

Action minimizing orbits in the n -body problem with simple choreography constraint

V. Barutello - S. Terracini *

Dipartimento di Matematica e Applicazioni

Università di Milano Bicocca

via Bicocca degli Arcimboldi, 8 - Milano

e-mail: vivina@matapp.unimib.it

suster@matapp.unimib.it

Abstract

In 1999 Chenciner and Montgomery found a remarkably simple choreographic motion for the planar 3-body problem (see [11]). In this solution 3 equal masses travel on a eight shaped planar curve; this orbit is obtained minimizing the action integral on the set of simple planar choreographies with some special symmetry constraints. In this work our aim is to study the problem of n masses moving in \mathbb{R}^d under an attractive force generated by a potential of the kind $1/r^\alpha$, $\alpha > 0$, with the only constraint to be a simple choreography: if $q_1(t), \dots, q_n(t)$ are the n orbits then we impose the existence of $x \in H_{2\pi}^1(\mathbb{R}, \mathbb{R}^d)$ such that

$$q_i(t) = x(t + (i-1)\tau), \quad i = 1, \dots, n, \quad t \in \mathbb{R},$$

where $\tau = 2\pi/n$. In this setting, we first prove that for every $d, n \in \mathbb{N}$ and $\alpha > 0$, the lagrangian action attains its absolute minimum on the planar circle. Next we deal with the problem in a rotating frame and we show a richer phenomenology: indeed while for some values of the angular velocity minimizers are still circles, for others the minima of the action are not anymore rigid motions.

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1 Introduction

In the recent years, several new periodic solutions to the n -body problem have been found by minimizing the action functional in space of loops having some

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special symmetries (see [3, 4, 5, 6, 7, 8, 10, 11, 12, 17, 19, 23, 24]). The present work is motivated by the results in [16] where the authors prove the existence of collisionless minimizers in very general classes of symmetric loops: a natural question is whether these minimizers are rigid motions corresponding to central configurations.

To be more precise we are interested in the 2π -periodic solutions to the dynamical systems of n bodies, q_1, \dots, q_n with masses m_1, \dots, m_n , interacting according to the Newton's law

$$-m_i \ddot{q}_i = \sum_{\substack{i, j=1 \\ i \neq j}}^n \nabla V_{ij}(q_i - q_j), \quad i = 1, \dots, n \quad (1)$$

where

$$V_{ij}(q_i - q_j) = -\frac{1}{|q_i - q_j|^\alpha}, \quad \alpha \in \mathbb{R}_+^* \quad (2)$$

is the potential that generates an attractive force between each pair of bodies. According to [2, 10] a *simple choreographies* is a motion in which the bodies lie on the same curve and exchange their mutual positions after a fixed time. If $q_1(t), \dots, q_n(t)$ are the n orbits then we impose the existence of $x \in H_{2\pi}^1(\mathbb{R}, \mathbb{R}^d)$ such that

$$q_i(t) = x(t + (i-1)\tau), \quad i = 1, \dots, n, \quad t \in \mathbb{R}, \quad (3)$$

where $\tau = 2\pi/n$. Moreover, we assume that our system consists of particles of equal masses. Without loss of generality, we may assume that each $m_i = 1$.

Solving (1) under the constraint (3) is equivalent to seek the critical points of the functional

$$\mathcal{A}(x) = \frac{1}{2} \sum_{h=0}^{n-1} \int_0^\tau |\dot{x}(t+h\tau)|^2 dt + \frac{1}{2} \sum_{\substack{h, l=0 \\ h \neq l}}^{n-1} \int_0^\tau \frac{dt}{|x(t+l\tau) - x(t+h\tau)|^\alpha} \quad (4)$$

on the set

$$\Lambda = \{x \in H_{2\pi}^1(\mathbb{R}, \mathbb{R}^d) : x(t) \neq x(t+h\tau), \forall t \in \mathbb{R}, h = 1, \dots, n-1\}.$$

The choreography constraint implies that the quantities $\int_0^\tau \frac{dt}{|x(t+l\tau) - x(t+h\tau)|^\alpha}$ only depend on $l-h$, since there are only $n-1$ distinct average on $[0, 2\pi]$ of distances between two bodies; we can then rewrite the action in (4) in the easier way

$$\mathcal{A}(x) = \frac{1}{2} \int_0^{2\pi} |\dot{x}(t)|^2 dt + \frac{1}{2} \sum_{h=1}^{n-1} \int_0^{2\pi} \frac{dt}{|x(t) - x(t+h\tau)|^\alpha}. \quad (5)$$

Recall that a *central configuration* is a configuration in which the acceleration vector of every particles is $\ddot{q}_i = -kq_i$, $k > 0$, and a *relative equilibrium motion* is a rigid and uniform rotation of a central configuration around the center of mass; our main result is the following

Theorem 1. *For every $\alpha \in \mathbb{R}_+^*$ and $d \geq 2$, the absolute minimum of \mathcal{A} on Λ is attained on a relative equilibrium motion associated to the regular n -gon.*

In particular Theorem 1 implies that, among simple choreographies:

- (i) the minimum is a planar motion;
- (ii) the minimum is a motion of relative equilibrium;
- (iii) the minimum has minimal period 2π .

In other words, we prove that the absolute minimum of the action integral among 2π -periodic loops in H^1 is the relative equilibrium solution whose central configuration is an absolute minimum of the potential among all the configurations that generate a simple choreography. This result agrees with the one of Chenciner and Desolneux in [9] in the sense that, among choreographies, the minimum of the action is the relative equilibrium motion correspondent to the minimum among central configurations; moreover it agrees with the work of Albouy and Chenciner in [2] since the minimum is a planar motion.

In literature we know few works where explicit minima of the action are determined on special classes of loops. From [9] we can understand the following assertion

If the minimal central configuration is planar and it is a regular polygon, then the correspondent motion of relative equilibrium (A) (i.e. the circle) is the absolute minimum of the action under the choreography constraint.

It's easy to prove that (A) is fulfilled for 3-body in any dimension. If $d \geq 3$ the minimal central configuration is never planar as soon as $n \geq 4$, therefore (A) never holds (see Correction to [9], [18], [20]). Indeed, F. Pacella in [20], proves that the planar central configurations of 4 equal masses are saddle points for the potential energy and R. Moeckel, in [18], shows that for every planar configuration of n different masses, there exists a variation on the direction perpendicular to the plane of the configuration that makes the potential decrease.

Concerning the flat central configurations we know from A. Albouy, in [1], that the square is the minimal central configuration for 4 bodies and again (A) holds true. On the hands, we recall that Palmore in 1976, in [21], claimed that the n -gon is the minimum among flat central configurations, but some years later Slaminka and Woerner in [22] show the n -gon is not the minimum for $n \geq 6$, n even. This result shows a strong obstruction to the fulfillment of (A) also in the planar case. Numerically, D. Ferrario, shows in [15] that the square and the pentagon are the absolute minima for 4 and 5 bodies respectively, while if $n \geq 6$ the n -gon is not anymore the absolute minimizer; but if we take into account just central configurations that generates a simple choreography, we have in [15] a confirmation of our result.

The above discussion shows that Theorem 1 relies more on the choreographical assumption than on the properties of the central configurations. In order

to obtain a better understanding of the structure underlying Theorem 1, we proceed with the analysis of the same problem in a rotating frame with angular velocity $\omega > 0$. In this case the lagrangian action is

$$\mathcal{A}(y) = \frac{1}{2} \int_0^{2\pi} |\dot{y}(t) + i\omega y(t)|^2 dt + \frac{1}{2} \sum_{h=1}^{n-1} \int_0^{2\pi} \frac{dt}{|y(t) - y(t+h\tau)|^\alpha},$$

where $y \in \Lambda$ is the loop in the rotating system on which the bodies move with the choreography constraint $y_{i+1}(t) = y_i(t + \tau)$. In order to deal with coercive functionals we shall choose $\omega \notin \mathbb{N}$. Our results show a strong dependence on the value of the angular velocity ω . Indeed consider the function $h : \mathbb{R}_+^* \rightarrow \mathbb{N}$, $h(\omega) = \min_{n \in \mathbb{N}} \frac{(\omega - n)^2}{n^2}$ and let $\omega^* := \max\{\omega : h(\omega) = 1\}$. We will prove that for $\omega \in (0, \omega^*) \setminus \{1\}$ Theorem 1 still holds true. In a similar manner, we shall show that for ω in appropriate ϵ -neighborhood of integers k coprime with n , $\omega \notin \mathbb{N}$, the minimum of the action is still attained on the rigid equilibrium motions associated to the n -gon, but the minimal period of the motion of each body is now $2\pi/k$ (see Theorem 2).

The assertion of Theorem 1 can not be valid for every value of ω : in fact if we consider values of ω close to an integer k such that $\text{g.c.d.}(k, n) = \tilde{k} > 1$, $\tilde{k} \neq n$, the minima of the action are more complex orbits that are not anymore rigid motions, but 2π -periodic curves with winding number k about the center of mass; furthermore this motions show a clustering phenomenon (see Theorems 3 and 4). Our rigorous results cover a range of ω close to the set of integers; for this reason we investigate numerically the behavior of the minimizers for values of ω close to an half integer. The numerical experiments show that the minimizers are not anymore rigid motions, moreover they are not planar provided the number of bodies is large enough ($n \geq 12$ fits) while they seem to be planar circles when $n \leq 9$. As a final remark non planar motions appear as mountain pass points for low number of bodies ($n = 3$).

The paper is structured as follows: the first four sections are devoted to the proof of Theorem 1: in Section 2 we recall its proof in the Keplerian case, while in Sections 3 and 4 there are all the elements to make the proof of Section 5. In Section 6 we study the minima of the action functional when the bodies lie in a rotating system with angular velocity $\omega > 0$. In the last section we show examples for Theorems 1, 2, 3, 4 and some non-planar saddle points we have found numerically.

2 Preliminaries

In this section we recall, for the reader's convenience, two classical inequalities and we show the equivalent result of Theorem 1 in the Keplerian case. This result follows from the application of Jensen Inequality as first remarked in [13] and [14].

Let $q \in H_{2\pi}^1(\mathbb{R}, \mathbb{R})$ such that $[q] = \frac{1}{2\pi} \int_0^{2\pi} q(t) dt = 0$, then

$$\int_0^{2\pi} |\dot{q}(t)|^2 dt \geq \int_0^{2\pi} |q(t)|^2 dt. \quad (6)$$

The inequality (6) is known as the *Poincaré Inequality* and to prove it, consider the Fourier representation of q , $q(t) = \sum_{n \in \mathbb{Z}^*} c_n e^{int}$, $c_{-n} = \bar{c}_n$; then $\dot{q}(t) = \sum_{n \in \mathbb{Z}^*} in c_n e^{int}$ and

$$\int_0^{2\pi} |\dot{q}(t)|^2 dt = 2\pi \sum_{n \in \mathbb{Z}^*} n^2 c_n^2 \geq 2\pi \sum_{n \in \mathbb{Z}^*} c_n^2 = \int_0^{2\pi} |q(t)|^2 dt;$$

the equality holds if and only if $c_n = 0$, for every $n \notin \{-1, +1\}$, i.e. $q(t) = c_1 e^{it} + c_{-1} e^{-it} = a \cos t + b \sin t$, $a, b \in \mathbb{R}$.

When we have a vector-valued function $q \in H_{2\pi}^1(\mathbb{R}, \mathbb{R}^d)$, $d \in \mathbb{N} \setminus \{0\}$, $q(t) = (q_1(t), \dots, q_n(t))$, $[q] = 0$, then inequality (6) have to be verified from every component $q_i(t)$, $i = 1, \dots, d$ and the equality is attained if and only if $q_i(t) = a_i \cos t + b_i \sin t$, $a_i, b_i \in \mathbb{R}$.

Consider now a convex function f on $(a, b) \subset \mathbb{R}$ and $g \in L^1(0, 2\pi)$ such that $a < g(t) < b$ for every $t \in (0, 2\pi)$, then the *Jensen Inequality* holds

$$f\left(\frac{1}{2\pi} \int_0^{2\pi} g(t) dt\right) \leq \frac{1}{2\pi} \int_0^{2\pi} f(g(t)) dt. \quad (7)$$

Observe that in equation (7) the equality holds if and only if $g(t) = \text{cost}$.

Take now an inertial system with a fixed body at the origin and a second one moving on a 2π -periodic orbit $q \in H_{2\pi}^1(\mathbb{R}, \mathbb{R}^d)$ according to the Newton's law

$$-\ddot{q}_i = \nabla V(q), \quad (8)$$

where $V(q) = -\frac{1}{|q|^\alpha}$, $\alpha \in \mathbb{R}_+^*$, is a generalization of the Newtonian potential.

To seek solutions of (8), we look for the critical points of the functional

$$\mathcal{A}(q) = \frac{1}{2} \int_0^{2\pi} |\dot{q}(t)|^2 dt + \int_0^{2\pi} \frac{dt}{|q(t)|^\alpha} \quad (9)$$

defined on the set

$$\Lambda_K = \{q \in H_{2\pi}^1(\mathbb{R}, \mathbb{R}^d) : [q] = 0, q(t) \neq 0, \forall t \in \mathbb{R}\}.$$

Using (6) and (7) we can state that the minimum of (9) is attained on a circle. In fact, applying first the Jensen Inequality to the convex function $1/s^{\alpha/2}$, we obtain

$$\frac{1}{2\pi} \int_0^{2\pi} \frac{dt}{|q(t)|^\alpha} \geq \left(\frac{1}{\frac{1}{2\pi} \int_0^{2\pi} |q(t)|^2 dt} \right)^{\alpha/2} = \left(\frac{2\pi}{\int_0^{2\pi} |q(t)|^2 dt} \right)^{\alpha/2},$$

then, using the Poincaré Inequality we have

$$\mathcal{A}(q) \geq \frac{1}{2} \int_0^{2\pi} |\dot{q}(t)|^2 dt + \frac{(2\pi)^{\alpha/2+1}}{\left[\int_0^{2\pi} |q(t)|^2 dt \right]^{\alpha/2}} \quad (10)$$

$$\geq \frac{1}{2} \int_0^{2\pi} |q(t)|^2 dt + \frac{(2\pi)^{\alpha/2+1}}{\left[\int_0^{2\pi} |q(t)|^2 dt \right]^{\alpha/2}}. \quad (11)$$

In particular

$$\mathcal{A}(q) \geq \min_{s>0} \{f(s)\}. \quad (12)$$

where $f(s) = \frac{s}{2} + \frac{(2\pi)^{\alpha/2+1}}{s^{\alpha/2}}$, $s > 0$. If there exists $\bar{q} \in \Lambda_K$ such that in (10), (11), (12) the equalities are reached, then \bar{q} is the absolute minimum of the action \mathcal{A} . Since the minimum of the function f is attained at the point $s_{min} = 2\pi\alpha^{\frac{2}{\alpha+2}}$, the absolute minimum of \mathcal{A} have to verify the conditions:

$$(H) \quad \begin{cases} \int_0^{2\pi} |\bar{q}(t)|^2 dt = s_{min}, \\ |\bar{q}(t)| = c \in \mathbb{R}_+^*, \\ \bar{q}_i(t) = a_i \sin t + b_i \cos t, \quad i = 1, \dots, d. \end{cases}$$

When $d = 3$, the only function that satisfies (H) is the planar circle $\bar{q}(t) = \alpha^{\frac{1}{\alpha+2}} e^{it}$. If $d = 4$, we identify $\mathbb{R}^4 \cong \mathbb{C}^2$ and we have that (H) are satisfied if and only if $q(t) = (r_1 e^{it}, r_2 e^{it})$, $r_1, r_2 \in \mathbb{R}_+^*$. Since $r_2 r_1 e^{it} - r_1 r_2 e^{it} = 0$, we deduce that q lies in \mathbb{C} and it is a circle. If $d > 4$, we can reduce the problem to the two cases we have already examined.

3 Some inequalities

To simplify the notation of this section, we define the integers $K := n - 1$, $N := K - 1$.

Proposition 1. *For every $\beta > 0$, fixed any $\mu_1, \dots, \mu_K \in \mathbb{R}_+^*$ the problem*

$$\min \sum_{h=1}^K \frac{1}{s_h^\beta}$$

has a unique solution on the set $\left\{ s_h : \sum_{h=1}^K \mu_h s_h = 1; s_1, \dots, s_K \in \mathbb{R}_+^ \right\}$.*

Proof. We want to prove that the function $f(s) := \sum_{h=1}^K \frac{1}{s_h^\beta}$ has a unique minimum under the constraint

$$\sum_{h=1}^K \mu_h s_h = 1,$$

that is equivalent to prove that the function

$$f_C(s_1, \dots, s_N) := \sum_{h=1}^N \frac{1}{s_h^\beta} + \mu_K^\beta \left(1 - \sum_{h=1}^N \mu_h s_h \right)^{-\beta},$$

has a unique minimum. This function is defined, continuous and coercive on the bounded set

$$\mathcal{S} = \left\{ (s_1, \dots, s_N) : s_i > 0, i = 1, \dots, N, \sum_{i=1}^N \mu_i s_i < 1 \right\}.$$

Moreover f_C is strictly convex on \mathcal{S} , since composition of a strictly convex function with a linear one. The uniqueness of the minimum follows. \square

Corollary 1. For every $\beta > 0$, fixed any $(\tilde{\mu}_1, \dots, \tilde{\mu}_K) \in (\mathbb{R}_+^*)^K$ the problem

$$\begin{aligned} \min \Phi(s), \quad s = (s_1, \dots, s_K) \in (\mathbb{R}_+^*)^K, \\ \Phi(s) := \left(\sum_{h=1}^K \frac{1}{s_h^\beta} \right) \left(\sum_{h=1}^K \tilde{\mu}_h s_h \right)^\beta \end{aligned} \quad (13)$$

has a unique solution up dilations, in the sense that if $m = \min \Phi(s)$ and $\Phi(\tilde{s}) = m$ then $\Phi(s) = m$ if and only if there exists a positive λ such that $s = \lambda \tilde{s}$. Moreover the minimum is attained on $\tilde{s} \in (\mathbb{R}_+^*)^K$ such that

$$\tilde{\mu}_h = \frac{1}{\tilde{s}_h^{\beta+1} \sum_{h=1}^K \tilde{s}_h^\beta} \quad (14)$$

and its value is $\sum_{h=1}^K \frac{1}{\tilde{s}_h^\beta}$.

Proof. The function Φ is homogeneous, for every $\lambda > 0$ and $s \in (\mathbb{R}_+^*)^K$, $\Phi(\lambda s) = \Phi(s)$. Problem (13) is equivalent to the one we have studied in Proposition 1, since, fixed $\tilde{\mu} \in (\mathbb{R}_+^*)^K$, then $(\mathbb{R}_+^*)^K = \bigcup_{\lambda > 0} \lambda s^1$, where $s^1 \in (\mathbb{R}_+^*)^K$ is such that $\sum_{h=1}^K \tilde{\mu}_h s_h^1 = 1$. The uniqueness of the solution in the sense specified follows then from Proposition 1.

To find an explicit solution for (13), we impose $\frac{\partial \Phi}{\partial s_h} = 0$, $h = 1, \dots, K$ and we obtain relation (14). \square

Proposition 2. For every $k \geq 2$, $k \in \mathbb{N}$ and $x \in (0, 2\pi)$ the following inequality holds

$$1 - \cos(kx) < k^2(1 - \cos x). \quad (15)$$

Proof. Let $f(x) := 1 - \cos(kx)$ and $g(x) := k^2(1 - \cos x)$, real functions defined on $(0, 2\pi)$. First of all observe that the function $\varphi := g - f$ is such that $\varphi(\pi - x) = \varphi(\pi + x)$, so we only have to prove (15) on the set $(0, \pi)$. Moreover, since

the function g is increasing on $(0, \pi)$, f increases in $(0, \pi/k)$ and $f(x) \leq 2 = f(\pi/k) < g(\pi/k)$ our claim is to show (15) only on the set $(0, \pi/k)$. To do that it is enough to remark that the function φ is positive on the interval $(0, \pi/k)$, in fact

$$\varphi(0) = 0, \quad \varphi'(0) = 0$$

and, if $x \in (0, \pi/k)$

$$\varphi''(x) = k^2(\cos x - \cos kx) > 0 \quad \text{for } k \geq 2. \quad (16)$$

□

4 An eigenvalue problem

For every $x \in \Lambda$ let $\xi^x = (\xi_1^x, \dots, \xi_{n-1}^x) \in (\mathbb{R}_+^*)^{n-1}$ defined as

$$\xi_h^x := \int_0^{2\pi} |x(t) - x(t+h\tau)|^2 dt, \quad h = 1, \dots, n-1,$$

and let $\bar{\xi}_h := \bar{\xi}_h^x = 8\pi \sin^2(\pi h/n)$, where \bar{x} is the circle of radius one centered at the origin.

The positive numbers $\bar{\mu}_h$ are defined as

$$\bar{\mu}_h = \frac{1}{c \bar{\xi}_h^{\alpha/2+1}}, \quad \text{where } c = \sum_{h=1}^{n-1} \frac{1}{\bar{\xi}_h^{\alpha/2}}.$$

In what follows $x_i, i = 0, \dots, n-1$, are n vectors of \mathbb{R}^d . If $j \geq n$ then $x_j = x_i$, where $i = j \bmod n$, so when we write x_j with $j \geq n$, we actually want to write the correspondent x_i .

Define the linear operator $\Delta^\mu : \mathbb{R}^{nd} \rightarrow \mathbb{R}^{nd}$ in the following way

$$(\Delta^\mu x)_i := \sum_{h=1}^{n-1} \bar{\mu}_h (2x_i - x_{i+h} - x_{i-h}), \quad (17)$$

where $x_i \in \mathbb{R}^d, i = 0, \dots, n-1$.

In the first part of this section our aim is to study the eigenvalues and the eigenvectors of Δ^μ . Remark that Δ^μ acts on each component of every $x_i \in \mathbb{R}^d$ in the same way

$$(\Delta^\mu x)_i^j := \sum_{h=1}^{n-1} \bar{\mu}_h (2x_i^j - x_{i+h}^j - x_{i-h}^j), \quad (18)$$

for $i = 0, \dots, n-1$ and $j = 1, \dots, d$. For this reason we can associate to Δ^μ the square matrix $nd \times nd$

$$\begin{bmatrix} D^\mu & & 0 \\ & \ddots & \\ 0 & & D^\mu \end{bmatrix}$$

where D^μ is the $n \times n$ matrix associated to the operator Δ^μ restricted to the first components. The eigenvalues of Δ^μ are the eigenvalues of the matrix D^μ and their multiplicity is d times the multiplicity of the ones of D^μ . We will have a complete description of Δ^μ by studying the eigenvalues and the eigenvectors of the matrix D^μ .

Lemma 1. *The matrix D^μ defined below has $[n/2] + 1$ eigenvalues*

$$\delta_0 = 0, \quad \delta_1 = \frac{1}{2\pi}, \quad \delta_l < \frac{l^2}{2\pi} \quad l = 2, \dots, [n/2]. \quad (19)$$

The multiplicity of δ_0 is 1 and the correspondent eigenspace is the one spanned by a constant vector of \mathbb{R}^n .

The multiplicity of $\delta_1, \dots, \delta_{[n/2]}$ is 2 except when n is even, in this case the multiplicity of $\delta_{n/2}$ is 1.

The eigenspace correspondent to the eigenvalue δ_l , $l = 1, \dots, [n/2]$, is spanned by the pair of vectors of \mathbb{R}^n

$$\left(\sin\left(\frac{2\pi}{n}il\right) \right)_{i=0, \dots, n-1}, \quad \left(\cos\left(\frac{2\pi}{n}il\right) \right)_{i=0, \dots, n-1}. \quad (20)$$

Remark 1. *If n is even, the multiplicity of $\delta_{n/2}$ is 1 since $(\sin(\frac{2\pi}{n}il))_i = (\sin(\pi i))_i = (0)_i$.*

Proof. $\delta_0 = 0$ and its multiplicity is 1 since the kernel of D^μ is not trivial and it is composed by the vectors $\lambda \mathbf{1}_{\mathbb{R}^n}$, where $\mathbf{1}_{\mathbb{R}^n} = (1, \dots, 1)$ and $\lambda \in \mathbb{R}$.

The vectors in (20), that we denote with v_l are eigenvectors for D^μ , in fact for every $l = 1, \dots, [n/2]$

$$(D^\mu v_l)_i = \left[2 \sum_{h=1}^{n-1} \bar{\mu}_h (1 - \cos\left(\frac{2\pi}{n}hl\right)) \right] (v_l)_i$$

and then

$$\delta_l = 2 \sum_{h=1}^{n-1} \bar{\mu}_h (1 - \cos\left(\frac{2\pi}{n}hl\right)), \quad l = 1, \dots, [n/2]. \quad (21)$$

We can calculate δ_1 :

$$\begin{aligned} 2 \sum_{h=1}^{n-1} \bar{\mu}_h (1 - \cos\left(\frac{2\pi}{n}h\right)) &= 2 \sum_{h=1}^{n-1} \frac{2 \sin^2(\pi h/n)}{c \bar{\xi}_h^{\beta+1}} = \frac{4}{c} \sum_{h=1}^{n-1} \frac{\bar{\xi}_h}{8\pi \bar{\xi}_h^{\beta+1}} \\ &= \frac{1}{2\pi c} \sum_{h=1}^{n-1} \frac{1}{\bar{\xi}_h^\beta} = \frac{1}{2\pi}; \end{aligned}$$

replacing the inequality of Proposition 2 in (21) we obtain the estimate

$$\delta_l < \frac{l^2}{2\pi}, \quad (22)$$

for $l = 2, \dots, [n/2]$.

The eigenspaces corresponding to $\delta_1, \dots, \delta_{[n/2]}$ are 2-dimensional (except the case n even, where the eigenspace correspondent to $\delta_{[n/2]}$ is 1-dimensional, see Remark 1) since

$$\sum_{i=1}^n \sin\left(\frac{2\pi}{n}il\right) \cos\left(\frac{2\pi}{n}il\right) = \frac{1}{2} \sum_{i=1}^n \sin\left(2\frac{2\pi}{n}il\right) = 0; \quad (23)$$

some terms in the second sum of (23) are zero ($\sin(k\pi)$), the others are pairs of opposite numbers of the kind

$$\sin\left(2l\frac{2\pi}{n}i\right) = -\sin\left(2l\frac{2\pi}{n}(n-i)\right).$$

Moreover the eigenvectors in (20) with $l = l_1$ are orthogonal to the ones with $l = l_2$ if $l_1 \neq l_2$. We conclude the proof, since we have found a set of n independent eigenvectors for the n -dimensional matrix D^μ . \square

Remark 2. *The eigenspace spanned by a pair of eigenvectors of the kind described in (20) can be interpreted as 2-dimensional subspaces of \mathbb{R}^n or as the following subspaces of \mathbb{C}^n*

$$\left\{ \left(z e^{\frac{2\pi}{n}Jil} + \bar{z} e^{-\frac{2\pi}{n}Jil} \right)_{i=0, \dots, n-1} : z \in \mathbb{C} \right\}, \quad l = 1, \dots, [n/2],$$

where J is the complex unit. When n is even, the multiplicity of $\delta_{n/2}$ is 1 and the eigenspace correspondent to the eigenvector $\delta_{n/2}$ is generated by the real vector $(1, -1, \dots, 1, -1)$.

A priori, the eigenvalues in (21) have not to be all distinct; if we call δ_L , $L \in [-[n/2], [n/2]] \cap \mathbb{Z}^*$ the nonzero eigenvectors of D^μ , then we know that $\delta_L = \delta_{-L}$ and the eigenspace of D^μ correspondent to $\delta_{\bar{L}}$, $\bar{L} \in [-[n/2], [n/2]] \cap \mathbb{Z}^*$ has the form

$$\left\{ \left(\sum_{L \in \mathcal{L}} z_L e^{\frac{2\pi}{n}JiL} \right)_{i=0, \dots, n-1} : z_L \in \mathbb{C} \right\}, \quad (24)$$

where $z_L, z_{-L} \in \mathbb{C}$ are conjugate and $\mathcal{L} = \{L \in [-[n/2], [n/2]] \cap \mathbb{Z}^* : \delta_L = \delta_{\bar{L}}\}$.

Lemma 2. *Consider $x_i : \mathbb{R} \rightarrow \mathbb{R}^d$, $i = 0, \dots, n-1$ and the system*

$$\begin{cases} -\ddot{x}_i(t) = \lambda(\Delta^\mu x(t))_i, \\ x_i \in H_{2\pi}^1(\mathbb{R}, \mathbb{R}^n), \\ x_i(t + 2\pi/n) = x_{i+1}(t), \end{cases} \quad i = 0, \dots, n-1 \quad (25)$$

where the linear operator Δ^μ is defined in (17). Then the minimal λ such that (25) has a nontrivial solution is $\lambda_{min} = 2\pi$. Moreover if $\lambda = \lambda_{min}$ the solutions has the form

$$x_i(t) = \vec{a} \cos\left(t + \frac{2\pi}{n}i\right) + \vec{b} \sin\left(t + \frac{2\pi}{n}i\right), \quad (26)$$

$$\vec{a}, \vec{b} \in \mathbb{R}^d, \quad i = 0, \dots, n-1.$$

Remark 3. Remark that the motion described in (26) is planar and lies on the plane generated by \vec{a}, \vec{b} .

Proof. The linear operator Δ^μ acts on each component of the functions x_i in the same way (see (18)), then we can study problem (25) in the case $d = 1$ and easily find a solution for $d \geq 2$.

If $d = 1$, the scalar periodic functions x_i can be written as

$$x_i(t) = \sum_{k \in \mathbb{Z}^*} a_{i,k} e^{Jkt} \quad (27)$$

where $a_{i,-k}$ and $a_{i,k}$ are complex conjugate numbers. Remark that for every $k \in \mathbb{Z}^*$, the vector $(a_{i,k})_{i=0,\dots,n-1} \in \mathbb{R}^n$ is an eigenvector for the operator Δ^μ ; in fact from

$$\begin{aligned} \sum_{k \in \mathbb{Z}^*} a_{i,k} k^2 e^{Jkt} &= -\ddot{x}_i = \sum_{k \in \mathbb{Z}^*} \lambda \sum_{h=1}^{n-1} \bar{\mu}_h (2a_{i,k} - a_{i+h,k} - a_{i-h,k}) e^{Jkt} \\ &= \sum_{k \in \mathbb{Z}^*} \lambda (\Delta^\mu(a_{i,k}))_i e^{Jkt}, \end{aligned}$$

we deduce that for every $k \in \mathbb{Z}^*$, there holds

$$\lambda (\Delta^\mu(a_{i,k}))_i = \frac{k^2}{\lambda} a_{i,k},$$

that implies the existence of an eigenvector δ_L (see Remark 2) such that

$$\lambda = \frac{k^2}{\delta_L}.$$

When we impose that $x_i(t + 2\pi/n) = x_{i+1}(t)$, from (27) we have

$$a_{i,k} e^{\frac{2\pi}{n} Jk} = a_{i+1,k}, \quad (28)$$

where $a_{n,k} = a_{0,k}$. Remark 2 implies that, since vectors $(a_{i,k})_i$ are eigenvectors for Δ^μ , we can write

$$a_{i,k} = \sum_{L \in \mathcal{L}} a_{L,k} e^{\frac{2\pi}{n} J i L},$$

where $\mathcal{L} = \left\{ L : \delta_L = \frac{k^2}{\lambda} \right\}$ and $a_{L,k} = \bar{a}_{-L,k}$; relation (28) then becomes

$$\sum_{L \in \mathcal{L}} a_{L,k} e^{\frac{2\pi}{n} J(i+1)L} = \sum_{L \in \mathcal{L}} a_{L,k} e^{\frac{2\pi}{n} J(iL+k)},$$

that is verified when $a_{L,k} = 0$ or, if $a_{L,k} \neq 0$, when $k = L + n\mathbb{Z}$. Since $L \in [-n/2, n/2] \cap \mathbb{Z}^*$, using (19), the minimal value for the parameter λ is

$$\lambda_{min} = \min_{k,L} \frac{k^2}{\delta_L} = \min_L \frac{L^2 + n\mathbb{Z}}{\delta_L} = \min_L \frac{L^2}{\delta_L} = \frac{1}{\delta_1} = 2\pi$$

and it is achieved for $|k| = |L| = 1$. The solution of (25) associated to λ_{min} then is

$$x_i(t) = \sum_{k=\pm 1} a_{i,k} e^{Jkt} = a \sin\left(\frac{2\pi}{n}i + t\right) + b \cos\left(\frac{2\pi}{n}i + t\right), \quad (29)$$

with $a, b \in \mathbb{R}$.

When $d \geq 2$, each component of the functions $x_i(t)$, $i = 0, \dots, n-1$ has the form (29) and then

$$x_i(t) = \vec{a} \sin\left(\frac{2\pi}{n}i + t\right) + \vec{b} \cos\left(\frac{2\pi}{n}i + t\right),$$

with $\vec{a}, \vec{b} \in \mathbb{R}^d$. □

Let $x \in \Lambda$ and define

$$x_i : \mathbb{R} \rightarrow \mathbb{R}^d, \quad x_i(t) = x(t + i\tau), \quad i = 0, \dots, n-1, \quad (30)$$

the motion of the $(i+1)$ -th body. Since $x \in \Lambda$, $x_i \in H_{2\pi}^1(\mathbb{R}, \mathbb{R}^d)$.

Corollary 2. For every $x \in \Lambda$

$$\int_0^{2\pi} |\dot{x}(t)|^2 dt \geq 2\pi \sum_{h=1}^{n-1} \bar{\mu}_h \xi_h^x \quad (31)$$

and the equality holds if and only if the functions x_i associated to x (see (30)) have the form (26).

Proof. First of all observe that if $x \in \Lambda$ is not the zero function, (31) is equivalent to

$$\frac{\sum_{i=0}^{n-1} \int_0^{2\pi} |\dot{x}_i(t)|^2 dt}{\sum_{i=0}^{n-1} \int_0^{2\pi} \langle (\Delta^\mu x)_i, x_i \rangle_{\mathbb{R}^d} dt} \geq 2\pi, \quad (32)$$

in fact

$$\begin{aligned} \sum_{i=0}^{n-1} \int_0^{2\pi} \langle (\Delta^\mu x)_i, x_i \rangle_{\mathbb{R}^d} dt &= \sum_{i=0}^{n-1} \sum_{h=1}^{n-1} \bar{\mu}_h \int_0^{2\pi} (2x_i^2 - x_i x_{i+h} - x_i x_{i-h}) dt \\ &= \sum_{i=0}^{n-1} \sum_{h=1}^{n-1} \bar{\mu}_h \int_0^{2\pi} (2x_i^2 - 2x_i x_{i+h}) dt \\ &= \sum_{i=0}^{n-1} \sum_{h=1}^{n-1} \bar{\mu}_h \int_0^{2\pi} |x_i - x_{i+h}|^2 dt = n \sum_{h=1}^{n-1} \bar{\mu}_h \xi_h^x \end{aligned}$$

and

$$\sum_{i=0}^{n-1} \int_0^{2\pi} |\dot{x}_i(t)|^2 dt = n \int_0^{2\pi} |\dot{x}(t)|^2 dt.$$

The functional

$$J(x) := \frac{\sum_{i=0}^{n-1} \int_0^{2\pi} |\dot{x}_i(t)|^2 dt}{\sum_{i=0}^{n-1} \int_0^{2\pi} \langle (\Delta^\mu x)_i, x_i \rangle_{\mathbb{R}^d} dt},$$

defined on Λ , is bounded from below (by a strictly positive constant) and, if $(x_n)_n$ is a minimizing sequence then

- $\|x_n\|_\Lambda = \int_0^{2\pi} |\dot{x}_n(t)|^2 dt$ is bounded,

- the compactness in L^2 of the space of 2π -periodic functions with zero mean value in $H^1(0, 2\pi)$ ensures that if $x_n \rightarrow x$ in L^2 then

$$\sum_{i=0}^{n-1} \int_0^{2\pi} \langle (\Delta^\mu x_n)_i, (x_n)_i \rangle_{\mathbb{R}^d} dt \rightarrow \sum_{i=0}^{n-1} \int_0^{2\pi} \langle (\Delta^\mu x)_i, x_i \rangle_{\mathbb{R}^d} dt,$$

- the boundedness of $\|x_n\|_\Lambda = \|\dot{x}_n\|_{L^2}$ ensures that $\dot{x}_n \rightarrow \dot{x}$ in L^2 and then (by the lower-semi continuity of the norm)

$$J(x) \leq \liminf_n J(x_n).$$

So we can state that the minimum of J exists. The minimal value of J corresponds to the first eigenvalue λ_{min} for the problem (25). From Lemma 2 the inequality follows. \square

5 Proof of Theorem 1

The idea of the proof is to find a functional $\bar{\mathcal{A}}$ defined on the set Λ such that:

- (i) $\mathcal{A}(x) \geq \bar{\mathcal{A}}(x)$ for every $x \in \Lambda$;
- (ii) $\bar{\mathcal{A}}(x)$ attains its minimum on the circles;
- (iii) $\mathcal{A}(x) = \bar{\mathcal{A}}(x)$ if and only if x is a circle.

With this aim we remember the definitions of ξ^x , $\bar{\xi}$, $\bar{\mu}$ and c given in Section 4 and we use the Jensen Inequality, applied to the convex function $f(s) = \frac{1}{s^{\alpha/2}}$ and $g(t) = |x(t) - x(t+h\tau)|^2$, to obtain

$$\mathcal{A}(x) \geq \frac{1}{2} \int_0^{2\pi} |\dot{x}(t)|^2 dt + \frac{1}{2} \sum_{h=1}^{n-1} \frac{(2\pi)^{\frac{\alpha}{2}+1}}{(\xi_h^x)^{\alpha/2}}. \quad (33)$$

Remark that in (33) the equality is attained if the quantities $|x(t) - x(t+h\tau)|^2$, $h = 1, \dots, n-1$, are constant.

Take now

$$\tilde{\mathcal{A}}(x) := \frac{1}{2} \int_0^{2\pi} |\dot{x}(t)|^2 dt + \tilde{c} \frac{1}{\left(\sum_{h=1}^{n-1} \bar{\mu}_h \xi_h^x \right)^{\alpha/2}} \quad (34)$$

where $\tilde{c} = \frac{(2\pi)^{\frac{\alpha}{2}+1}}{2} c$. Corollary 1 ensures the inequality

$$\mathcal{A}(x) \geq \tilde{\mathcal{A}}(x). \quad (35)$$

The quantities $\bar{\mu}_h$ are defined in such a way that the minimum of the functional $\left(\sum_{h=1}^{n-1} \frac{1}{(\xi_h^x)^{\alpha/2}}\right) \left(\sum_{h=1}^{n-1} \bar{\mu}_h \xi_h^x\right)^{\alpha/2}$ is obtained if $\xi_h^x = \bar{\xi}_h$ and its value is c , so if x is a circle the equality in (35) is attained.

From Lemma 2 follows that if we define

$$\bar{\mathcal{A}}(x) := 2\pi \sum_{h=1}^{n-1} \bar{\mu}_h \xi_h^x + \tilde{c} \frac{1}{\left(\sum_{h=1}^{n-1} \bar{\mu}_h \xi_h^x\right)^{\alpha/2}}, \quad x \in \Lambda, \quad (36)$$

then

$$\tilde{\mathcal{A}}(x) \geq \bar{\mathcal{A}}(x), \quad (37)$$

and the equality holds if x satisfies (26).

To study the minima of the functional $\bar{\mathcal{A}}$ let us consider the function $g : \mathbb{R}_+^* \rightarrow \mathbb{R}$, $g(y) = 2\pi y + \frac{\tilde{c}}{y^{\alpha/2}}$, which has a unique minimum at $\bar{y} = \left(\frac{\alpha\tilde{c}}{4\pi}\right)^{\frac{1}{\frac{\alpha}{2}+1}}$; $\bar{\mathcal{A}}$ attains its absolute minimum on the functions $x \in \Lambda$ such that $\sum_{h=1}^{n-1} \bar{\mu}_h \xi_h^x = \left(\frac{\alpha\tilde{c}}{4\pi}\right)^{\frac{1}{\frac{\alpha}{2}+1}}$, among others on the circle of radius

$$R = \left(\frac{\alpha\tilde{c}}{4\pi}\right)^{\frac{1}{\alpha+2}}. \quad (38)$$

Condition (ii) is then verified. To conclude the proof we show condition (iii), i.e. that the only $x \in \Lambda$ such that the quantities $|x(t) - x(t+h\tau)|^2$, $h = 1, \dots, n-1$, are constant and, at the same time verify (26) are the circles. We prove this in the following Proposition.

Proposition 3. *For every $x \in \Lambda$ such that $|x(t) - x(t+h\tau)|^2$ are constant, $h = 1, \dots, n-1$, and $x(t) = \vec{a} \cos t + \vec{b} \sin t$, then $\vec{a} \cdot \vec{b} = 0$ and $|\vec{a}| = |\vec{b}|$.*

Proof. Let $\tau_h = h\tau$ and consider

$$\begin{aligned} x(t) - x(t + \tau_h) &= \vec{a} (\cos t - \cos(t + \tau_h)) + \vec{b} (\sin t - \sin(t + \tau_h)) \\ &= \vec{a} (\cos t(1 - \cos \tau_h) + \sin t \sin \tau_h) + \vec{b} (\sin t(1 - \cos \tau_h) - \sin \tau_h \cos t) \\ &= 2 \sin\left(\frac{\tau_h}{2}\right) \left[\vec{a} \left(\cos t \sin \frac{\tau_h}{2} + \sin t \cos \frac{\tau_h}{2}\right) + \vec{b} \left(\sin t \sin \frac{\tau_h}{2} - \cos t \cos \frac{\tau_h}{2}\right) \right] \end{aligned}$$

then the square of the distances between each pair of bodies are

$$\begin{aligned} |x(t) - x(t + \tau_h)|^2 &= 4 \sin^2\left(\frac{\tau_h}{2}\right) \left[|\vec{a}|^2 \sin^2\left(t + \frac{\tau_h}{2}\right) + |\vec{b}|^2 \cos^2\left(t + \frac{\tau_h}{2}\right) \right. \\ &\quad \left. - 2\vec{a} \cdot \vec{b} \sin\left(t + \frac{\tau_h}{2}\right) \cos\left(t + \frac{\tau_h}{2}\right) \right] \\ &= 4 \sin^2\left(\frac{\tau_h}{2}\right) \left[|\vec{a}|^2 + (|\vec{b}|^2 - |\vec{a}|^2) \cos^2\left(t + \frac{\tau_h}{2}\right) - \vec{a} \cdot \vec{b} \sin(2t + \tau_h) \right] \\ &= 4 \sin^2\left(\frac{\tau_h}{2}\right) \left[\frac{|\vec{a}|^2 + |\vec{b}|^2}{2} + (|\vec{b}|^2 - |\vec{a}|^2) \cos(2t + \tau_h) - \vec{a} \cdot \vec{b} \sin(2t + \tau_h) \right]. \end{aligned}$$

these quantities do not depend on t if and only if $\vec{a} \cdot \vec{b} = 0$ and $|\vec{a}| = |\vec{b}|$. \square

6 Minima in rotating systems

In this section we try to generalize the previous results to the case when the n particles move in a non-inertial system, more precisely in a system rotating with angular velocity of intensity ω . We shall prove that Theorem 1 still holds in some range of ω . This is not a technical obstruction: indeed the assertion of Theorem 1 does not hold true for many angular velocities ω .

Suppose $d = 2$, let $x(t)$, the orbit on which the bodies move in the inertial system and $y(t)$ the orbit in the non-inertial one. We suppose that the two systems are linked by a rotation with angular velocity of intensity $\omega > 0$, then the relation between the orbit in the inertial system and the one in the the rotating system is

$$x(t) = e^{J\omega t}y(t).$$

We want to minimize the action among 2π -periodic loops in the rotating system, $y \in \Lambda$, which satisfies the choreography constraint $y_{i+1}(t) = y_i(t + \tau)$, where $y_i(t) = y(t + i\tau)$, $i = 0, \dots, n-1$, is the motion of the $(i+1)$ -th body and $y_n(t) = y_0(t)$. In the inertial system the motion verifies the following conditions

$$x(t + 2\pi) = e^{2\pi J\omega}x(t), \quad x_{i+1}(t) = e^{-J\omega\tau}x_i(t + \tau). \quad (39)$$

We seek the loop that minimize the action integral in the rotating system. The mutual distances among bodies are invariant under a rotation, and, since the potential depends only on these quantities, when we write the action as a function of the orbit in the rotating frame, $y(t)$, we have the only difference in the kinetic part

$$\mathcal{A}(y) = \frac{1}{2} \int_0^{2\pi} |\dot{y}(t) + J\omega y(t)|^2 dt + \frac{1}{2} \sum_{h=1}^{n-1} \int_0^{2\pi} \frac{dt}{|y(t) - y(t + h\tau)|^\alpha}. \quad (40)$$

Proposition 4. *If $\omega = k \in [2, n-1] \cap \mathbb{N}$ and if k, n are coprime, then the action does not admit any minimum.*

Proof. Consider the loops $y_\nu(t) = R_\nu e^{Jkt}$, $R_\nu \rightarrow +\infty$ if $\nu \rightarrow +\infty$; $(y_\nu(t))_\nu$ form a not converging minimizing sequence for \mathcal{A} . □

Proposition 5. *If $\omega = n$, then the action admits a continuum of minima.*

Proof. From (39), we have that the orbit in the inertial system is 2π -periodic and verifies the choreography condition $x_{i+1}(t) = x_i(t + \tau)$. Since the center of mass is not fixed, all 2π -periodic circle with radius expressed in (38) are minima in the inertial system. In the rotating system the minima are $y(t) = (R(\alpha, n)e^{\pm Jt} + c)e^{-Jnkt}$, $c \in \mathbb{C}$. □

Proposition 6. *If $\omega = k \in [2, n-1] \cap \mathbb{N}$ and $g.c.d.(k, n) = \tilde{k} > 1$, $\tilde{k} \neq n$, let $j := n/\tilde{k}$ and $\tilde{j} := k/\tilde{k}$. Then*

(i) the action does not achieve its infimum;

(ii) any minimizing sequence $(y^{(\nu)})_\nu$ has the form

$$y_i^{(\nu)} = e^{-2\pi J \frac{i}{j}} \left(c_m^{(\nu)} + \eta(t + i\tau) \right) e^{Jkt} + o(1), \quad i = 0, \dots, n-1$$

$$i \equiv m \pmod{j} \quad \text{and} \quad m = 0, \dots, j-1$$

where $|c_m^{(\nu)}| \rightarrow +\infty$ if $\nu \rightarrow +\infty$ for $m = 0, \dots, j-1$ and the function η is the minimum of the action described in Theorem 1 with $n = \tilde{k}$.

Proof. Conditions (39) in the inertial system imply that

$$x(t + 2\pi) = x(t), \quad x_{i+1}(t) = e^{-2\pi J \tilde{j}/j} x_i(t + \tau); \quad (41)$$

then we have j equal systems (up to rotations) with \tilde{k} bodies each in which the bodies are linked by a choreography constraint. We can split the action as the sum of two part

$$\mathcal{A} = j\mathcal{A}_{\tilde{k}} + \mathcal{A}_{int},$$

where $\mathcal{A}_{\tilde{k}}$ is the action of each subsystem with \tilde{k} bodies and \mathcal{A}_{int} is the sum of the interactions between each pair of bodies in different subsystems. Therefore the infimum of the action is j -times the action of the minimal motion with \tilde{k} bodies.

On every minimizing sequence $\mathcal{A}_{int} \rightarrow 0$, so that the center of mass of every subsystem will go to infinity in a radial direction and the angle between two of such directions is a multiple of $2\pi/j$; moreover the motion on every subsystem tends to the minimal orbit with \tilde{k} bodies we have found in Theorem 1. \square

Consider the function $h : \mathbb{R}_+^* \rightarrow \mathbb{N}$, $h(\omega) = \min_{n \in \mathbb{N}^*} \frac{(\omega-n)^2}{n^2}$ and let $\omega^* := \max\{\omega : h(\omega) = (\omega-1)^2\}$, ($\omega^* = \frac{4}{3}$). We can state the following

Theorem 2. *If $\omega \in (0, \omega^*) \setminus \{+1\}$, then the functional in (40) attains its minimum on a circle with minimal period 2π and radius depending on n , α and ω .*

Proof. Following the proof of Theorem 1, we would like to show the existence of a constant depending on ω , $c(\omega)$, such that the following inequality holds

$$\int_0^{2\pi} |\dot{y}(t) + J\omega y(t)|^2 dt \geq c(\omega) \sum_{h=1}^{n-1} \bar{\mu}_h \xi_h^y, \quad (42)$$

for every loop $y \in \Lambda$. Using the same arguments of Lemma 2 we have that (42), is equivalent to

$$\frac{\sum_{i=0}^{n-1} \int_0^{2\pi} |\dot{y}_i(t) + J\omega y_i(t)|^2 dt}{\sum_{i=0}^{n-1} \int_0^{2\pi} \langle (\Delta^\mu y)_i, y_i \rangle_{\mathbb{R}^d} dt} \geq c(\omega), \quad (43)$$

where $y_i : \mathbb{R} \rightarrow \mathbb{R}^2$ describes the motion of the $(i+1)$ -th body $y_i(t) = y(t+i\tau)$, $i = 0, \dots, n-1$. Following the proof of Corollary 2, the functional at the left side of (43) attains its minimum on the set Λ and the minimum corresponds to the first eigenvalue for the problem

$$\begin{cases} -\ddot{y}_i(t) + 2J\omega\dot{y}_i + \omega^2 y_i = \lambda(\Delta^\mu y(t))_i, \\ y_i \in H_{2\pi}^1(\mathbb{R}, \mathbb{R}^2), \quad i = 0, \dots, n-1 \\ y_i(t+\tau) = y_{i+1}(t) \end{cases} \quad (44)$$

where $\lambda \in \mathbb{R}$ and $\Delta^\mu : \mathbb{R}^{2n} \rightarrow \mathbb{R}^{2n}$ is defined in (17). To find the first eigenvalue for the periodic problem (44), we write the Fourier series

$$y_i(t) = \sum_{k \in \mathbb{Z}^*} b_{i,k} e^{Jkt}, \quad i = 0, \dots, n-1$$

and, as in the proof of Lemma 2, we obtain that the vectors $(b_{i,k})_{i=0, \dots, n-1} \in \mathbb{R}^n$ are eigenvectors for the linear operator Δ^μ

$$(k-\omega)^2 b_{i,k} = \lambda \sum_{h=1}^{n-1} \bar{\mu}_h (2b_{i,k} - b_{i+h,k} - b_{i-h,k}) = \lambda (\Delta^\mu (b_{i,k}))_i.$$

It follows that there exist δ_L , eigenvalue for Δ^μ such that $\delta_L = \frac{(k-\omega)^2}{\lambda}$. When we impose that $y_i(t + \frac{2\pi}{n}) = y_{i+1}(t)$, we obtain the relation $k = L + n\mathbb{Z}$ and, since $w \in (0, \omega^*) \setminus \{1\}$ and $\delta_L < l^2/2\pi$, the minimal value for λ is verified for $L = 1$. Then we obtain that the minimal eigenvalue for the problem is

$$c(\omega) = 2\pi(\omega - 1)^2.$$

As in the proof of Theorem 1, we can conclude that the minimal loop for the functional in (40) is a 2π -periodic circle with radius $R_\omega = \left(\frac{\alpha\bar{c}}{2c(\omega)}\right)^{1/(\alpha+2)}$. \square

Remark 4. *When the bodies move in a d -dimensional space, we can use the same arguments of Lemma 2 to show that the orbit is planar and, precisely, that it lies on the plane orthogonal to the direction of the rotation. In fact, we can decompose the motion in $[d/2]$ 2-dimensional spaces (if d is odd, there is no motion in the last dimension since the choreography constraint could not be verified) such that the first of them is the plane of the rotation. The eigenvalues correspondent to the motions in the 2-dimensional subspaces are all 2π except the one of the plane orthogonal to the rotation that is $c(\omega)$; since $c(\omega) < 2\pi$ the motion is a planar circle on the rotating plane.*

Theorem 3. *Suppose that n and k are coprime. Then there exist $\epsilon = \epsilon(\alpha, n, k)$ such that if $\omega \in (k - \epsilon, k + \epsilon)$ the minimum of the action is attained on a circle with minimal period $2\pi/k$ that lies in the rotating plane with radius depending on n , α and ω .*

Proof. Take $\omega \in (\omega^*, n - 1/2)$, $\omega \notin \mathbb{Z} + 1/2$ and let $[\omega]_n = k \in [2, n - 1] \cap \mathbb{N}$, the integer closest to ω . Suppose k coprime with n . We define a new linear operator $\Delta^{\mu, k}$ that on the two components of the rotating plane acts in the following way

$$(\Delta^{\mu, k} y)_i^j := \sum_{h=1}^{n-1} \bar{\mu}_h^k (2y_i^j - y_{i+h}^j - y_{i-h}^j), \quad (45)$$

where $\bar{\mu}_h^k = \frac{1}{c^k (\xi_h^k)^{\alpha/2+1}}$, $\bar{\xi}_h^k = 8\pi \sin^2(\pi kh/n)$ and $c^k = \sum_1^{n-1} \frac{1}{(\xi_h^k)^{\alpha/2}}$. On the other components $\Delta^{\mu, k}$ acts in the same way as Δ^μ . The eigenvalues for the operator $\Delta^{\mu, k}$ corresponding to the components on the rotating frame are the same eigenvalues of the matrix D^μ , but rearranged in a different way, we will call them δ_l^k . It is easy to calculate that the smallest eigenvalue for $\Delta^{\mu, k}$ is $\delta_k^k = 1/2\pi$. As in Theorems 1 and 2, to prove that the minimum of the action is a circle with minimal period $2\pi/k$ in the rotating plane we would like to verify that

$$\min_l \frac{(\omega - l)^2}{\delta_l^k} = \frac{(\omega - k)^2}{\delta_k^k} = 2\pi(\omega - k)^2. \quad (46)$$

In fact, since on the components orthogonal to the rotating plane, the minimal eigenvalue is 2π (see Remark 4) and $(\omega - k)^2 < 1$ (k is the closest integer to ω) the minimal eigenvalue on the rotating plane is smaller than the ones on the orthogonal directions and the motion will lie on the rotating plane. Since the minimal eigenvalue is $1/2\pi$, condition (46) is implied by

$$\delta_{\max}^k \leq \frac{1}{2\pi(\omega - k)^2} \min_k ((\omega - k + 1)^2, (\omega - k - 1)^2), \quad (47)$$

where δ_{\max}^k is the largest eigenvalue of the linear operator $\Delta^{\mu, k}$. Fixed n and α , (47) is verified if we take ω in an appropriate ϵ -neighborhood of k . We can then assert that the minimum is the planar circle with minimal period $2\pi/k$ that lies in the rotating plane. Moreover the radius of a minimal circle is $R_\omega = \left(\frac{\alpha \bar{c}}{2c(\omega)}\right)^{1/(\alpha+2)}$, where $c(\omega) = 2\pi(\omega - k)^2$. □

Remark 5. When $n = 3$, $\delta_{\max}^2 = 1/2\pi$ and condition (47) is always verified. In this particular case if $\omega \in \{1, 2\}$, the action does not admit a minimum; if $\omega = 3/2$, the minima are all circles with minimal period π or 2π ; if $\omega \in (0, 3/2) \setminus \{1\}$ the minimum is attained on circles with period 2π and if $\omega \in (3/2, 3) \setminus \{2\}$ the minimum is on circles with period π .

Remark 6. Calculating the values of δ_{\max}^k , we remark that if $4 \leq n \leq 9$, condition (47) is verified for every $\alpha > 0$, this implies that taking k coprime with n , then for every $\omega \in (k - 1/2, k + 1/2)$ the minimum of the action is attained on a circle with minimal period $2\pi/k$ that lies in the rotating plane.

An interesting situation appears when the integer closest to the angular velocity is not coprime with the number of bodies. In this case we prove that

the minimal orbit is planar, but it is no more a circle, as the following theorem states.

Theorem 4. *Take $k \in \mathbb{N}$ and $\text{g.c.d.}(k, n) = \tilde{k} > 1$, $\tilde{k} \neq n$. Then there exists $\epsilon = \epsilon(\alpha, n, k) > 0$ such that if $\omega \in (k - \epsilon, k + \epsilon) \setminus \{k\}$ the minimum of the action is attained on a planar 2π -periodic orbit with winding number k which is not a relative equilibrium motion.*

Proof. Let $(\omega_\nu)_\nu$, $\omega_\nu \rightarrow k$ as $\nu \rightarrow +\infty$. If $m(\omega_\nu)$ is the minimal value of the action correspondent to ω_ν then $m(\omega_\nu) \rightarrow \inf j\mathcal{A}_{\tilde{k}}$ (see Proposition 6) for the continuity of the action in ω . Then there exists ν_0 such that when $\nu > \nu_0$, $m(\omega_\nu)$ is attained on a orbit of the kind described in (ii) of Proposition 6. \square

Remark 7. *With the same argument we used in the proof of Theorem 4, we can assert that there exists $\epsilon = \epsilon(\alpha, n) > 0$ such that if $\omega \in (n - \epsilon, n + \epsilon) \setminus \{n\}$ then the minimum of the action is attained on a orbit of the kind $y(t) = (R(\alpha, n)e^{\pm Jt} + c)e^{-Jnkt}$, $c \in \mathbb{C}$ (see Proposition 5).*

In the theorems above we have examined the minimizers of the action functional when the angular velocity of the rotating system is in the interval $(0, n - 1 + \epsilon)$, where $\epsilon = \epsilon(\alpha, n, n - 1)$ according with Proposition 4, 6 and Theorems 2, 3, 4, or when $\omega \in (n - \xi, n + \xi)$, $\xi = \xi(\alpha, n)$ according with Proposition 5 and Remark 7. Consider now $\omega > n$, such that

$$\omega = \bar{\omega} + ln, \quad l \in \mathbb{N}^*, \quad \bar{\omega} \in (0, n).$$

When $\bar{\omega} = 0$ then we are in the same situation described in Proposition 5. If $\bar{\omega} = k$, Propositions 4 and 6 still hold true replacing ω with $\bar{\omega}$. When $\bar{\omega}$ is not an integer, then Theorems 2, 3, 4 still hold replacing ω with $\bar{\omega}$ and with the following differences: in Theorems 2 and 3 the minimal period of the minimizing circles are in this case respectively $2\pi/ln$ and $2\pi/(ln + k)$; in Theorem 4 the winding number of the minimizing orbit is $k + ln$.

7 Numerical results and further comments

We give now some numerical examples of the planar orbits we described in Theorem 4. The algorithm we use to minimize the action functional acts following the steepest descendant flow, for this reason it could happen that the orbits in Figures 1, 2 are just local minima for the action and not the global ones; anyway their qualitative behavior is the one described in Theorem 4. In Figure 1 there are some examples of the orbits we have described in Theorem 4 when the angular velocity ω is closed to an integer that divides the number of bodies. In Figure 2 the we consider values of the angular velocity that are close to an integer that is not coprime with n and that does not divide it.

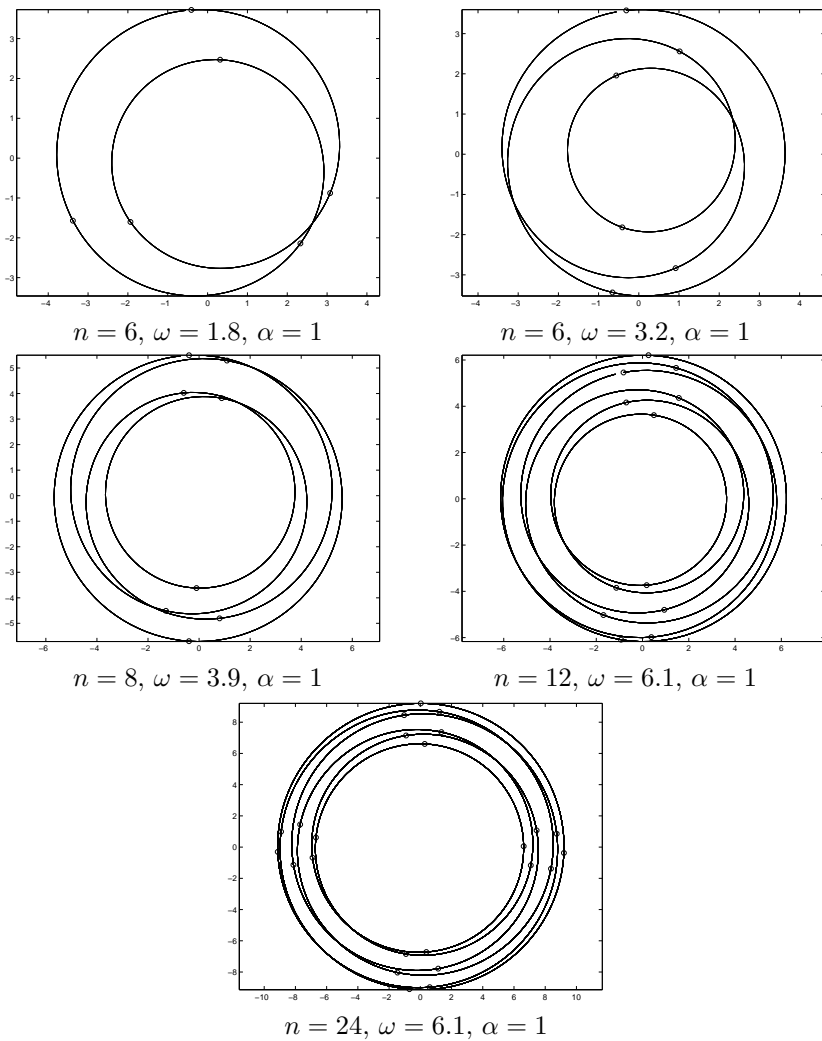


Figure 1: Examples for Theorem 4, ω close to an integer that divides n .

Theorems 2, 3 and 4 examine the minima of the action integral for values of ω close to integers; this limitation is not just a technical obstruction, we can in fact find numerical examples of minima that are not the ones of Theorems 2, 3 and 4 when ω is close to the half of an integer. In the left column of Figure 3 we show some examples of non planar minima correspondent to values of ω close to the half on an integer; on the right side we show their projections on the plane $x_3 = 0$. Remark that in the example with 12 bodies, the projection of the curve on the horizontal plane is still a circle with winding number 7, that is the integer closest to ω ; this orbit has a similar behavior to the “hip-hop”

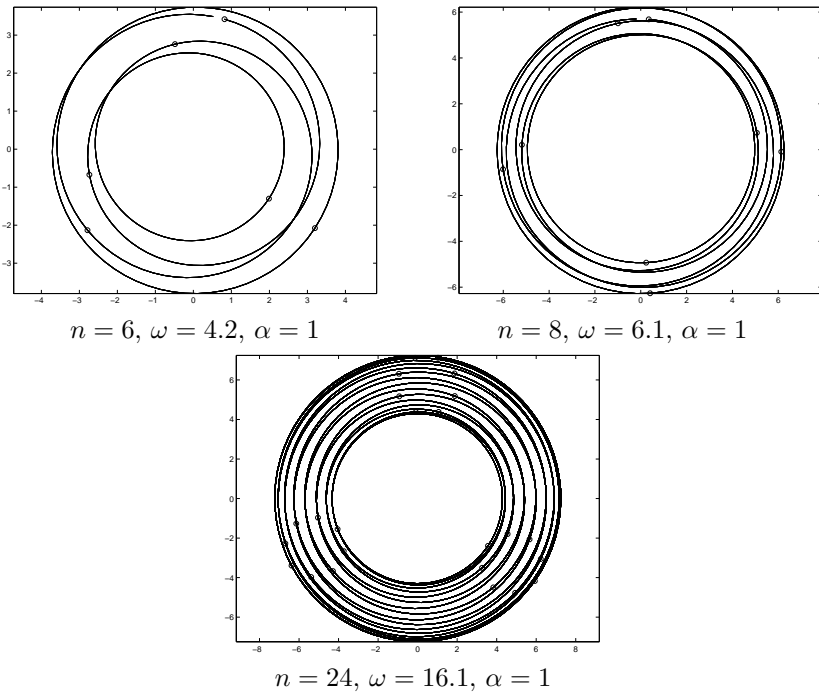


Figure 2: Examples for Theorem 4, ω close to an integer that divides n .

found by A. Chenciner and A. Venturelli in [12]. In the other two pictures, with 15 and 17 bodies respectively, the projections on the $x_3 = 0$ plane are orbits similar to the one described on line by Simò with winding number equal to the integer closest to ω .

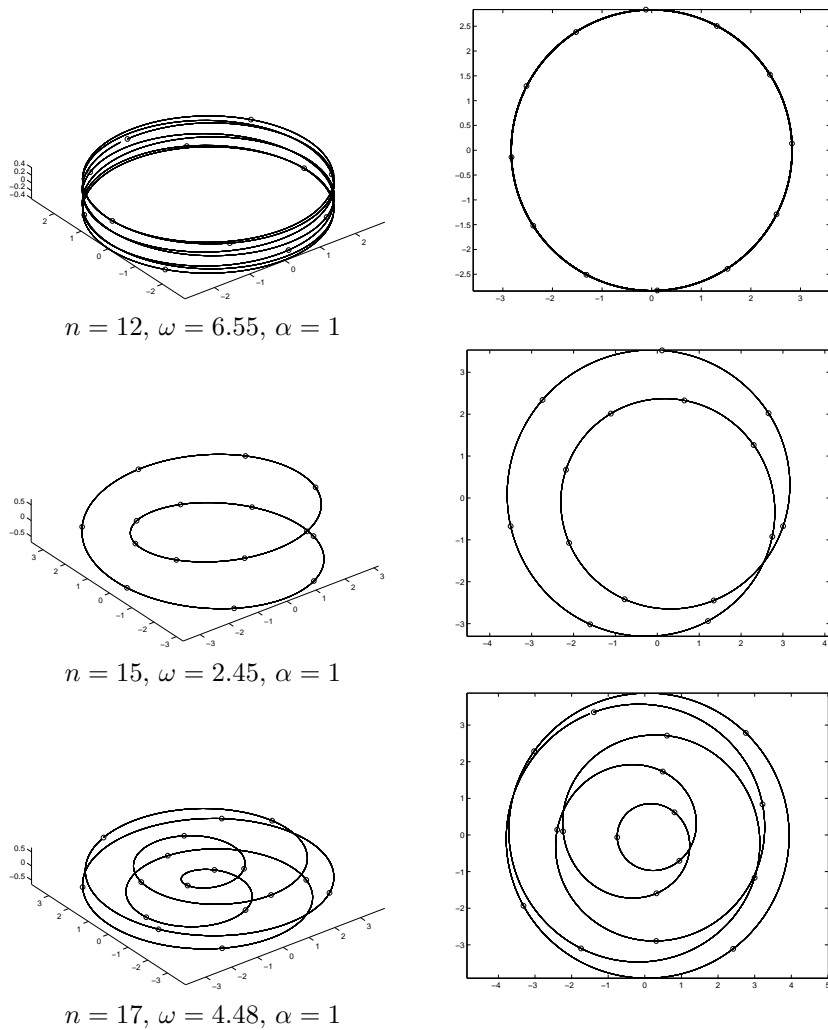


Figure 3: Minima of the action with angular velocities close to the half on an integer.

The analysis of the minimizers of the action in a rotating system suggests the existence of saddle critical points; in fact in many cases we have an infinity of absolute minima, in other we proved the existence of absolute minima, but we can compute numerically other relative minima. By a Mountain Pass algorithm we have found some example of such saddle critical points. In Figure 4 we show a saddle point for the 3-body problem with angular velocity $\omega = 1.5$, between the circles with minimal period 2π and π (see Remark 5). The second picture is a saddle point for the 4-body problem with angular velocity $\omega = 2.2$, between the circle with minimal period $2\pi/3$ (that is a local minimum) and orbit described

in Theorem 4 for $k = 2$. The last picture of Figure 4 is a saddle point for the 5-body problem with $\omega = 2.5$, between the circles with minimal period π and $2\pi/3$. Remark that these saddle points are collisionless.

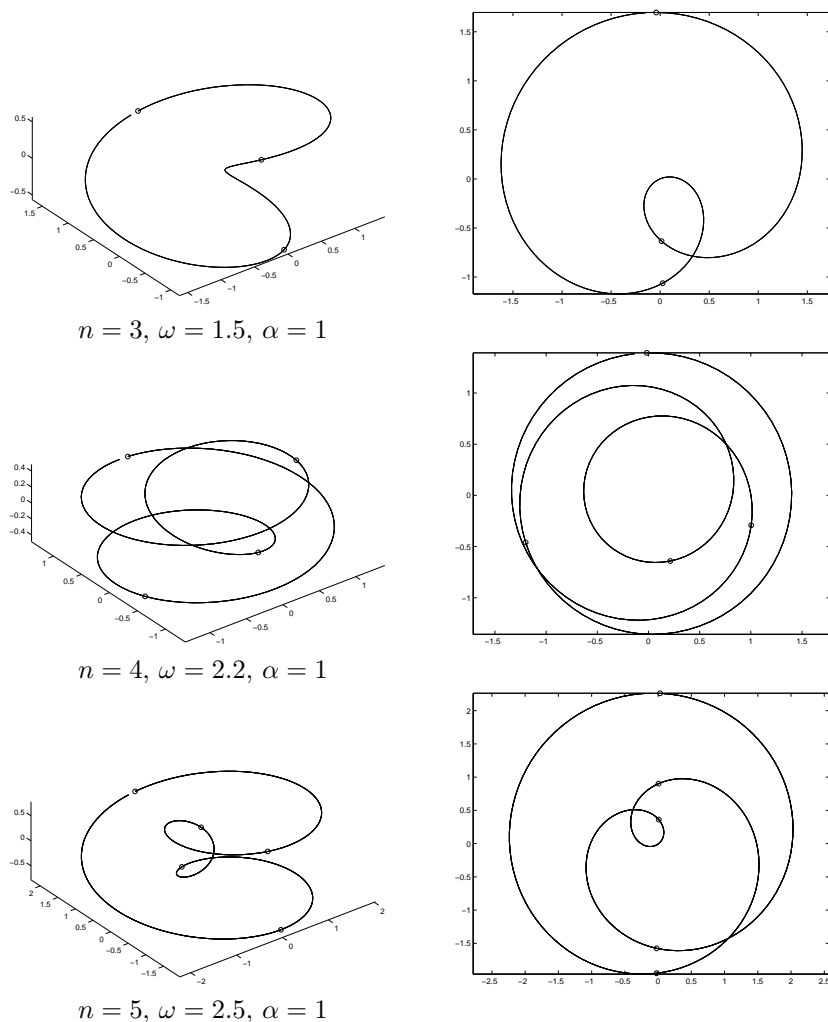


Figure 4: Saddle points.

In this setting, we suggest a new way to explain the eight shaped curve that Chenciner and Montgomery found in [11] as a minimum of the function integral among simple choreographies with some special symmetry constraints. Consider n bodies, n odd, moving in \mathbb{R}^3 on a loop $x(t) = (x_1(t), x_2(t), x_3(t))$ in

the set Λ . We impose the following symmetry constraints

$$\begin{aligned}x_1(t) &= x_1(t + \pi), & x_1(t) &= -x_1(-t), \\x_2(t) &= x_2(t + 2\pi), & x_2(t) &= -x_2(-t), \\x_3(t) &= x_3(t + 2\pi), & x_3(t) &= x_3(-t).\end{aligned}\tag{48}$$

Remark that these constraints are satisfied by a eight in the plane $x_3 = 0$ or by a circle in the plane $x_1 = 0$. In the set Λ with (48), we have two circles that attains the minimum

$$(0, R \sin t, R \cos t) \quad \text{and} \quad (0, R \sin t, -R \cos t)$$

where R is determined in (38). This two curves are isolated minima; the Mountain Pass Theorem, ensures the existence of a saddle point, which is the minimum among the maximal points of the paths joining these two minima. Using a numerical mountain pass algorithm, we find that the eight in the plane $x_3 = 0$ is a saddle points between the two circles.

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