Advances in Differential Equations Volume 1, Number 2, March 1996, pp. 241 – 264

ON POSITIVE ENTIRE SOLUTIONS TO A CLASS OF EQUATIONS WITH A SINGULAR COEFFICIENT AND CRITICAL EXPONENT

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(Submitted by: Jean-Michel Coron)

Abstract. We prove some results about existence, uniqueness and qualitative behavior of positive solutions to equations of the type

$$-\Delta u = a(x/|x|)\frac{u}{|x|^2} + f(x, u) \qquad \text{in } \mathbf{R}^n \setminus \{0\}\,, \tag{0.1}$$

depending on the behavior of the function a of the angular variable $x/|x|$. Our main results concern the critical nonlinearity $f(s) = s^{(n+2)/(n-2)}$. The proofs are based on variational arguments and the moving plane method.

0. Introduction and statement of the results. In this paper we are concerned with equations of the form

$$-\Delta u = a(x/|x|)\frac{u}{|x|^2} + f(x, u) \qquad \text{in } \mathbf{R}^n \setminus \{0\}\,, \tag{0.1}$$

where $a \in C^1(S^{n-1}; \mathbf{R})$. We shall always assume $n \geq 3$. Equations of this type arise in the study of nonlinear Schrödinger operators when the field presents a (possibly) nonisotropic singularity at the origin (the coefficient a being a function of the angle only). It is known [14, X] that $\alpha = -2$ is the smallest exponent that gives sense to the linear operator $-\Delta - a(x/|x|)|x|^{-\alpha}$ at the origin. As to the case $\alpha > -2$, the Dirichlet problem for equations involving critical nonlinearity and singular coefficients has been studied by Egnell in [8], for bounded domains. On the other hand, in $\mathbf{R}^n \setminus \{0\}$, the Kelvin transform $u(x) \to |x|^{2-n}u(x/|x|^2)$ maps the operator $-\Delta - a(x/|x|)|x|^{-\alpha}$ into the operator $|x|^{-n-2}(-\Delta - a(x/|x|)|x|^{-4+\alpha})$ showing a relation between the cases $\alpha < -2$ and $\alpha > -2$. Moreover, a Pohozaev–type identity implies some nonexistence results when $\alpha \neq 2$ (see Remark 1.5).

Among the nonlinear functions f, the case

$$-\Delta u = a(x/|x|)\frac{u}{|x|^2} + u^{2^*-1} \qquad \text{in } \mathbf{R}^n \setminus \{0\}\,, \tag{0.2}$$

Received for publication June 1995.

*This work was performed while the author was visiting the University of Southern California. She would like to thank USC's Department of Mathematics for the kind hospitality.

AMS Subject Classifications: 35J20, 35B99, 35J10.

where $2^* = 2n/(n-2)$, is of special interest. Indeed this equation is invariant under scaling $u(x) \to R^{(2-n)/2}u(x/R)$ and also with respect to weighted composition with Kelvin's map $u(x) \to |x|^{2-n}u(x/|x|^2)$. Moreover, necessary conditions suggest that the exponent $2^* - 1$ is the more likely to admit solutions having nice behavior near the origin and at infinity (see Theorem 0.1 below). Equations like (0.1) and (0.2) have been widely studied when $a \equiv 0$; see [1–4, 6, 9–13, 15–16]. In particular,

$$-\Delta u = u^{2^*-1} \tag{0.3}$$

is the Euler–Lagrange equation corresponding to the minimization of the Sobolev quotient $\|\nabla u\|_2^2/\|u\|_{2^*}^2$ over the completion $\mathcal{D}^{1,2}(\mathbf{R}^n)$ of C_0^∞ with respect to the norm $\|\nabla\varphi\|_2$. It was shown by Talenti [18] that the best Sobolev constant is attained by a positive, radially symmetric function, besides all the functions obtained by rescaling it. On the other hand, Gidas, Ni and Nirenberg [9, 10] and Caffarelli, Gidas and Spruck [7] proved that positive continuous solutions to (0.3) are radially symmetric about a point; this fact implies uniqueness (up to rescaling and translations of the domain) among the positive solutions that are regular either at zero or at infinity and, in particular, of Talenti's minimizers. In addition, they showed that the same equation but with an exponent $1 < \theta < 2^* - 1$ admits no continuous positive solution, without assuming any decay condition at infinity. The main tools in proving radial symmetry of positive solutions were the moving plane method and a suitable use of Kelvin's inversion map.

The main purpose of this paper is to show how equation (0.2) may share with (0.3) some of the features described above, depending on the behavior of the function a. To begin with, we associate to the linear part in (0.1) the quadratic form

$$Q(u) := \int_{\mathbf{R}^n} |\nabla u|^2 - a(x/|x|)\frac{u^2}{|x|^2} \tag{0.4}$$

and its "first eigenvalue" (see Section 1)

$$\lambda_1(a) = \inf_{u\in\mathcal{D}^{1,2}(\mathbf{R}^n)\setminus\{0\}} \frac{Q(u)}{\int_{\mathbf{R}^n}\frac{u^2}{|x|^2}}.$$

When $a \equiv 0$, Hardy's inequality yields $\lambda_1(0) = (n-2)^2/4$. We shall see that, if $\lambda_1(a) > 0$, then $Q^{1/2}$ defines an equivalent norm in $\mathcal{D}^{1,2}(\mathbf{R}^n)$. We first show how the coercivity of the quadratic form Q affects the existence of solutions to (0.1).

Theorem 0.1. (i) *If* $\lambda_1(a) \le 0$ *and* $f > 0$, *then* (0.1) *has no positive solutions in the space* $\mathcal{D}^{1,2}(\mathbf{R}^n)$. (ii) *There exists no solution* $u \in \mathcal{D}^{1,2}(\mathbf{R}^n) \cap L^{\theta+1}$ *of* (0.1) *when* $f(s) = s^\theta$ *and* $\theta \ne 2^* - 1$.

The above result leads us to examine the existence of solutions to (0.1) in cases when $\lambda_1(a) > 0$ and $f(s) = s^{2^*-1}$. Our first attempt goes through the minimization of a Sobolev type quotient.

Theorem 0.2. *Assume* $\lambda_1(a) > 0$ *and* $\max_{S^{n-1}} a > 0$, *if* $n \geq 4$, *and* $\int_{S^{n-1}} a \geq 0$ *if* $n = 3$. *Then*

$$\inf_{u \in \mathcal{D}^{1,2}(\mathbf{R}^n)\setminus\{0\}} \frac{Q(u)}{\|u\|_{2^*}^2}$$

is achieved. Therefore (0.2) *admits a positive solution in* $\mathcal{D}^{1,2}(\mathbf{R}^n)$.

As usual (compare with the results in [6]) the dimension $n = 3$ shows some peculiarity. It is worthwhile noticing that the above infimum cannot be achieved when $a \leq 0$, except for $a \equiv 0$, though, as shown in Theorem 0.5, this fact does not prevent the existence of positive $\mathcal{D}^{1,2}(\mathbf{R}^n)$–solutions to the equation. Also, we point out that solutions in $\mathcal{D}^{1,2}(\mathbf{R}^n)$ do not exhaust all the possible solutions, as shown by the next result.

Theorem 0.3. *There is a positive solution* $u \notin \mathcal{D}^{1,2}(\mathbf{R}^n)$ *of equation (0.2) which is homogeneous of degree* $(2-n)/2$ *if and only if* $\lambda_1(a) > 0$.

Besides the minimization of Sobolev type quotients, another interesting similarity between (0.2) and (0.3) concerns qualitative properties of positive solutions.

Theorem 0.4. *Assume* $a(x) \equiv A \in [0, (n-2)^2/4)$. (i) *The positive solution of* (0.2) *is unique (up to rescaling) in the class of functions* $L^{2^*}(B_1) \cup L^{2^*}(\mathbf{R}^n \setminus B_1)$. (ii) *When* $f(s) = s^\theta$, $1 < \theta < 2^* - 1$, *there are no positive solutions of* (0.1) *belonging to* $L^{2^*}(B_1)$.

We point out that, when $A > 0$, $W^{1,2}_{loc}$–solutions are not continuous at the origin, and neither are they regular at infinity (i.e., $u(x) = \mathcal{O}(|x|^{2-n})$); the assumption on the L^{2^*}–integrability in a neighborhood either of zero or of infinity plays here the same role as regularity did in [9, 10].

Theorem 0.4 follows from a general result about radial symmetry of positive solutions (Theorem 2.1) and from the study of the ODE associated to (0.2). The proof of Theorem 2.1 is based upon a suitable application of the moving plane method as used in [9, 10, 12]. Due to the already mentioned lack of regularity, the decay estimates, that played a central role in [9, 10], are no longer available. We shall show how the integral estimates can successfully replace the pointwise decay properties. The analysis given in Sections 2, 3 and 4 shows that, through the results on radial symmetry, the problem of positive solutions to (0.2) is equivalent to the same problem for (0.3) (that is with $a \equiv 0$), at least among functions satisfying suitable integrability conditions and provided $a(x) \equiv A \in [0, (n-2)^2/4)$. This is not the case when $A < 0$; indeed the following result holds.

Theorem 0.5. *Assume* $a(x) \equiv A < 0$. *There exists* $A^* < 0$ *such that, when* $A < A^*$, (0.2) *admits at least two positive solutions in* $\mathcal{D}^{1,2}(\mathbf{R}^n)$ *which are distinct modulo rescaling. Moreover, one is radially symmetric while the second is not.*

This type of result does not come unexpected, given the Bahri and Coron existence results [1] about the Dirichlet problem for (0.3) on a class bounded domain having nontrivial topology. Indeed, very roughly speaking, as $A \to -\infty$, the effect of the larger singularity amounts to a richer topology of some sublevel set. We shall show that, though the minimization of the Sobolev quotient in $\mathcal{D}^{1,2}(\mathbf{R}^n)$ fails, some constrained minimization problems are solvable, in spaces of functions enjoying suitable symmetry properties.

The paper is organized as follows: in Section 1 we study the basic properties of the quadratic form Q and we prove Theorems 0.1 and 0.3. In Section 2 we prove a general symmetry result for positive solutions to (0.1) that will be applied in section 3 to nonlinearities of the type $f(s) = s^\theta$. Section 4 contains some analysis of the behavior of positive spherically symmetric solutions to (0.1). Theorem 0.4 easily follows from the results of Sections 2, 3 and 4. Sections 5 and 6 are devoted to the proof of Theorems 0.2 and 0.5 respectively.

Notations. Throughout this paper, $\mathcal{D}^{1,2}(\mathbf{R}^n)$ will denote the completion of $C_0^\infty(\mathbf{R}^n)$ with respect to the norm $\left(\int_{\mathbf{R}^n} |\nabla\varphi|^2\right)^{1/2}$. When $n \geq 3$, this space is embedded into $L^{2^*}(\mathbf{R}^n)$, where $2^* = 2n/(n-2)$. S will denote the best constant in Sobolev inequality:

$$S =: \inf_{u\in\mathcal{D}^{1,2}(\mathbf{R}^n)\setminus\{0\}} \frac{\int_{\mathbf{R}^n} |\nabla u|^2}{\|u\|_{2^*}^2} .$$

1. The quadratic form and the first eigenvalue problem. Let $a \in C^1(S^{n-1}; \mathbf{R})$, $n \geq 3$; to keep simpler notations, in the following we shall identify a with its positively homogeneous extension of degree zero. We associate with a the bilinear and quadratic forms

$$Q(u,v) = \int_{\mathbf{R}^n} \nabla u\cdot\nabla v - a(x)\frac{uv}{|x|^2}, \quad Q(u) := Q(u,u) = \int_{\mathbf{R}^n} |\nabla u|^2 - a(x)\frac{u^2}{|x|^2} . \tag{1.1}$$

Thanks to Hardy's inequality,

$$\frac{(n-2)^2}{4}\int_{\mathbf{R}^n} \frac{u^2}{|x|^2} \leq \int_{\mathbf{R}^n} |\nabla u|^2 , \tag{1.2}$$

the above forms are continuous in $\mathcal{D}^{1,2}(\mathbf{R}^n) \times \mathcal{D}^{1,2}(\mathbf{R}^n)$ and $\mathcal{D}^{1,2}(\mathbf{R}^n)$ respectively. Therefore there exists a unique bounded symmetric operator $L_Q \in \mathcal{L}(\mathcal{D}^{1,2}(\mathbf{R}^n))$ such that $\langle L_Q u, v\rangle_{\mathcal{D}^{1,2}(\mathbf{R}^n)} = Q(u,v)$.

To study the sign of Q, we define the first eigenvalue of Q as

$$\lambda_1(a) = \inf_{u\in\mathcal{D}^{1,2}(\mathbf{R}^n)\setminus\{0\}} \frac{Q(u)}{\int_{\mathbf{R}^n}\frac{u^2}{|x|^2}} . \tag{1.3}$$

The next result motivates this choice.

Lemma 1.1.

$$\lambda_1(a) = \inf_{\varphi\in H^1(S^{n-1})\setminus\{0\}} \frac{\int_{S^{n-1}} |\nabla\varphi|^2 + \left(\frac{(n-2)^2}{4} - a(x)\right)\varphi^2}{\int_{S^{n-1}}\varphi^2} . \tag{1.4}$$

Proof. Let us call ℓ the right hand side of (1.4). We first prove that $\lambda_1(a) \le \ell$. Let $\xi : \mathbf{R}^+ \to [0, 1]$ be a smooth cut–off function such that $\xi(s) = 0$, for $s \in [0, 1/2]$, and $\xi(s) = 1$ for $s \ge 1$; moreover for $\varepsilon \in (0, 1)$ we define the function

$$\xi_\varepsilon(s) = \begin{cases} \xi(s/\varepsilon) & \text{if } s \le 1 \\ \xi(1/\varepsilon s) & \text{if } s \ge 1. \end{cases}$$

In this way $\xi_\varepsilon(s) = \xi_\varepsilon(1/s)$, for every positive s. Now let $w_\varepsilon \in \mathcal{D}^{1,2}(\mathbf{R}^n)$ be defined by

$$w_\varepsilon(x) = |x|^{(2-n)/2}\xi_\varepsilon(|x|)\varphi_1(x/|x|),$$

where $\varphi_1 \in H^1(S^{n-1})$ is one positive eigenfunction associated to ℓ.

We remark that $w_\varepsilon(x) = \varepsilon^{(2-n)/2} w_1(x/\varepsilon)$ if $|x| \le \varepsilon$ while $w_\varepsilon(x) = \varepsilon^{(n-2)/2} w_1(\varepsilon x)$ if $|x| \ge \varepsilon^{-1}$; therefore by scale invariance of the integrals below, for a constant $C > 0$ independent of ε, we have

$$\int_{\{|x|\le\varepsilon\}\cup\{|x|\ge\varepsilon^{-1}\}} |\nabla w_\varepsilon|^2 + \frac{1}{|x|^2} w_\varepsilon^2 = \int_{\mathbf{R}^n} |\nabla w_1|^2 + \frac{1}{|x|^2} w_1^2 \le C\,. \tag{1.5}$$

From the definition of $\lambda_1(a)$ and from estimate (1.5) we find a constant C_1 independent of ε such that

$$\lambda_1(a) \le \frac{\int_{\mathbf{R}^n} |\nabla w_\varepsilon|^2 - a(x)\frac{w_\varepsilon^2}{|x|^2}}{\int_{\mathbf{R}^n} \frac{w_\varepsilon^2}{|x|^2}} \le \frac{C_1 + \int_{\varepsilon\le|x|\le\varepsilon^{-1}} |\nabla w_\varepsilon|^2 - a(x)\frac{w_\varepsilon^2}{|x|^2}}{\int_{\varepsilon\le|x|\le\varepsilon^{-1}} \frac{w_\varepsilon^2}{|x|^2}}$$
$$= \frac{C_1 + 2\log\varepsilon^{-1}\int_{S^{n-1}} |\nabla\varphi_1|^2 + \left(\frac{(n-2)^2}{4} - a(x)\right)\varphi_1^2}{2\log\varepsilon^{-1}\int_{S^{n-1}} \varphi_1^2}\,.$$

As $\varepsilon \to 0$ the last term tends to ℓ; thus $\lambda_1(a) \le \ell$. To prove the reverse inequality, we associate to each $w \in C_0^\infty(\mathbf{R}^n \setminus \{0\})$ its transform $\tilde{w}$, homogeneous of degree $(2-n)/2$, defined

$$\tilde{w}(x) = \left(\int_0^\infty \frac{1}{R^{n-1}} w^2(x/R)dR\right)^{1/2}. \tag{1.6}$$

Standard manipulations of multiple integrals yield

$$\int_{S^{n-1}} \tilde{w}^2 = \int_{\mathbf{R}^n} \frac{w^2}{|x|^2}\,, \qquad \int_{S^{n-1}} a(x)\tilde{w}^2 = \int_{\mathbf{R}^n} a(x)\frac{w^2}{|x|^2}\,.$$

On the other hand, by differentiating in (1.6) and using Hölder inequality we obtain

$$|\nabla\tilde{w}(x)| \le \left(\int_0^\infty \frac{1}{R^{n+1}}|\nabla w(x/R)|^2 dR\right)^{1/2};$$

so that

$$\int_{S^{n-1}} |\nabla\tilde{w}|^2 \le \int_{\mathbf{R}^n} |\nabla w|^2\,.$$

Let now define $\varphi(x) = \tilde{w}(x/|x|)$. Since $\tilde{w}$ is homogeneous of degree $(2-n)/2$ we have

$$\begin{aligned} \ell &\leq \frac{\int_{S^{n-1}} |\nabla\varphi|^2 + \left(\frac{(n-2)^2}{4} - a(x)\right)\varphi^2}{\int_{S^{n-1}} \varphi^2} = \frac{\int_{S^{n-1}} |\nabla\tilde{w}|^2 - a(x)\tilde{w}^2}{\int_{S^{n-1}} \tilde{w}^2} \\ &\leq \frac{\int_{\mathbf{R}^n} |\nabla w|^2 - a(x)\frac{w^2}{|x|^2}}{\int_{\mathbf{R}^n} \frac{w^2}{|x|^2}} . \end{aligned}$$

Therefore, by density of $C_0^\infty(\mathbf{R}^n \setminus \{0\})$ (remember that $n \geq 3$), the above inequality holds for every $w \in \mathcal{D}^{1,2}(\mathbf{R}^n)$ and $\ell \leq \lambda_1(a)$; thus the proof is complete.

Remark 1.2. Since S^{n-1} is a compact manifold, $\ell = \lambda_1(a)$ is achieved by a positive function φ_1 satisfying the equation

$$-\Delta_{S^{n-1}}\varphi + \left(\frac{(n-2)^2}{4} - a(x)\right)\varphi = \lambda_1(a)\varphi,$$

where $\Delta_{S^{n-1}}$ is the Laplace–Beltrami operator on S^{n-1}. Now one easily sees that

$$w_1(x) = |x|^{(2-n)/2}\varphi_1(x/|x|)$$

solves the eigenvalue problem

$$-\Delta w = (a(x) + \lambda_1(a))\frac{w}{|x|^2} , \qquad \text{in } \mathbf{R}^n \setminus \{0\} . \tag{1.7}$$

Obviously, due to its homogeneity, $w_1 \notin \mathcal{D}^{1,2}(\mathbf{R}^n)$. On the other hand, it is easy to prove that the infimum in (1.3) cannot be achieved, and that (1.7) has no solution in $\mathcal{D}^{1,2}(\mathbf{R}^n)$.

Proposition 1.3. *Assume* $\lambda_1(a) > 0$. *Then* $(Q(u))^{1/2}$ *is an equivalent norm in* $\mathcal{D}^{1,2}(\mathbf{R}^n)$.

Proof. Assume by contradiction that, corresponding to every $\varepsilon > 0$ there is $u_\varepsilon \in \mathcal{D}^{1,2}(\mathbf{R}^n)$ such that $Q(u_\varepsilon) < \varepsilon\|u_\varepsilon\|^2_{\mathcal{D}^{1,2}(\mathbf{R}^n)}$. We then deduce that $\lambda_1((1-\varepsilon)^{-1}a) < 0$. From the characterization given in Lemma 1.1, λ_1 is clearly continuous with respect to the $C(S^{n-1})$-norm of the coefficient a. Therefore, as $\varepsilon \to 0$, we find that $\lambda_1(a) \leq 0$, a contradiction.

Proof of Theorem 0.1, part (i). Assume by the contrary that there is a positive solution $u \in \mathcal{D}^{1,2}(\mathbf{R}^n)$ of (0.1). Let $A_{r,R}$ denote the annulus $\{x \in \mathbf{R}^n \ : \ r \leq |x| \leq R\}$. Taking the cross $L^2(A_{r,R})$ products of (0.1) by w_1 and of (1.7) by u, by the divergence theorem we obtain

$$\int_{A_{r,R}} f(u)w_1 - \lambda_1(a)\frac{uw_1}{|x|^2} = \int_{\partial A_{r,R}} w_1\frac{\partial u}{\partial\nu} - u\frac{\partial w_1}{\nu} .$$

Since the left hand term is in any case positive, for $\lambda_1(a) \leq 0$, the proof will be done when we show that we can choose sequences of $r_\nu \to 0$ and $R_\nu \to +\infty$ such that the integrals over the boundaries $\partial A_{r_\nu, R_\nu}$ tend to zero as ν goes to $+\infty$. To see this we estimate, for any positive ρ,

$$\int_{|x|=\rho} u\left|\frac{\partial w_1}{\nu}\right| + w_1\left|\frac{\partial u}{\partial \nu}\right| \leq \|u\|_{L^{2^*}(\rho S^{n-1})} \|\nabla w_1\|_{L^{(2^*)'}(\rho S^{n-1})} + \|w_1\|_{L^2(\rho S^{n-1})} \|\nabla u\|_{L^2(\rho S^{n-1})}.$$

By the homogeneity of w_1, the above expression can be estimated with

$$C_1\Big(\rho \int_{|x|=\rho} u^{2^*}\Big)^{1/2^*} + C_2\Big(\rho \int_{|x|=\rho} |\nabla u|^2\Big)^{1/2},$$

where C_1 and C_2 do not depend on ρ. Since $u \in \mathcal{D}^{1,2}(\mathbf{R}^n)$, the integrals

$$\int_0^{+\infty} d\rho \int_{|x|=\rho} u^{2^*} \quad \text{and} \quad \int_0^{+\infty} d\rho \int_{|x|=\rho} |\nabla u|^2$$

converge. Thus, since ρ^{-1} is not integrable at the origin and at infinity, there are sequences $r_\nu \to 0$ and $R_\nu \to +\infty$ such that, when $\rho_\nu = r_\nu$ and $\rho_\nu = R_\nu$,

$$\rho_\nu \int_{|x|=\rho_\nu} u^{2^*} + \rho_\nu \int_{|x|=\rho_\nu} |\nabla u|^2 \to 0.$$

Proof of Theorem 0.3. It is easy to see that $u(x) = |x|^{(2-n)/2}\varphi(x/|x|)$ solves equation (0.2) if and only if φ is a solution of the equation

$$-\Delta_{S^{n-1}}\varphi + \frac{(n-2)^2}{4}\varphi = a(x)\varphi + \varphi^{2^*-1}. \tag{1.8}$$

To solve (1.8), we consider the minimization problem

$$\inf_{\varphi \in H^1(S^{n-1})\setminus\{0\}} \frac{\int_{S^{n-1}} |\nabla\varphi|^2 + \left(\frac{(n-2)^2}{4} - a(x)\right)\varphi^2}{\left(\int_{S^{n-1}} |\varphi|^{2^*}\right)^{2/2^*}}.$$

Since $\lambda_1(a) > 0$, by Lemma 1.1 the value of the infimum is positive. Moreover, $H^1(S^{n-1})$ is compactly embedded in $L^p(S^{n-1})$, for every $p < 2(n-1)/(n-3)$. Therefore, since $2^* = 2n/(n-2) < 2(n-1)/(n-3)$, the minimization problem (and thereby (1.8)) admits a solution. On the other hand, if $\lambda_1(a) \leq 0$, (1.8) does not admit any positive solution (just take the L^2 product of (1.8) by the positive first eigenfunction).

Remark 1.4. Using the variational theory for even functionals, one can easily prove that (1.8) has infinitely many solutions (see, for instance, Struwe [17, II.5]). This means that equation (0.2) has an infinity of solutions homogeneous of degree $(2-n)/2$ that may change sign.

Proof of Theorem 0.1, part (ii). The proof relies on a Pohozaev type identity [13]; the reader can also find the proof in [17, II.1]. We multiply (0.1) by $\nabla u(x) \cdot x + \frac{n-2}{2}u$ and, integrating by parts over the annulus $A_{r,R} = \{r \leq |x| \leq R\}$, we obtain

$$\frac{2n-(n-2)(\theta+1)}{2(\theta+1)} \int_{A_{r,R}} u^{\theta+1} = \int_{\partial A_{r,R}} \{\frac{n-2}{2} u \frac{\partial u}{\partial \nu} - \frac{1}{2}|\nabla u|^2 x \cdot \nu + \nabla u \cdot x \frac{\partial u}{\partial \nu} + \frac{1}{2} a(x) \frac{u^2}{|x|^2} x \cdot \nu + \frac{1}{\theta+1} u^{\theta+1} x \cdot \nu\} .$$

It is not hard to prove, arguing as in the proof of part (i), that the boundary integrals tend to zero, at least for suitable sequences of r_ν, R_ν. Again, since the left hand side of the above identity does not vanish, we obtain a contradiction.

Remark 1.5. In a similar way one can prove that, when $\alpha \neq 2$ and a has a constant sign, the equation

$$-\Delta u = a(x/|x|)\frac{u}{|x|^\alpha} + u^{2^*-1}$$

has no solution in $\mathcal{D}^{1,2}(\mathbf{R}^n) \cap L^2(\mathbf{R}^n, \frac{dx}{|x|^\alpha})$.

2. The moving planes method. We are concerned with positive solutions to the family of equations

$$-\Delta u = A\frac{u}{|x|^2} + f(|x|, u) , \qquad \text{in } \mathbf{R}^n \setminus \{0\}. \tag{2.1}$$

Our goal is to prove symmetry about the origin of positive solutions to (2.1), under suitable assumptions on A, f and u. Our arguments are based on the moving plane method. This technique has been used by several authors (we quote Serrin [13], Gidas, Ni and Nirenberg [9,10], Li [12], see also references therein) to prove symmetry and monotonicity of positive solutions to various nonlinear elliptic problems, both in bounded and unbounded domains. Roughly speaking, the moving plane method consists of two main steps: first in reflecting the domain about a fixed hyperplane (say $x_1 = \lambda \ll 0$) and proving that the value of the solution at each reflected point is larger than the value at the point itself and secondly in moving the hyperplane to a critical position; finally the solution results to be symmetric with respect to this limit hyperplane. When the domain is unbounded, a major difficulty consists in checking the first step; this was done in [9, 10, 12] by exploiting estimates on the pointwise decay of the positive solutions. Due to the lack of regularity at zero and at infinity, these estimates are no longer available when $A > 0$; instead, we shall use integral decay estimates.

Consider the following assumptions:

$(A1)$ $A \in [0, (n-2)^2/4)$,

$(f1)$ $f : \mathbf{R}^+ \setminus \{0\} \times \mathbf{R}^+ \to \mathbf{R}$ is locally Lipschitz in s and nonincreasing in $|x|$,

$(f2)$ $\exists\ \rho(x) \geq 0$, $\rho \in L^\infty_{loc}(\mathbf{R}^n \setminus \{0\})$ such that $\forall 0 < s < t$, $\frac{f(|x|,s)-f(|x|,t)}{s-t} \leq \rho(x)t^\mu$,

$(u1)$ $\rho u^\mu \in L^{n/2}(\mathbf{R}^n \setminus B_r)$, $\forall r > 0$,

$(u2)$ $u \in W^{1,2}_{loc}(\mathbf{R}^n \setminus \{0\}) \cap L^\infty_{loc}(\mathbf{R}^n \setminus \{0\})$,

$(u3)$ $u \in L^{2^*}(\mathbf{R}^n \setminus B_1)$.

Our main goal is the following result.

Theorem 2.1. *Under the assumptions above, every positive solution of* (2.1) *is radially symmetric about a point.*

Remark 2.2. When there is an actual dependence on $|x|$, or when the solution has a nonremovable singularity at zero, the center of symmetry will be, of course, the origin. Otherwise, the problem is invariant under translations of the domain and symmetry may occur about any point.

We shall use the moving plane method to prove symmetry in each given direction. To this aim, for $\lambda \in \mathbf{R}$ we consider the reflection

$$x = (x_1, \dots, x_\nu) \longrightarrow x^\lambda = (2\lambda - x_1, \dots, x_\nu),$$

where $x \in \Sigma^\lambda = \{x \in \mathbf{R}^n : x_1 < \lambda\}$, and we put

$$u^\lambda(x) = u(x^\lambda) ; \tag{2.2}$$

of course we have that

$$u^\lambda(x) = u(x) , \qquad \text{for } x \in \partial\Sigma^\lambda = \{x \in \mathbf{R}^n \; : \; x_1 = \lambda\}. \tag{2.3}$$

The proof of Theorem 2.1 is based on the following result.

Theorem 2.3. *Let u be a positive solution of* (2.1). *Under the above assumptions we have that either $u^\lambda > u$ on Σ^λ, for every $\lambda < 0$, or there exists $\lambda^* < 0$ such that $u^{\lambda^*} = u$ on Σ^{λ^*}.*

Indeed, an easy consequence is the following result (see [19]).

Corollary 2.4. *If either u has a nonremovable singularity at zero, or the nonlinear $(A/|x|^2)s + f(|x|, s)$ is strictly decreasing in $|x|$, for every s, then u is symmetric about the origin. Otherwise u is symmetric about some point $\bar{x} \in \mathbf{R}^n$.*

Preliminaries. Let us write $v^\lambda = u^\lambda - u$; then v^λ satisfies a linear boundary problem of the type

$$\begin{cases} -\Delta v^\lambda = \big(\frac{A}{|x|^2} + a_\lambda(x)\big)v^\lambda + h^\lambda(y) & \text{in } \Sigma^\lambda \setminus \{0^\lambda\} \\ v^\lambda(x) = 0 & \text{on } \partial\Sigma^\lambda, \end{cases} \tag{2.4}$$

where

$$a_\lambda(x) = \frac{f(|x|, u^\lambda) - f(|x|, u)}{u^\lambda - u},$$

$$h^\lambda(x) = A\big(\frac{1}{|x^\lambda|^2} - \frac{1}{|x|^2}\big)u^\lambda + f(|x^\lambda|, u^\lambda) - f(|x|, u^\lambda) .$$

A key argument in proving positiveness of the v^λ's will be the following result.

Lemma 2.5. *Let w satisfy*

$$\begin{cases} -\Delta w \geq \big(\frac{A}{|x|^2} + a(x)\big)w & \text{in } \Sigma^\lambda \setminus \{x_0\} \\ w(x) = 0 & \text{on } \partial\Sigma^\lambda, \end{cases} \tag{2.5}$$

and assume that $w \in W^{1,2}_{loc}(\Sigma^\lambda \setminus \{x_0\})$ and that $w^- \in L^{2^*}(\Sigma^\lambda)$. Then, denoting $\Omega^- = \{x \in \Sigma^\lambda : w(x) < 0\}$,

$$\text{either } \mu(\Omega^-) = 0, \text{ or } \quad \|a^+\|_{L^{n/2}(\Omega^-)} \geq S\big(1 - \frac{4A}{(n-2)^2}\big)\,. \tag{2.6}$$

Proof. Let $0 < \varepsilon \ll 1$, and let $\eta_\varepsilon : \mathbf{R}^n \to [0, 1]$ be a Lipschitz continuous cut–off function satisfying

$$\begin{aligned} &\eta_\varepsilon(x) = 0 && \text{if either } |x - x_0| < \varepsilon \text{ or } |x| > 2\varepsilon^{-1}, \\ &\eta_\varepsilon(x) = 1 && \text{if } |x - x_0| > 2\varepsilon \text{ and } |x| < \varepsilon^{-1}, \\ &|\nabla\eta_\varepsilon(x)| = \varepsilon^{-1} && \text{if } \varepsilon < |x - x_0| < 2\varepsilon, && (2.7) \\ &|\nabla\eta_\varepsilon(x)| = \varepsilon && \text{if } \varepsilon^{-1} < |x| < 2\varepsilon^{-1}. && (2.8) \end{aligned}$$

Define

$$\varphi = \eta_\varepsilon^2 w^-, \qquad \psi = \eta_\varepsilon w^-\,.$$

Then we have

$$\int_{\Omega^-} |\nabla\psi|^2 = \int_{\Omega^-} \nabla w^- \cdot \nabla\varphi + |\nabla\eta_\varepsilon|^2 (w^-)^2$$

so that, testing (2.5) with φ and using Hölder and Hardy inequalities, we obtain

$$\begin{aligned} \int_{\Omega^-} |\nabla\psi|^2 &\leq \int_{\Omega^-} \big(\frac{A}{|x|^2} + a(x)\big)\psi^2 + |\nabla\eta_\varepsilon|^2 (w^-)^2 \\ &\leq \frac{4A}{(n-2)^2} \int_{\Omega^-} |\nabla\psi|^2 + \|a^+\|_{L^{n/2}(\Omega^-)} \|\psi\|^2_{L^{\frac{2n}{n-2}}(\Omega^-)} + \int_{\Omega^-} |\nabla\eta_\varepsilon|^2 (w^-)^2\,. \end{aligned}$$

Finally, using Sobolev's inequality, we infer

$$\big(1 - \frac{4A}{(n-2)^2}\big) \int_{\Omega^-} |\nabla\psi|^2 \leq S^{-1} \|a^+\|_{L^{n/2}(\Omega^-)} \int_{\Omega^-} |\nabla\psi|^2 + \int_{\Omega^-} |\nabla\eta_\varepsilon|^2 (w^-)^2\,.$$

Assuming that $\mu(\Omega^-) \neq 0$ and dividing the above inequality by $\|\nabla\psi\|_2^2$, we obtain the desired inequality for $\varepsilon \to 0$; indeed, since $(w^-)^{2^*}$ is integrable, we have, by Hölder's inequality and (2.7)–(2.8),

$$\int_{\Omega^-} |\nabla\eta_\varepsilon|^2 (w^-)^2 \leq C \|w^-\|^2_{L^{2^*}(\{\varepsilon \leq |x - x_0| \leq 2\varepsilon\} \cup \{\varepsilon^{-1} \leq |x| \leq 2\varepsilon^{-1}\})} \longrightarrow 0 \qquad \text{as } \varepsilon \to 0.$$

Proof of Theorem 2.3. Looking at (2.4), by (f1) we have

$$h^\lambda(x) \geq 0 \qquad \text{in } \Sigma^\lambda. \tag{2.9}$$

Thus the v^λ's solve problems of the type (2.5). We shall use the alternative (2.6) in order to prove that the v^λ's are nonnegative. To this aim, denoting $\Omega_\lambda^- = \{x \in \Sigma^\lambda : v^\lambda(x) < 0\}$, we remark that, by assumption (f2),

$$a_\lambda{}^+(x) \leq \rho(x)u^\mu(x) \qquad \forall x \in \Omega_\lambda^- \, ; \tag{2.10}$$

hence, by (u1) we have the following a priori estimates on the $L^{n/2}$–norm of the coefficients a_λ^+ on the set where the v^λ's are negative:

$$\|a_\lambda{}^+(x)\|_{L^{n/2}(\Omega_\lambda^-)} \leq \|\rho u^\mu\|_{L^{n/2}(\Omega_\lambda^-)} \leq \|\rho u^\mu\|_{L^{n/2}(\Sigma^\lambda)} \, . \tag{2.11}$$

Step 1. We claim that there is $\bar\lambda < 0$ such that $v^\lambda > 0$ in $\Sigma^\lambda \setminus \{0^\lambda\}$, for every $\lambda < \bar\lambda$. To this aim, we show that Lemma 2.5 is applicable. Indeed, as already pointed out, the v^λ's solve problems of the type (2.5). Moreover, as $\lambda < 0$, the only possible singular point of $v^\lambda = u^\lambda - u$ in Σ^λ is $0^\lambda = (2\lambda, 0, \ldots, 0)$, where (possibly) only u^λ is singular; therefore, by (u2), $v^\lambda \in W^{1,2}_{loc}(\Sigma^\lambda \setminus \{0^\lambda\})$. Finally, from (u3), we have that $(v^\lambda)^- \in L^{2^*}(\Sigma^\lambda)$; indeed $(v^\lambda)^-$ is bounded by u itself and, since λ is negative, u is in $L^{2^*}(\Sigma^\lambda)$.

Thanks to (u1) there exists $\bar\lambda$ such that, for every $\lambda \leq \bar\lambda$,

$$\|\rho u^\mu\|_{L^{n/2}(\Sigma^\lambda)} < S\big(1 - \frac{4A}{(n-2)^2}\big) \, .$$

Putting $\Omega_\lambda^- = \{x \in \Sigma^\lambda \, : \, v^\lambda(x) < 0\}$ we then obtain from (2.11) that, for $\lambda \leq \bar\lambda$,

$$\|a_\lambda{}^+(x)\|_{L^{n/2}(\Omega_\lambda^-)} < S\big(1 - \frac{4A}{(n-2)^2}\big) \, .$$

Therefore the alternative (2.6) of Lemma 2.5 yields $v^\lambda \geq 0$ almost everywhere in Σ^λ, for $\lambda \leq \bar\lambda$. By the strong maximum principle we then deduce that $v^\lambda > 0$ in $\Sigma^\lambda \setminus \{0^\lambda\}$, for $\lambda \leq \bar\lambda$.

Step 2. Let us define $\lambda^* = \sup\{\bar\lambda < 0 : v^\lambda > 0 \text{ in } \Sigma^\lambda \setminus \{0^\lambda\} \, , \forall \, \lambda < \bar\lambda\}$. If either $\lambda^* = 0$ or $v^{\lambda^*} \equiv 0$ in Σ^{λ^*} the proof is done. Arguing by contradiction, let us assume that $\lambda^* < 0$ and $v^{\lambda^*} \not\equiv 0$. Since we clearly have $v^{\lambda^*} \geq 0$, we deduce that $v^{\lambda^*} > 0$ in $\Sigma^{\lambda^*} \setminus \{0^{\lambda^*}\}$, by the strong maximum principle.

From (u1), for every $\varepsilon > 0$, there is $\delta_0 > 0$ such that

$$\|\rho u^\mu\|_{L^{n/2}(\Sigma^\lambda \setminus \Sigma^{\lambda^*})} < \varepsilon \, , \qquad \forall \, \lambda \in [\lambda^*, \lambda^* + \delta_0) \, .$$

On the other hand, since v^{λ^*} is positive in $\Sigma^{\lambda^*} \setminus \{0^{\lambda^*}\}$, by convergence almost everywhere and thereby in measure of the v^λ's to v^{λ^*} in Σ^{λ^*}, we have that $\|\rho u^\mu\|_{L^{n/2}(\Omega_\lambda^- \cap \Sigma^{\lambda^*})}$ converges to zero as λ tends to λ^*. Therefore there exists $0 < \delta_1 \leq \delta_0$ such that

$$\|\rho u^\mu\|_{L^{n/2}(\Omega_\lambda^- \cap \Sigma^{\lambda^*})} < S\big(1 - \frac{4A}{(n-2)^2}\big) - \varepsilon \, , \qquad \forall \, \lambda \in [0, \, \lambda^* + \delta_1).$$

In this way, using again estimate (2.11), we obtain

$$\|a_\lambda{}^+\|_{L^{n/2}(\Omega_\lambda^-)} < S\big(1 - \frac{4A}{(n-2)^2}\big)\,, \qquad \forall\lambda \in [0,, \lambda^* + \delta_1)\,.$$

Then applying once again Lemma 2.5 we obtain $v^\lambda \geq 0$ and therefore $v^\lambda > 0$ in $\Sigma^\lambda \setminus \{0^\lambda\}$, for every $\lambda \in [\lambda^*, \lambda^* + \delta_1)$, in contradiction with the definition of λ^*.

3. Symmetry properties of positive solutions. Consider the equation

$$-\Delta u = A\frac{u}{|x|^2} + u^\theta\,, \qquad u \in W^{1,2}_{loc}(\mathbf{R}^n \setminus \{0\}) \cap L^\infty_{loc}(\mathbf{R}^n \setminus \{0\})\,. \tag{3.1}$$

Theorem 3.1. *Assume $A \in (0, (n-2)^2/4)$ and $\theta > 1$. A positive solution u of (3.1) is radially symmetric about the origin in the following cases.*

(i) $\theta = 2^* - 1$ *and either* $u \in L^{2^*}(B_1)$ *or* $u \in L^{2^*}(\mathbf{R}^n \setminus B_1)$;

(ii) $\theta < 2^* - 1$ *and either* $u \in L^{2^*}(B_1)$ *or* $u \in \big(L^{(\theta-1)n/2}(\mathbf{R}^n \setminus B_1) \cap L^{2^*}(\mathbf{R}^n \setminus B_1)\big)$;

(iii) $\theta > 2^* - 1$ *and* $u \in L^{2^*}(\mathbf{R}^n \setminus B_1) \cap L^{(\theta-1)n/2}(\mathbf{R}^n \setminus B_1)$.

Proof. We first prove the assertion under the integrability conditions at infinity. In all three cases assumptions (A1), (f1), (u2) and (u3) of Theorem 2.1 are clearly satisfied. As to (f2) and (u2), we remark that, by convexity of f,

$$\frac{f(s) - f(t)}{s - t} \leq \theta t^{\theta - 1}\,, \qquad \forall\, 0 < s < t$$

and by assumption, $u^{\theta-1} \in L^{n/2}(\mathbf{R}^n \setminus \{B_1\}) \cap L^{n/2}_{loc}(\mathbf{R}^n \setminus \{0\})$ (note that, when $\theta = 2^* - 1$, $(\theta - 1)n/2 = 2^*$).

To prove the theorem under the L^{2^*} assumption at the origin, we use the conformal equivariance of the Laplacian and we define

$$v(x) = \frac{1}{|x|^{n-2}} u\big(\frac{x}{|x|^2}\big)\,.$$

Then $v \in W^{1,2}_{loc}(\mathbf{R}^n \setminus \{0\})$ and, as $\Delta v(x) = |x|^{-n-2}\Delta u\big(x/|x|^2\big)$, it solves the equation

$$-\Delta v = A\frac{v}{|x|^2} + |x|^{\theta(n-2)-n-2} v^\theta\,.$$

Note that the new nonlinear $f(|x|, s) = |x|^{\theta(n-2)-n-2} s^\theta$ satisfies (f1) only if $n + 2 - \theta(n-2) \geq 0$, that is $\theta \leq 2^* - 1$. Moreover, since $\|v\|_{L^{2^*}(\mathbf{R}^n \setminus B_1)} = \|u\|_{L^{2^*}(B_1)}$ and $\|\,|x|^{\theta(n-2)-n-2} v^{\theta-1}\|_{L^{n/2}(\mathbf{R}^n \setminus B_1)} = \|u^{\theta-1}\|_{L^{n/2}(B_1)} \leq C\|u\|_{L^{2^*}(B_1)}$, we can conclude that all the assumptions of Theorem 2.1 are fulfilled.

Remark 3.2. When $\theta \leq 2^* - 1$ the assumption $u \in L^\infty_{loc}(\mathbf{R}^n \setminus \{0\})$ is actually redundant, for every $W^{1,2}_{loc}(\mathbf{R}^n \setminus \{0\})$ solution is locally bounded. Concerning the regularity of solutions to (3.1) at infinity, one can show that, when $\theta \leq 2^* - 1$, every solution in $L^{(\theta-1)n/2}(\mathbf{R}^n \setminus B_1) \cap L^{\delta}(\mathbf{R}^n \setminus B_1)$, where $\delta = \max(2, (\theta-1)n/2)$, is actually in $\mathcal{D}^{1,2}(\mathbf{R}^n \setminus B_1)$ and therefore in $L^{2^*}(\mathbf{R}^n \setminus B_1)$.

4. Analysis of the radially symmetric case. In this section we are concerned with positive entire radially symmetric solutions of (3.1). Writing $u(x) = \varphi(|x|)$, $|x| = r$ and denoting by " $'$ " the derivation with respect to r, then φ solves the ordinary differential equation

$$\varphi'' + \frac{n-1}{r}\varphi' + \frac{A}{r^2}\varphi + \varphi^{\theta} = 0\,, \qquad r > 0\,. \tag{4.1}$$

Equations of the above form with $A = 0$ have been studied in [11]. We are going to show that the above class of equations is actually equivalent to that with $A = 0$. To this aim we put $\psi(s) = e^{\alpha s}\varphi(e^s)$, and, denoting by " $\dot{}$ " the derivation with respect to s, we find that (4.1) is equivalent to

$$\ddot{\psi} + h\dot{\psi} + k\psi + \psi^{\theta} = 0 \tag{4.2}$$

when $\alpha = 2/(\theta-1)$, $2\alpha + 1 + h = n - 1$ and $\alpha^2 + h\alpha + k = A$. Putting

$$\bar{\theta} = \frac{\theta(n-2) - n - 2}{2(\theta-1)}\,, \tag{4.3}$$

the above relations become

$$\alpha = \frac{n-2}{2} - \bar{\theta}, \quad h = 2\bar{\theta}, \quad k = \bar{\theta}^2 + A - \frac{(n-2)^2}{4}\,.$$

Note that, as $\theta > 1$, the sign of $\bar{\theta}$ (and of h) is the same as that of $\theta - 2^* + 1$. Thus equation (4.1) is conservative if and only if $\theta = 2^* - 1$, is accretive when $\theta < 2^* - 1$ and it is dissipative when $\theta > 2^* - 1$.

Case $\theta = 2^* - 1$: Since $\bar{\theta} = 0$, (4.2) becomes

$$\ddot{\psi} - \Big(\frac{(n-2)^2}{4} - A\Big)\psi + \psi^{2^*-1} = 0\,. \tag{4.4}$$

Looking at the phase–plane, positive orbits of (4.4) are those at energy level $E(\psi) = (1/2)(\dot{\psi})^2 - (1/2)\big((n-2)^2/4 - A\big)\psi^2 + (1/2^*)\psi^{2^*}$ less than or equal to zero. For $E = 0$ we have one homoclinics whereas negative value of E correspond to periodic solutions. The minimum value of the energy is achieved by the stationary point $\psi = \big((n-2)^2/4 - A\big)^{1/(2^*-2)}$. We point out that, as long as $A < (n-2)^2/4$, equation (4.4) is equivalent to

$$\ddot{\xi} - \frac{(n-2)^2}{4}\xi + \xi^{2^*-1} = 0\,, \tag{4.5}$$

by means of the substitution $\psi(s) = \eta^{(n-2)/2}\xi(\eta s)$, where $\eta = \big(1 - 4A/(n-2)^2\big)^{1/2}$. As the function $\xi(s) = \big(n(n-2)\big)^{(n-2)/4} e^{(n-2)s/2}(1 + e^{2s})^{(2-n)/2}$ generates the unique homoclinic orbits of (4.5), the expression of the unique (up to shifts) homoclinic solution to (4.4) is easily obtained.

Recalling our initial substitution $\varphi(r) = r^{(2-n)/2}\psi\big(\log(r)\big)$, we can classify all the positive entire solutions of (4.1), when $\theta = 2^* - 1$:

- one solution homogeneous of degree $(2-n)/2$, corresponding to the nonzero constant solution of (4.4);
- a two parameter family of solutions behaving like $\mathcal{O}(|x|^{(2-n)/2})$ near the origin and at infinity (which are not homogeneous), corresponding to the periodic solutions of (4.4);
- the solution

$$u_A(x) = \frac{\big(n(n-2)\eta_A^2\big)^{(n-2)/4}}{\big(|x|^{1-\eta_A}(1+|x|^{2\eta_A})\big)^{(n-2)/2}}, \tag{4.6}$$

  where

$$\eta_A = \Big(1 - \frac{4A}{(n-2)^2}\Big)^{1/2},$$

  together with all its rescaled $u_R(x) = R^{(2-n)/2}u(x/R)$.

Note that this last family describes all the regular solutions (either at the origin or at infinity; i.e., belonging to $L^{2^*}(B_1) \cap L^{2^*}(\mathbf{R}^n \setminus B_1)$).

Case $\theta < 2^* - 1$: In this case $\bar{\theta} < 0$; thus $\alpha > (n-2)/2$ and $h < 0$. Following [11], when $k \leq 0$ there are no positive solutions. When $k > 0$, every positive orbit must connect the two stationary points $O_1 = (0,0)$ and $O_2 = ((-k)^{1/(\theta-1)}, 0)$. Since the system is accretive ($h < 0$), we obtain that every positive solution satisfies $\lim_{s\to-\infty}\psi(s) = (-k)^{1/(\theta-1)}$ and $\lim_{s\to+\infty}\psi(s) = 0$, as $(-k)^{1/(\theta-1)}$ minimizes the energy. Going back to (4.1) and looking at the corresponding $\varphi(r) = r^{-\alpha}\psi(\log(r))$, we obtain that every positive entire solution of (4.1) behaves like $r^{-\alpha}$ at the origin. Since $\theta < 2^* - 1$ implies that $\alpha > (n-2)/2$, such a function cannot be $L^{2^*}(B_1)$.

5. The minimization problem. We are concerned with the minimization problem

$$S(a) = \inf_{u\in\mathcal{D}^{1,2}(\mathbf{R}^n)\setminus\{0\}} \frac{Q(u)}{\|u\|_{2^*}^2}. \tag{5.1}$$

Note that the best constant in the usual Sobolev inequality is $S = S(0)$. As $\lambda_1(a) > 0$, the quadratic form defines equivalent norm and inner product in $\mathcal{D}^{1,2}(\mathbf{R}^n)$; therefore $S(a) > 0$.

Proposition 5.1. *Let $(u_\nu)_\nu$ be a minimizing sequence, weakly converging to $u_0 \not\equiv 0$. Then u_0 is a minimum and the convergence holds in the strong $\mathcal{D}^{1,2}(\mathbf{R}^n)$–topology.*

Proof. The argument resembles that of [6, Lemma 1.2]. We first remark that the weak convergence implies that $Q(u_\nu) = Q(u_0) + Q(u_\nu - u_0) + o(1)$; indeed, keeping the notations of section 1 we have $Q(u_\nu) = Q(u_0 + u_\nu - u_0) = Q(u_0) + Q(u_\nu - u_0) + 2Q(u_0, u_\nu - u_0)$ and clearly $Q(u_0, u_\nu - u_0) = < L_Q u_0, u_\nu - u_0 >_{\mathcal{D}^{1,2}(\mathbf{R}^n)} \to 0$.