''^2 + \gamma v'^2) + \alpha(v+1)^2]\, dx.$$

Due to convexity, the infimum of

$$\int_{-\infty}^{x_1} [\frac{1}{2}(v''^2 + \gamma v'^2) + \alpha(v+1)^2]\, dx$$

on $\mathcal{E}_{\infty,1}$ is achieved by taking $v(x) = -1 + \hat{v}(x)$, where $\hat{v}$ satisfies the linear equation (4) together with the conditions

$$\hat{v}(x_1) = u(x_1) + 1,\ \hat{v}'(x_1) = u'(x_1) \ \text{ and } \ \hat{v}(-\infty) = 0.$$

We now infer from Lemma 8 that

$$\inf_{\mathcal{D}_{\infty,1}} \mathcal{F}_1 \leq L(\hat{v}^2(x_1) + \hat{v}'^2(x_1)) \leq 2L\varepsilon^2 < s\frac{\eta a}{4N}.$$

Conclusion of Step 1 – There exists a function $v \in \mathcal{D}_{\infty,1}$ such that $\mathcal{F}(v) \leq \mathcal{F}(u)$.

Step 2 - Modification of v in $[x_4, \infty[$. We proceed as in Step 1 to build a function $w \in \mathcal{E}$ such that

$$\forall x \leq x_1,\ |w(x) + 1| \leq a, \quad \forall x \geq x_4,\ |w(x) - 1| \leq a,$$

and $\mathcal{F}(w) \leq \mathcal{F}(u)$.

Step 3 - Estimates on time intervals. Define

$$x_2 = \inf\{x \geq x_1 \mid |w(x)| \leq \varepsilon \text{ and } |w'(x)| \leq \varepsilon\}$$

and

$$x_3 = \sup\{x \leq x_4 \mid |w(x)| \leq \varepsilon \text{ and } |w'(x)| \leq \varepsilon\}.$$

Notice that x_2 and x_3 need not exist. In this step, we estimate $x_2 - x_1$ and $x_4 - x_3$ in case x_2 and x_3 exist and $x_4 - x_1$ if these points do not exist.

Consider the interval $[x_1, x_2]$ and define the sets

$$A = \{x \in [x_1, x_2] \mid w(x) \in \,]-\infty, -1-\varepsilon] \cup [-1+\varepsilon, -\varepsilon] \cup [\varepsilon, 1-\varepsilon] \cup [1+\varepsilon, +\infty[\},$$

and

$$B = \{x \in [x_1, x_2] \mid w(x) \in \,]-1-\varepsilon, -1+\varepsilon[\, \cup \,]-\varepsilon, \varepsilon[\, \cup \,]1-\varepsilon, 1+\varepsilon[\}.$$

It is easy to see that B is the union of intervals I_i on which $|w'(x)| \geq \varepsilon$. Here we assume the intervals I_i are of maximal length. Further except maybe for the last one, each of these intervals is followed by an interval $J_i = [c_i, d_i]$ that we also suppose to be of maximal length and which is so that one of the following conditions holds for all $x \in [c_i, d_i]$:

$$\begin{array}{llll}
(a) & w(x) \geq 1+\varepsilon, & w'(c_i) \geq \varepsilon, & w'(d_i) \leq -\varepsilon, \\
(b) & \varepsilon \leq w(x) \leq 1-\varepsilon, & |w'(c_i)| \geq \varepsilon, & |w'(d_i)| \geq \varepsilon, \\
(c) & -1+\varepsilon \leq w(x) \leq -\varepsilon, & |w'(c_i)| \geq \varepsilon, & |w'(d_i)| \geq \varepsilon, \\
(d) & w(x) \leq -1-\varepsilon, & w'(c_i) \leq -\varepsilon, & w'(d_i) \geq \varepsilon.
\end{array}$$

Claim 1 – $\text{meas}(A) \leq \frac{C}{s\, r_\varepsilon}$, *where*

$$r_\varepsilon = \min\{f(w) \mid w \in \,]-\infty, -1-\varepsilon[\, \cup \,]-1+\varepsilon, -\varepsilon[\, \cup \,]\varepsilon, 1-\varepsilon[\, \cup \,]1+\varepsilon, +\infty[\}.$$

This follows from the inequalities

$$C \geq \mathcal{F}(w) \geq s \int_A f(w(x))\, dx \geq s\, r_\varepsilon \text{meas}(A).$$

Claim 2 – $\text{meas}(I_i) \leq 2$. Indeed, on any interval $\bar{I}_i = [a_i, b_i]$, we have $|w'(x)| \geq \varepsilon$ and

$$2\varepsilon \geq |w(b_i) - w(a_i)| = |\int_{a_i}^{b_i} w'(x)\, dx| \geq \varepsilon(b_i - a_i).$$

Claim 3 – *The number n of intervals J_i is bounded.* Consider first an interval $J_i = [c_i, d_i]$ such that (a) holds. We can write

$$2\varepsilon \leq |w'(d_i) - w'(c_i)| = |\int_{c_i}^{d_i} w''(x)\, dx| \leq \|w''\|_{L^2(c_i, d_i)} (d_i - c_i)^{1/2}$$

and

$$\int_{c_i}^{d_i} [\frac{1}{2}(w'')^2 + f(w)]\, dx \geq \frac{2\varepsilon^2}{d_i - c_i} + r_\varepsilon (d_i - c_i) \geq 2\varepsilon\sqrt{2r_\varepsilon}.$$

Assume now that the interval $J_i = [c_i, d_i]$ is such that (b) holds with

$$w'(c_i) \geq \varepsilon \quad \text{and} \quad w'(d_i) \geq \varepsilon.$$

We know from Lemma 6 that $\|w'\|_\infty \leq N$. Hence, we compute

$$1 - 2\varepsilon = \int_{c_i}^{d_i} w'(x)\, dx \leq N(d_i - c_i)$$

and

$$\int_{c_i}^{d_i} [\frac{1}{2}(w'')^2 + f(w)]\,dx \geq r_\varepsilon(d_i - c_i) \geq r_\varepsilon(1-2\varepsilon)/N.$$

Similar arguments hold if J_i is an interval of the other types. It follows then that

$$C \geq \mathcal{F}(w) \geq s\sum_i \int_{c_i}^{d_i} [\frac{1}{2}(w'')^2 + f(w)]\,dx \geq ns\min\{2\varepsilon\sqrt{2r_\varepsilon}, r_\varepsilon(1-2\varepsilon)/N\}.$$

Conclusion of Step 3 – We deduce from the previous claims that

$$x_2 - x_1 = \text{ meas}(A) + \text{ meas}(B) \leq \frac{C}{s\,r_\varepsilon} + 2(n+1).$$

A similar computation holds for $x_4 - x_3$ and $x_4 - x_1$ if x_2 and x_3 do not exist.

Step 4 - Modification in $[x_2, x_3]$. If x_2 and x_3 do not exist, this step can of course be skipped.

Claim 1 – There exists a function $\bar{w}$ *which minimizes* $\mathcal{F}$ *in*

$$\mathcal{E}_{2,3} = \{v \in \mathcal{E} \mid v = w \text{ on } \mathbb{R} \setminus [x_2, x_3]\}.$$

Indeed, we can replace w by a local minimizer $\bar{w}$ of $\mathcal{F}$ on $\mathcal{E}_{2,3}$. It is well-known that such a minimizer exists and is such that $\mathcal{F}(\bar{w}) \leq \mathcal{F}(w)$. Further, it satisfies the differential equation (3) on $[x_2, x_3]$ together with the boundary conditions

$$\bar{w}(x_2) = w(x_2),\ \bar{w}'(x_2) = w'(x_2),\ \bar{w}(x_3) = w(x_3),\ \bar{w}'(x_3) = w'(x_3).$$

Claim 2 – If $x_3 - x_2 \geq \max(S_0, 8\tau_0)$*, where* S_0 *and* τ_0 *are given from Lemma 7, the function* $\bar{w}$ *can be replaced by another function* $\hat{w} \in \mathcal{E}$ *so that the interval* $[x_1, x_4]$ *is clipped out to an interval of length smaller than* $(x_2 - x_1) + \max(S_0, 8\tau_0) + (x_4 - x_3)$*. Further, we have* $\mathcal{F}(\hat{w}) \leq \mathcal{F}(\bar{w})$*.*

Define

$$x_2' = \max\{x \leq x_2 \mid |\bar{w}(x)| = \delta_0\} \quad \text{and} \quad x_3' = \min\{x \geq x_3 \mid |\bar{w}(x)| = \delta_0\}.$$

It follows from Claim 1 of this step and Lemma 7 that $|w(x)| \leq \delta_0$ for all $x \in [x_2, x_3]$ and therefore also for all $x \in [x_2', x_3']$.

Suppose first that $\bar{w}(x_2') = -\delta_0$ and $\bar{w}(x_3') = \delta_0$. Define

$$\begin{aligned} s_2 &= \min\{x \in [x_2', x_3'] \mid \bar{w}'(x) = 0 \text{ and } \bar{w}(x) \geq 0\}, \\ s_4 &= \max\{x \in [x_2', x_3'] \mid \bar{w}(x) = \bar{w}(s_2)\}, \\ s_3 &= \max\{x \in [s_2, s_4] \mid \bar{w}'(x) = 0\} \end{aligned}$$

and take

$$s_1 = \max\{x \in [x_2', s_2] \mid \bar{w}(x) = \bar{w}(s_3)\}.$$

Observe that Lemma 7 ensures the existence of s_2 and moreover $s_2 \in [x_2', x_2 + 2\tau_0]$. Also, $s_3 \in [x_3 - 2\tau_0, x_3']$. Further, it is clear that $\bar{w}$ is invertible on $[s_3, s_4]$. If $\bar{w}$ is invertible on $[s_1, s_2]$, we can directly apply Lemma 9 to the function $\bar{w}$ on $[x_2', x_3']$. In the contrary, we define

$$\tilde{s}_2 = \min\{x \in [s_1, s_2] \mid \bar{w}'(x) = 0\}$$

and

$$\tilde{s}_4 \in [s_3, s_4]$$

such that $\bar{w}(\tilde{s}_4) = \bar{w}(\tilde{s}_2)$. Now that $\bar{w}$ is invertible on $[s_1, \tilde{s}_2]$ and $[s_3, \tilde{s}_4]$, we are able to apply the clipping procedure. ***In any case we can discard the restriction of $\bar{w}$ to some interval $[a_1, b_1]$ containing $[s_2, s_3]$ and join the two remaining pieces to define a new function $\hat{w} \in \mathcal{E}$ as in Lemma 9. Namely, we define

$$\hat{w}(x) = \begin{cases} w(x) & \text{if } x \in [a, a_1] \\ w(x + b_1 - a_1) & \text{if } x \in (a_1, b - (b_1 - a_1)). \end{cases}$$

Letting $x_4^* = x_4 - (b_1 - a_1)$, we have

$$x_4^* - x_1 \leq (x_4 - x_3) + 4\tau_0 + (x_2 - x_1).$$

If $\bar{w}(x_2') = \delta_0$ and $\bar{w}(x_3') = -\delta_0$, we use the same argument.

Assume now that $\bar{w}(x_2') = -\delta_0$ and $\bar{w}(x_3') = -\delta_0$. Let $q \in [x_2', x_3']$ be such that $\max_{x \in [x_2', x_3']} \bar{w}(x) = \bar{w}(q)$. Using Lemma 7, we know that $\bar{w}(q) > 0$ and we can apply the preceeding argument to each of the intervals $[x_2', q]$ and $[q, x_3']$. Here, denoting by x_4^{**} the point into which x_4 is transformed, we have after clipping, $x_4^{**} - x_1 \leq (x_4 - x_3) + 8\tau_0 + (x_2 - x_1)$.

Conclusion of step 4 – There exist a function $\hat{w} \in \mathcal{E}$ ($\hat{w} = \bar{w}$ if $x_3 - x_2 \leq \max(S_0, 8\tau_0)$) which satisfies

$$|\hat{w}(x) + 1| \leq a \quad \forall x \leq x_1,$$
$$|\hat{w}(x) - 1| \leq a \quad \forall x \geq x_4',$$
$$\mathcal{F}(\hat{w}) \leq \mathcal{F}(\bar{w})$$

for some $x_4' \in \mathbb{R}$ such that

$$x_4' - x_1 \leq \frac{2C}{sr_\varepsilon} + 4(n+1) + \max(S_0, 8\tau_0).$$

Conclusion – Taking

$$T = \frac{C}{sr_\varepsilon} + 2(n+1) + \frac{1}{2}\max(S_0, 8\tau_0)$$

and using a time-translation if necessary, we finally construct a function $z \in \mathcal{E}$ that satisfies Proposition 2. Notice that in case x_2 and x_3 are not defined, we can choose

$$T = \frac{C}{sr_\varepsilon} + 2(n+1) \geq \frac{1}{2}(x_4 - x_1).$$

*** □

5. Existence of a minimizer

We are now able to prove Theorem 1.

Proof. Let $m := \inf_{u \in \mathcal{E}} \mathcal{F}(u)$ and choose a minimizing sequence $(u_n)_n \subset \mathcal{E}$ such that $\mathcal{F}(u_n) \leq m + 1/n$, for any $n \in \mathbb{N}$.

The modified sequence. Let $T > 0$ be given by Proposition 2 for $C = m + 1$. According to Proposition 2, there exist sequences

$$(v_n)_n \subset \mathcal{E}$$

that satisfy

$$|v_n(x) + 1| \leq a \quad \forall x \leq -T$$
$$|v_n(x) - 1| \leq a \quad \forall x \geq T$$

and

$$\mathcal{F}(v_n) \leq \mathcal{F}(u_n) \leq m + \frac{1}{n}.$$

Convergence. Estimates in Lemma 4 and Lemma 6 imply that $(v_n)_n$ has a subsequence (still written $(v_n)_n$ for simplicity) such that for some function v

$$v_n \overset{\mathcal{C}^1_{loc}(\mathbb{R})}{\longrightarrow} v, \quad v_n'' \overset{L^2(\mathbb{R})}{\rightharpoonup} v''.$$

To see that v is a minimizer of $\mathcal{F}$ it suffices to note that, since $g(v_n)$ is bounded below and positive on $]-\infty,-T]\cup[T,+\infty[$, Fatou's lemma is applicable and

$$\int_{-\infty}^{+\infty} g(v(x))v'(x)^2\,dx \le \liminf_{n\to\infty}\int_{-\infty}^{+\infty} g(v_n(x))v_n'(x)^2\,dx.$$

We then conclude that $\mathcal{F}(v) = m$. To complete the proof it remains to show that $v' \in L^\infty(\mathbb{R})$ and

$$\lim_{x\to\pm\infty} v(x) = \pm 1.$$

It is clear that $|v(x)-1| \le a$ for $x \ge T$ and $|v(x)+1| \le a$ for $x \le -T$. As v_n' converges uniformly to v' on every compact subset of $\mathbb{R}$, it follows from Lemma 6 that v' is bounded. Consequently, using the arguments of the proof of Proposition 2 (Step 1), it follows that if either $\limsup_{x\to+\infty} v(x) > 1$ or $\liminf_{x\to+\infty} v(x) < 1$, then $\mathcal{F}(v) = +\infty$. We derive a similar contradiction in case $\limsup_{x\to-\infty} v(x) > -1$ or $\liminf_{x\to-\infty} v(x) < -1$. This completes the proof. □

Remark. Suppose that the hypotheses of Theorem 1 are satisfied and assume moreover that f satisfies for some $0 < b < 1/2$ and $\beta > 0$,

$$(F1') \qquad \begin{aligned} \frac{f(u)}{(u-1)^2} &\ge \beta, \quad \forall u \in (1-b, 1+b),\\ \frac{f(u)}{(u+1)^2} &\ge \beta, \quad \forall u \in (-1-b, -1+b). \end{aligned}$$

Then it is easily seen that a minimizer u of $\mathcal{F}$ in $\mathcal{E}$ satisfies $u+1 \in L^2(\mathbb{R}^-)$ and $u-1 \in L^2(\mathbb{R}^+)$. Now, as we also have $u'' \in L^2(\mathbb{R})$, we infer by interpolation that $u' \in L^2(\mathbb{R})$ and thus the minimizer u actually satisfies $u+1 \in H^2(\mathbb{R}^-)$ and $u-1 \in H^2(\mathbb{R}^+)$. Notice that the assumptions $(F1)$ and $(F1')$ are satisfied if -1 and $+1$ are nondegenerate minima of f as for the model potential $f(u) = (u^2-1)^2u^2$.

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