MULTIVARIATE COMPACTLY SUPPORTED BIORTHOGONAL SPLINE WAVELETS

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Abstract

We study biorthogonal bases of compactly supported wavelets constructed from box splines in $\mathbb{R}^N$ with any integer dilation factor. For a suitable class of box splines we write explicitly dual low–pass filters of arbitrarily high regularity and indicate how to construct the corresponding high–pass filters (primal and dual).

1 Introduction

Biorthogonal bases of univariate compactly supported wavelets of arbitrarily high regularity were first constructed in 1992 by Cohen, Daubechies and Feauveau [7]. The innovation consisted in greater flexibilty in the filters design and in avoiding the asymmetry and the numerical complexity of the coefficients of Daubechies orthonormal wavelets. Since then, aside from the direct tensor product construction, there have been several extensions to higher dimensions and/or to general dilation matrices: see e.g.[4], [6], [9], [11], [12], [13], [16], [20], [21], [23]. As in the unidimensional case, special emphasis was put on compactly supported wavelets constructed from multivariate box splines, because spline functions are fairly well understood, allow explicit computations and possess additional properties (see [2] for a general reference on box splines).

A general framework for the construction of biorthogonal wavelets is provided by multiresolution analyses. One starts with a refinable box spline φ (with respect to an expanding matrix S, $S\mathbb{Z}^N \subset \mathbb{Z}^N$), generating a multiresolution analysis with low–pass filter m_0 (i.e, the symbol of the refinement mask of φ); then one tries to construct a polynomial dual filter $\widetilde{m}_0$, which in turn defines a compactly supported dual distribution $\widetilde{\varphi}$ by means of an infinite product. If φ and $\widetilde{\varphi}$ have Fourier transforms with enough decay at infinity and have mutually orthogonal translates, then the construction of biorthogonal wavelets amounts to solving a matrix extension problem. It was shown in [20], [16] that the Quillen–Suslin algorithm provides a constructive solution to this problem.

Therefore, the concrete construction of a filter $\widetilde{m}_0$ giving rise to a compactly supported dual function $\widetilde{\varphi}$ of (arbitrarily) high regularity is a major issue in the theory of biorthogonal wavelets, and this problem was addressed in several of the above mentioned papers. Generally speaking, there are only few constructions avalaible for dual pairs. In some cases a single pair is produced by explicit computation, but in many other cases it is possible to construct a whole family of filters which are dual to a given primal filter and generate a functions $\widetilde{\varphi}$ of arbitrarily high regularity. As in the univariate case [7], one technique consists in constructing an appropriate filter, satisfying an interpolatory condition, which can be written as the product of a given filter and its dual. A general method for this construction appears in [16] and several applications are given. General as it may be, this method cannot guarantee arbitrarily high regularitry in many important cases. Using a different method to construct the relevant polynomials, He and Lai produced explicitly the dual filters for bivariate box spline in such a way that $\widetilde{\varphi}$ has arbitrarily high regularity.

This paper was inspired by the above mentioned paper [13], and can be considered as an extension of the results and the methods therein contained. Namely He and Lai paper deals with bivariate box splines and scaling matrix $2I$, while we study biorthogonal compactly supported spline wavelets in any dimension N with scaling matrix $M \cdot I_N$, where $M \geq 2$ is an integer, I_N is the identity matrix and the underlying lattice is $\mathbb{Z}^N$. While their construction is based on univariate Daubechies polynomials, our construction uses the multivariate Bernstein polynomials. We refer to [3] and [17] for the relation between Bernstein and Daubechies polynomials.

We assume that $\varphi = \varphi_\Xi$, is a box spline on $\mathbb{R}^N$, with full rank unimodular matrix Ξ having integer entries. Such a spline is obviously refinable with respect to $M \cdot I_N$. In section 2 we notice that, arguing along the lines of [21], there always exists a dual filter $\widetilde{m}_0$ such that the corresponding $\widetilde{\varphi}$ is the convolution of a box spline and a compactly supported distribution. If $\widehat{\widetilde{\varphi}}$ has sufficient decay along the directions of Ξ, then we can prove that $\widetilde{\varphi}$ and φ have orthonormal translates. Therefore the high–pass filters obtained by means of the Quillen–Suslin algorithm give rise to biorthogonal compactly supported wavelets.

Thus the problem arises of investigating which box splines (with unimodular Ξ) admit a dual polynomial filter $\widetilde{m}_0$ such that $\widehat{\widetilde{\varphi}}$ has the required decay. In section 3 we prove that this is the case for box splines with Fourier transform of the type

$$\widehat{\varphi}(\omega)=\prod_{k=1}^{N}\left(\frac{1-\exp(-i\omega_k)}{i\omega_k}\right)^{a_k}\prod_{k=2}^{N}\left(\frac{1-\exp(-i(\omega_1+\omega_2+\cdots+\omega_k))}{i(\omega_1+\omega_2+\cdots+\omega_k)}\right)^{b_k} \tag{1}$$

where $a_k>0$ and $b_k\geq 0$ are integers. Setting

$$f(t)=\frac{1+e^{-it}+e^{-i2t}+\cdots+e^{-i(M-1)t}}{M} \tag{2}$$

we have that

$$m_0(\omega)=\frac{\widehat{\varphi}(M\omega)}{\widehat{\varphi}(\omega)}=\prod_{k=1}^{N}f(\omega_k)^{a_k}\prod_{k=2}^{N}f(\omega_1+\omega_2+\cdots+\omega_k)^{b_k}. \tag{3}$$

For such splines, we will construct explicitly a dual filter $\widetilde{m}_0$ of the form

$$\widetilde{m}_0(\omega)=\prod_{k=1}^{N}\overline{f}(\omega_k)^{\alpha_k-a_k}\prod_{k=2}^{N}\overline{f}(\omega_1+\omega_2+\cdots+\omega_k)^{\beta_k-b_k}\overline{r}(\omega), \tag{4}$$

where $r(\omega)$ is a trigonometric polynomial. In the last section we will prove that the exponents α_k and β_k can be chosen in such a way that $\widetilde{\varphi}$, defined via its Fourier transform as

$$\widehat{\widetilde{\varphi}}(\omega)=\prod_{j=1}^{\infty}\widetilde{m}_0(M^{-j}\omega), \tag{5}$$

has arbitrarily high regularity. It follows that every box spline of the form (1) gives rise to compactly supported biorthogonal wavelet bases and the dual wavelets can be constructed of arbitrarily high regularity.

2 Biorthogonal spline wavelets

We assume in this section that $\Xi=\left[\begin{array}{cccc}\xi_1, & \xi_2, & \ldots & \xi_K\end{array}\right]$, where $K\geq N$, is a full rank unimodular matrix (i.e., all the bases in Ξ have determinant ± 1) with integer entries. The box spline φ_Ξ associated with Ξ has Fourier transform

$$\widehat{\varphi}_\Xi(\omega)=\prod_{j=1}^{K}\left(\frac{1-\exp(-i\langle\xi_j,\omega\rangle)}{i\langle\xi_j,\omega\rangle}\right).$$

In the following we will drop the subscript Ξ and we will write simply φ to denote this box spline. It is known that there exist constants C_1, $C_2 > 0$

$$C_1 \leq \sum_{k\in\mathbb{Z}^N} |\widehat{\varphi}(\omega + 2\pi k)|^2 \leq C_2$$

and that φ is the scaling function of a multiresolution analysis for any dilation factor $M \geq 2$ and translation lattice $\mathbb{Z}^N$ (see e.g. [15]). If f is defined as in (2), then we set as above

$$m_0(\omega) = \frac{\widehat{\varphi}(M\omega)}{\widehat{\varphi}(\omega)} = \prod_{j=1}^{K} f(\langle \xi_j, \omega\rangle). \tag{6}$$

Let now $p(\omega)$ be any trigonometric polynomial. To $p(\omega)$ one associates a Laurent polynomial $P(z)$, $z = (z_1, \ldots, z_N) \in (\mathbb{C}/0)^N$, in such a way that $P(e^{i\omega}) = p(\omega)$, where we made $e^{i\omega} = (e^{i\omega_1}, \ldots, e^{i\omega_N})$. In particular, we will denote by P_0 the Laurent polynomial such that $P_0(e^{i\omega}) = m_0(\omega)$. Clearly

$$P_0(z) = \prod_{j=1}^{K} F(z^{\xi_j}) \tag{7}$$

where we have adopted the usual multiindex notation and where

$$F(t) = \frac{1 + t^{-1} + t^{-2} + \cdots + t^{-(M-1)}}{M}.$$

Throughout the paper we will use the following notation. We set $E = \dfrac{\mathbb{Z}^N}{M\mathbb{Z}^N} = \{0, 1, \ldots, M-1\}^N$ (with sum modulo M) and $\vartheta = \dfrac{2\pi}{M}$. Furthermore we set

$$z \exp(i\vartheta\varepsilon) = (z_1 \exp(i\vartheta\varepsilon_1), \ldots, z_N \exp(i\vartheta\varepsilon_N)).$$

We start our analysis with the following proposition.

Proposition 1 *Let P_0 be as in (7). Then, for every integer $c_j > 0$, $j = 1, \ldots, K$, there exists a Laurent polynomial $R(z)$ such that*

$$\widetilde{P}_0(z) = \prod_{j=1}^{K} F(z^{\xi_j})^{c_j} R(z) \tag{8}$$

satisfies the relation

$$\sum_{\varepsilon\in E} P_0(z\exp(i\vartheta\varepsilon))\widetilde{P}_0(z\exp(i\vartheta\varepsilon)) = 1. \tag{9}$$

Proof. Let

$$Q(z)=\prod_{j=1}^{K}F(z^{\xi_j})^{c_j+1}.$$

Clearly, the matrix Ξ' in which the j–th column ξ_j is repeated c_j+1 times is still unimodular. Then, it was proved in [21, Proposition 1.11] (for $M=2$, but the proof is exactly the same for general M) that the set of polynomials

$$\{Q(z\exp(i\vartheta\varepsilon))\}_{\varepsilon\in E}$$

do not have common zeros. Therefore by Hilbert Nullstellensatz (exactly as in [9, Lemma 3.2.1], or in [21, Proposition 1.11]), there exists a Laurent polynomial R such that

$$\sum_{\varepsilon\in E}Q(z\exp(i\vartheta\varepsilon))R(z\exp(i\vartheta\varepsilon))=1.$$

This implies immediately (9). ■

The trigonometric polynomial

$$\widetilde{m}_0(\omega)=\overline{\widetilde{P}_0}(e^{i\omega})=\prod_{j=1}^{K}\overline{f}(\langle\xi_j,\omega\rangle)^{c_j}\overline{r}(\omega), \tag{10}$$

where $r(\omega)=R(e^{i\omega})$, is the dual filter of m_0 and satisfies (15) below (with $t=s=0$). Now we will proceed to construct the wavelet filters.

First we notice that, if P and $\widetilde{P}$ are Laurent polynomials represented by means of polyphase components in the forms

$$P=M^{-N/2}\sum_{\varepsilon\in E}P_\varepsilon(z^M)z^\varepsilon \quad \text{and} \quad \widetilde{P}=M^{-N/2}\sum_{\varepsilon\in E}\widetilde{P}_\varepsilon(z^M)z^{-\varepsilon} \tag{11}$$

(where $z^M=(z_1^M,\ldots,z_N^M)$), then clearly

$$\sum_{\varepsilon\in E}P(z\exp(i\vartheta\varepsilon))\widetilde{P}(z\exp(i\vartheta\varepsilon))=\sum_{\varepsilon\in E}P_\varepsilon(z^M)\widetilde{P}_\varepsilon(z^M). \tag{12}$$

In particular we will denote by $P_{0,\varepsilon}$ and $\widetilde{P}_{0,\varepsilon}$ the polyphase components (11) of the polynomials P_0 and $\widetilde{P}_0$ of Proposition 1. The following extension result is proved in [20] (see also [16]), but, for the sake of completeness, we will give a short account of the proof.

Proposition 2 *Let* P_0 *and* $\widetilde{P}_0$ *satisfy (9) and denote by* $P_{0,\varepsilon}$ *and* $\widetilde{P}_{0,\varepsilon}$ *their polyphase components (11). Then there exist* $M^N \times M^N$ *matrices* $\left[P_{s,\varepsilon}\right]$ *and* $\left[\widetilde{P}_{s,\varepsilon}\right]$*, whose entries are Laurent polynomials and whose first rows are* $[P_{0,\varepsilon}]$ *and* $\left[\widetilde{P}_{0,\varepsilon}\right]$*, such that* $\left[P_{s,\varepsilon}\right] \cdot \left[\widetilde{P}_{s,\varepsilon}\right]^T = I$.

Proof. We use the same notation to denote the polynomials P_0 and $\widetilde{P}_0$ and the vectors $[P_{0,\varepsilon}]$ and $\left[\widetilde{P}_{0,\varepsilon}\right]$. Taking into account (9) and (12) we can apply the Quillen–Suslin algorithm [18] to the vectors P_0 and $\widetilde{P}_0$. Then there are $M^N - 1$ vectors Q_s, having polynomial components, such that the matrix with first row P_0 and s–th row Q_s is not singular on $(\mathbb{C}/0)^N$. Set

$$P_s = Q_s - \left(Q_s \cdot \widetilde{P}_0^T\right) P_0.$$

Then $P_s = [P_{s,\varepsilon}]$ is orthogonal to $\widetilde{P}_0$. The matrix whose rows are P_0 and P_s is still nonsingular, since the equation $\lambda_0 P_0 + \sum_s \lambda_s P_s = 0$ implies

$$\left(\lambda_0 - \sum_{s=1}^{M^N-1} \left(Q_s \cdot \widetilde{P}_0^T\right)\right) P_0 + \sum_{s=1}^{M^N-1} \lambda_s Q_s = 0,$$

whence $\lambda_s = 0$ for $s = 0, 1, \ldots, M^N - 1$. Thus its determinant is a Laurent monomial, and the elements of its inverse $[H_{s,\varepsilon}]^T$ are Laurent polynomials. The first row in this matrix is necessarily $\widetilde{P}_0$. Thus, letting $\widetilde{P}_{s,\varepsilon} = H_{s,\varepsilon}$ for $s = 1, \ldots, M^N - 1$ we have the thesis. ■

Remark 1. The above proof is based on Theorem 1.1 in [18], whose proof is algorithmic, and each step of the algorithm can be implemented by means of computer algebra systems.

Let $P_{s,\varepsilon}$ and $\widetilde{P}_{s,\varepsilon}$ be as in Proposition 2. We set

$$m_s(\omega) = M^{-N/2} \sum_{\varepsilon \in E} P_{s,\varepsilon}(e^{iM\omega}) e^{i\langle \varepsilon, \omega \rangle} \tag{13}$$

$$\widetilde{m}_s(\omega) = M^{-N/2} \sum_{\varepsilon \in E} \overline{\widetilde{P}_{s,\varepsilon}}(e^{iM\omega}) e^{i\langle \varepsilon, \omega \rangle}. \tag{14}$$

Taking into account (12) and Propositions 1 and 2, we see that the filters just defined satisfy for all $\omega \in \mathbb{R}^N$ the system

$$\sum_{\varepsilon \in E} m_s(\omega + \vartheta\varepsilon) \overline{\widetilde{m}_t(\omega + \vartheta\varepsilon)} = \delta_{s,t}, \quad t, s = 0, 1, \ldots, M^N - 1 \tag{15}$$

or, equivalently,

$$\sum_{s=0}^{M^N-1} m_s(\omega)\overline{\widetilde{m}_s(\omega+\vartheta\varepsilon)} = \delta_{0,\varepsilon}, \qquad \varepsilon\in E. \tag{16}$$

In order to interpret these filters as arising from dual multiresolution analyses and biorthogonal wavelets, we need m_0 and $\widetilde{m}_0$ to be low–pass and m_s and $\widetilde{m}_s$, for $s=1,\dots,M^N-1$, to be high–pass, in the sense made precise by the following lemma.

Lemma 1 *Let m_0 be defined as in (6) and $\widetilde{m}_0$ as in (10). Moreover let m_s and $\widetilde{m}_s$ be defined by means of Proposition 2 and (13), (14) for $s\geq 1$. Then*

$$\widetilde{m}_0(0) = m_0(0) = 1, \tag{17}$$

$$\widetilde{m}_0(\vartheta\varepsilon) = m_0(\vartheta\varepsilon) = 0 \ \ \textit{for every}\ \varepsilon\neq 0, \tag{18}$$

$$\widetilde{m}_s(0) = m_s(0) = 0, \ \ \textit{for every} \quad s=1,\dots,M^N-1. \tag{19}$$

Proof. Equation $m_0(0)=1$ is obvious. To prove (18) observe that if, say, $\xi_{j_1},\dots,\xi_{j_N}$ are columns of Ξ forming a basis, then, by the unimodularity assumption, the matrix $[\xi_{j_1},\dots,\xi_{j_N}]$ can be transformed into the unit matrix by means of a change of variables with integer coefficients and determinant ±1. It follows that for every $\varepsilon\in E$ there exists a j_k such that $f(\langle\xi_{j_k},\vartheta\varepsilon\rangle)=0$, whence (18). Now, using (15) with $s=t=0$ and $\omega=0$, we get also $\widetilde{m}_0(0)=1$. Thus (17) and (18) hold. Now, we make $\omega=0$ in the system (16), and we express the $m_s(0)$ by means of the $\overline{\widetilde{m}_s(\vartheta\varepsilon)}$. We see immediately that any minor corresponding to $s>0$ and $\varepsilon=0$ must vanish, since it contains the row $\{\widetilde{m}_0(\vartheta\varepsilon)\}_{\varepsilon\neq 0}$. Thus $m_s(0)=0$. The analogous statement for $\widetilde{m}_s(0)$ follows exchanging the roles of m_s and $\widetilde{m}_s$. ■

Define $\widehat{\widetilde{\varphi}}(\omega)$ as in (5), with $\widetilde{m}_0$ as in (10). Clearly $\widetilde{\varphi}$ is a compactly supported distribution with $\widehat{\widetilde{\varphi}}(0)=1$, which is the convolution of the box spline with Fourier transform

$$\prod_{j=1}^{K}\left(\frac{1-\exp(-i\,\langle\xi_j,\omega\rangle)}{i\,\langle\xi_j,\omega\rangle}\right)^{c_j} \tag{20}$$

and the compactly supported distribution with Fourier transform

$$\prod_{j=1}^{\infty}\overline{r}(M^{-j}\omega) \tag{21}$$

where $r(\omega) = R(e^{i\omega})$. We set, for every $s = 1, \dots, M^N - 1$

$$\widehat{\psi}_s(M\omega) = m_s(\omega)\widehat{\varphi}(\omega), \qquad \widehat{\widetilde{\psi}}_s(M\omega) = \widetilde{m}_s(\omega)\widehat{\widetilde{\varphi}}(\omega). \tag{22}$$

Clearly we have

$$|\widehat{\varphi}(\omega)| \leq C \prod_{j=1}^{K} (1 + |\langle \xi_j, \omega \rangle|)^{-1}$$

where C is a constant, and $\widehat{\psi}_s$ has the same decay. On the other hand, the decay at infinity of (20) is majorized by $\prod_{j=1}^{K}(1 + |\langle \xi_j, \omega\rangle|)^{-c_j}$.

In the rest of this section we will assume that the Fourier transform (21) does not increase too much at infinity. Namely we assume that

$$\widetilde{\varphi} \in L^2(\mathbb{R}^N) \tag{23}$$

and that there exist $\sigma > 0$ and $C > 0$ such that for every sufficiently large n one has for all $\omega \in \mathbb{R}^N$

$$\prod_{h=1}^{n} \left| r(M^{-h}\omega) \right| \leq C \prod_{j=1}^{K} (1 + |\langle \xi_j, \omega \rangle|)^{c_j - \sigma} \ . \tag{24}$$

Obviously, since Ξ is full rank, if (24) holds with $\sigma > 1/2$ then (23) is satisfied as well.

Remark 2. Assumptions (23), (24) seem to be hard to verify. Nevertheless, in the following sections we will show that for a remarkable class of box splines we can prove much better estimates (Theorem 3 below), making it possible to construct square integrable $\widetilde{\varphi}$ of arbitrarily high regularity.

Let ψ and $\widetilde{\psi}$ be as in (22). We set for every $j \in \mathbb{Z}$, $k \in \mathbb{Z}^N$ and $s = 1, \dots, M^N - 1$

$$\psi_{j,k,s}(x) = M^{j/2}\psi_s(M^j x - k), \qquad \widetilde{\psi}_{j,k,s}(x) = M^{j/2}\widetilde{\psi}_s(M^j x - k). \tag{25}$$

Theorem 1 *Let φ be any box spline in $\mathbb{R}^N$ with full rank unimodular matrix Ξ having integer entries. Let $\widehat{\widetilde{\varphi}}(\omega)$ as in (5), with $\widetilde{m}_0$ as in (8) and (10). Assume that (23) and (24) hold. Then the families $\{\psi_{j,k,s}\}$ and $\left\{\widetilde{\psi}_{j,k,s}\right\}$ in (25) are biorthogonal bases of compactly supported wavelets in $L^2(\mathbb{R}^N)$.*

Proof. By (24), (17) and (19) we may apply Theorem 2.11 of Chui and Li in [4]. Thus we conclude that both systems $\{\psi_{j,k,s}\}$ and $\left\{\widetilde{\psi}_{j,k,s}\right\}$ are Bessel systems i.e., for every function $g \in L^2(\mathbb{R}^N)$

$$\sum_{j,k,s} |\langle g, \psi_{j,k,s}\rangle|^2 \le C \, \|g\|_2^2$$

$$\sum_{j,k,s} \left|\left\langle g, \widetilde{\psi}_{j,k,s}\right\rangle\right|^2 \le C \, \|g\|_2^2$$

for some constant C. Then we may apply Ron and Shen's Theorem 3.9 in [22] (or we can repeat the proof in [7, pp. 500-506]) to conclude that the two families are fundamental frames which are dual to each other. Therefore, to obtain the thesis it will be sufficient to prove that the families are biorthogonal i.e.,

$$\left\langle \psi_{j,k,s}, \widetilde{\psi}_{i,h,t}\right\rangle = \delta_{j,i}\delta_{k,h}\delta_{s,t}.$$

To this end, by [4, Theorem 2.2] (or repeating the arguments in [7, Lemma 3.7]) we have only to prove the analogous relation for the scaling functions i.e.,

$$\int_{\mathbb{R}^N} \varphi(x)\overline{\widetilde{\varphi}(x-k)}dx = \delta_{0,k}. \tag{26}$$

We will prove (26) arguing along the lines of [7]. Let us denote by Ω the fundamental domain for the box spline

$$\widehat{\varphi^\sharp}(\omega) = \prod_{j=1}^{K} \left(\frac{1-\exp(-i\,\langle \xi_j, \omega\rangle)}{i\,\langle \xi_j, \omega\rangle}\right)^{c_j+1}.$$

We recall [2, p. 106] that Ω is defined as

$$\Omega = \left\{\omega : \left|\widehat{\varphi^\sharp}(\omega + 2\pi k)\right| < \left|\widehat{\varphi^\sharp}(\omega)\right| \quad \text{for all } k \in \mathbb{Z}^N,\, k \neq 0\right\}.$$

It turns out that $\mathrm{meas}(\Omega) = (2\pi)^N$, the translates $\Omega + 2\pi k$ are pairwise disjoint and tessellate $\mathbb{R}^N$ up to a set of measure 0. Moreover $\overline{\Omega}$ is compact and, since $\varphi^\sharp$ has unimodular matrix,

$$\left|\widehat{\varphi^\sharp}(\omega)\right| \ge \kappa > 0 \quad \text{for every} \quad \omega \in \overline{\Omega}.$$

As a consequence we obtain that

$$\prod_{j=1}^{K} |\sin(\langle \xi_j, \omega/2\rangle)|^{c_j+1} \ge \kappa \prod_{j=1}^{K} |\langle \xi_j, \omega\rangle|^{c_j+1} \qquad \text{for every} \quad \omega \in \overline{\Omega}. \tag{27}$$

Define now $\widehat{\varphi}_0(\omega) = \widehat{\widetilde{\varphi}}_0(\omega) = \chi(\omega)$, where χ is the characteristic function of Ω, and, for $n = 1, 2, \ldots$

$$\widehat{\varphi}_n(\omega) = \chi(M^{-n}\omega) \prod_{h=1}^{n} m_0(M^{-h}\omega)$$
$$\widehat{\widetilde{\varphi}}_n(\omega) = \chi(M^{-n}\omega) \prod_{h=1}^{n} \widetilde{m}_0(M^{-h}\omega).$$

Since, for any periodic function g, $(2\pi)^{-N} \int_\Omega g d\omega = \int_{\mathbb{T}^N} g d\omega$ (where $\mathbb{T}^N$ is the N–dimensional torus), a standard argument based on (15) with $s = t = 0$ (cfr. [7, p. 528]) shows that

$$\int_{\mathbb{R}^N} \varphi_n(x)\overline{\widetilde{\varphi}_n(x-k)}dx = (2\pi)^{-N} \int_{\mathbb{R}^N} \widehat{\varphi}_n(\omega)\overline{\widehat{\widetilde{\varphi}}_n(\omega)} \exp(ik\omega)d\omega = \delta_{0,k}.$$

Now we look for an integrable majorant in order to apply the Lebesgue theorem. We have that $M|f(t)| = |1 - \exp(-iMt)| \, |1 - \exp(-it)|^{-1}$. Hence, by (24) and (27), we see that for some constant C

$$\begin{aligned}\left|\widehat{\varphi}_n(\omega)\widehat{\widetilde{\varphi}}_n(\omega)\right| &= \chi(M^{-n}\omega) \prod_{j=1}^{K} \left| \frac{\sin(\langle \xi_j, \omega/2 \rangle)}{M^n \sin(\langle \xi_j, M^{-n}\omega/2 \rangle)} \right|^{c_j+1} \prod_{j=1}^{n} \left| r(M^{-j}\omega) \right| \\ &\leq \kappa^{-1} \prod_{j=1}^{K} \left| \frac{\sin(\langle \xi_j, \omega/2 \rangle)}{\langle \xi_j, \omega \rangle} \right|^{c_j+1} \prod_{j=1}^{n} \left| r(M^{-j}\omega) \right| \\ &\leq C \prod_{j=1}^{K} \left(1 + |\langle \xi_j, \omega \rangle|\right)^{-1-\sigma}.\end{aligned}$$

As $\varphi_n \to \varphi$ and $\widehat{\varphi}_n \to \widehat{\varphi}$ pointwise, (26) follows from Lebesgue theorem. ■

3 Construction of the dual filter

Proposition 1 provides an abstract construction of the dual filter, but does not give information on the form of the "residual filter" r nor on the regularity of the dual scaling function. On the other hand, Theorem 1 proves that $\{\psi_{j,k,s}\}$ and $\left\{\widetilde{\psi}_{j,k,s}\right\}$ in (25) are biorthogonal bases provided that a decay condition on the Fourier transform side is satisfied. In this section we restrict the scope of our investigation, considering only spline filters of the form (3). For such filters we construct dual filters $\widetilde{m}_0$ of the form (4), by giving explicitly the expression of the trigonometric polynomial r. In the next section we will show that for such filters, the assumptions of Theorem 1 are satisfied, thus obtaining biorthogonal compactly supported wavelets. At the same time, we will show that the resulting dual scaling function $\widetilde{\varphi}$ can be constructed of arbitrarily high regularity.

Our starting point are the polynomials studied in [24] and [23] in the context of univariate compactly supported M–band wavelets (cfr. also [16]). We set

$$\begin{aligned} S_M &= \{(s_0, \ldots, s_{M-1}) \ : \ s_i \geq 0, \ s_0 + \cdots + s_{M-1} = 1\} \\ T_M &= \{(s_0, \ldots, s_{M-1}) \in S_M \ : \ s_0 \geq 1/M\}, \end{aligned}$$

and we denote by χ_M the characteristic function of T_M. We define the function g on S_M as

$$\begin{aligned} & g(s_0, \ldots, s_{M-1}) \\ & = \frac{\chi_M(s_0, \ldots, s_{M-1})}{\chi_M(s_0, \ldots, s_{M-1}) + \chi_M(s_1, s_2, \ldots, s_0) + \cdots + \chi_M(s_{M-1}, s_0 \ldots, s_{M-2})}. \end{aligned}$$

Note that the value of g at $(s_0, \cdots, s_{M-1}) \in T_M$ is the reciprocal of the number of variables s_i not smaller than $1/M$. We observe the following properties of g on S_M

$$g(s_0, \ldots, s_{M-1}) + g(s_{M-1}, s_{0,} \ldots, s_{M-2}) + \cdots + g(s_1, s_2, \ldots, s_0) = 1 \tag{28}$$

$$\begin{aligned} \frac{1}{M} \leq g(s_0, \ldots, s_{M-1}) \leq 1 \text{ if } s_0 \geq 1/M, \\ g(s_0, \ldots, s_{M-1}) = 0 \text{ if } s_0 < 1/M. \end{aligned} \tag{29}$$

For every integer $k \geq 2$ we define the polynomials Q_k as

$$\begin{aligned} & Q_k(y_0, \ldots, y_{M-1}) \\ & = \sum_{|j|=M(k-1)+1} \tbinom{M(k-1)+1}{j} g\left(\tfrac{j_0}{M(k-1)+1}, \ldots, \tfrac{j_{M-1}}{M(k-1)+1}\right) y_0^{j_0-k} y_1^{j_1} \cdots y_{M-1}^{j_{M-1}} \end{aligned} \tag{30}$$

(where $j = (j_0, \ldots, j_{M-1})$, $|j|$ is the length of the multindex j and $j! = j_0! \cdots j_{M-1}!$).

Taking (28) and (29) into account it is not difficult to check that

$$\begin{aligned} (y_0 + \cdots + y_{M-1})^{M(k-1)+1} = y_0^k Q_k(y_0, \ldots, y_{M-1}) + \\ + y_1^k Q_k(y_1, y_2, \ldots, y_0) + \cdots + y_{M-1}^k Q_k(y_{M-1}, y_0, \ldots, y_{M-2}). \end{aligned} \tag{31}$$

Actually, a moment's reflection shows that for $y_0 + \cdots + y_{M-1} = 1$ the polynomial $y_0^k Q_k(y_0, \ldots, y_{M-1})$ is nothing else but the Bernstein polynomial of order $M(k-1)+1$ of the function g on the M–dimensional simplex [19, p. 51]. In the case $M = 2$ we have

$$\begin{aligned} Q_k\left(1-y, y\right) &= \sum_{j=k}^{2k-1} \binom{2k-1}{j} (1-y)^{j-k} y^{2k-1-j} \\ &= \sum_{j=0}^{k-1} \binom{k-1+j}{j} y^j \end{aligned} \tag{32}$$

so that the polynomial $P_k(y) = Q_k\left(1-y, y\right)$ is the well known Daubechies polynomial [10]. The relation of the Daubechies polynomials with the Bernstein polynomials was pointed out in [3] and [17].

Let f be as in (2), and let, as in the foregoing section, $\vartheta = 2\pi/M$. Let us set

$$y_0(t,u) = f(t)f(u) = M^{-2} \sum_{a,b=0}^{M-1} \exp[-i(at+bu)] \tag{33}$$

and, for every $\ell = 1, \ldots, M-1$,

$$y_\ell(t,u) = y_0(t+\ell\vartheta,\, u+(M-\ell)\vartheta) = f\left(t+\ell\vartheta\right) f(u+(M-\ell)\vartheta). \tag{34}$$

Finally, we set

$$A_k(t,u) = Q_k(y_0(t,u), \ldots, y_{M-1}(t,u)), \tag{35}$$

where Q_k is as in (30). Using (33)–(34), the A_k can be written explicitly as

$$\begin{aligned} A_k(t,u) = &\sum_{|j|=M(k-1)+1} \tbinom{M(k-1)+1}{j} g\left(\tfrac{j_0}{M(k-1)+1}, \ldots, \tfrac{j_{M-1}}{M(k-1)+1}\right) \times \\ &\times \left(\frac{e^{-i(M-1)(t+u)/2}}{M^2} \frac{\sin\frac{Mt}{2}}{\sin\frac{t}{2}} \frac{\sin\frac{Mu}{2}}{\sin\frac{u}{2}} \right)^{j_0-k} \times \\ &\times \prod_{\ell=1}^{M-1} \left(\frac{e^{-i(M-1)(t+u)/2}}{M^2} \frac{\sin\left(\frac{Mt}{2}\right)}{\sin\left(\frac{t}{2}+\frac{\ell\pi}{M}\right)} \frac{\sin\left(\frac{Mu}{2}\right)}{\sin\left(\frac{u}{2}-\frac{\ell\pi}{M}\right)} \right)^{j_\ell} \end{aligned}$$

since $f(t) = M^{-1} e^{-i(M-1)t/2} \sin\frac{Mt}{2} / \sin\frac{t}{2}$.

We have the following lemma.

Lemma 2 *With notation as above we have for every* $k \geq 2$

$$\sum_{\ell=0}^{M} \left[f\left(t+\ell\vartheta\right) f(u+(M-\ell)\vartheta)\right]^{k} A_k(t{+}\ell\vartheta,\ u{+}(M{-}\ell)\vartheta) = [f(t+u)]^{M(k-1)+1}\,.$$

Proof. The Lemma is a consequence of (31). For every $t, u \in \mathbb{R}$ and $\ell = 1, \ldots, M-1$ we have

$$M^2 f\left(t+\ell\vartheta\right) f(u+(M-\ell)\vartheta) = \sum_{a,b=0}^{M-1} \exp[-i(at+a\ell\vartheta+bu+b(M-\ell)\vartheta)].$$

Summing up all these relations we obtain

$$\begin{aligned} & M^2 \sum_{\ell=0}^{M-1} f\left(t+\ell\vartheta\right) f(u+(M-\ell)\vartheta) \\ & = \sum_{a,b=0}^{M-1} \exp[-i(at+bu)] \sum_{\ell=0}^{M-1} \exp[-i\vartheta(a-b)\ell)] \qquad (36) \\ & = M \sum_{a=0}^{M-1} \exp[-ia(t+u)] = M^2 f(t+u). \end{aligned}$$

Now, since $y_h(t+\ell\vartheta, u+(M-\ell)\vartheta) = y_{\ell+h}(t,u)$ (where the sum $\ell+h$ in the subscript is $\bmod M$), we have

$$A_k(t+\ell\vartheta,\ u+(M-\ell)\vartheta) = Q_k(y_\ell(t,u), y_{\ell+1}(t,u), \ldots, y_{\ell-1}(t,u)).$$

Thus the thesis of the lemma follows from (31) and (36). ■

Let us introduce some more notation. First we have that

$$f(t)^2 = \frac{1}{M^2}\frac{\sin^2\left(\frac{Mt}{2}\right)}{\sin^2\left(\frac{t}{2}\right)} \exp[-i(M-1)t] = \left|f(t)\right|^2 \exp[-i(M-1)t]$$

and that $\sum_{j=0}^{M-1} \left|f(t+j\vartheta)\right|^2 = 1$. Then we set

$$\begin{aligned} B_k(t) &= \exp[ik(M-1)t] Q_k(\left|f(t)\right|^2, \ldots, \left|f(t+(M-1)\vartheta)\right|^2) \\ &= e^{i(M-1)kt} \sum_{|j|=M(k-1)+1} \tbinom{M(k-1)+1}{j} g\left(\tfrac{j_0}{M(k-1)+1}, \ldots, \tfrac{j_{M-1}}{M(k-1)+1}\right) \times \qquad (37) \\ &\times \left(\frac{\sin^2\left(\frac{Mt}{2}\right)}{M^2\sin^2\left(\frac{t}{2}\right)}\right)^{j_0-k} \prod_{\ell=1}^{M-1} \left(\frac{\sin^2\left(\frac{Mt}{2}\right)}{M^2\sin^2\left(\frac{t}{2}+\frac{\ell\pi}{M}\right)}\right)^{j_\ell}. \end{aligned}$$

Using (31) and arguing much in the same way as in the preceding lemma, one proves easily the following result.

Lemma 3 *With notation as above we have for every* $k \geq 2$

$$\sum_{\ell=0}^{M-1} f\,(t+\ell\vartheta)^{2k}\, B_k(t+\ell\vartheta) = 1.$$

Remark 4. If $M = 2$, then $B_k(t) = \exp(ikt)P_k\left(\sin^2 t/2\right)$, where P_k is as in (32). Hence, the filters $f(t)^{2k}B_k(t)$ can be considered as an extension to the case $M > 2$ of Daubechies filters.

Now we can state the main result of this section.

Theorem 2 *Let* $N > 1$ *and*

$$m_0(\omega) = \prod_{k=1}^{N} f(\omega_k)^{a_k} \prod_{k=2}^{N} f(\omega_1+\cdots+\omega_k)^{b_k} \tag{38}$$

where f *is as in (2),* $a_k > 0$ *and* $b_k \geq 0$ *are integers. Set*

$$\widetilde{m}_0(\omega) = \prod_{k=1}^{N} \overline{f}(\omega_k)^{\alpha_k - a_k} \prod_{k=2}^{N} \overline{f}(\omega_1+\cdots+\omega_k)^{\beta_k - b_k}\overline{r}(\omega) \tag{39}$$

with

$$r(\omega) = B_{\alpha_{N+1}}(\omega_1+\cdots+\omega_N) \prod_{k=1}^{N-1} A_{\alpha_{k+1}}(\omega_1+\cdots+\omega_k, \omega_{k+1}) \tag{40}$$

where α_k,$\beta_k > 0$ *are integers and the trigonometric polynomials* A *and* B *are defined as in (35) and (37) respectively. Then for every* $\omega \in \mathbb{R}^N$*,*

$$\sum_{\varepsilon \in E} m_0(\omega+\vartheta\varepsilon)\overline{\widetilde{m}_0(\omega+\vartheta\varepsilon)} = 1$$

provided that

$$\begin{cases} \alpha_1 = \alpha_2 \\ \alpha_{k+1} = \beta_k + M(\alpha_k - 1) + 1 & \text{if } 2 \leq k \leq N-1 \\ 2\alpha_{N+1} = \beta_N + M(\alpha_N - 1) + 1. \end{cases} \tag{41}$$

Proof. Let

$$\begin{aligned} h(\omega) &= m_0(\omega)\overline{\widetilde{m}_0(\omega)} \\ &= \prod_{k=1}^{N} f(\omega_k)^{\alpha_k} \prod_{k=2}^{N} f(\omega_1+\cdots+\omega_k)^{\beta_k} \times \\ &\times B_{\alpha_{N+1}}(\omega_1+\cdots+\omega_N) \prod_{k=1}^{N-1} A_{\alpha_{k+1}}(\omega_1 \cdots + \omega_k, \omega_{k+1}). \end{aligned}$$

Then

$$h(\omega) = D(\omega) \prod_{k=1}^{N-1} C_k(\omega)$$

where, for every $k = 1, \dots, N-1$,

$$C_k(\omega) = f(\omega_1 + \cdots + \omega_k)^{\beta_k} f(\omega_{k+1})^{\alpha_{k+1}} A_{\alpha_{k+1}}(\omega_1 + \cdots + \omega_k, \omega_{k+1})$$

and

$$D(\omega) = f(\omega_1 + \cdots + \omega_N)^{\beta_N} B_{\alpha_{N+1}}(\omega_1 + \cdots + \omega_N).$$

Here we set $\beta_1 = \alpha_1$. For every $k \geq 2$ we consider the elements $\varepsilon^{(k)}$ of E with the first coordinate equal to 1, the $k-$th equal to $M-1$ and the remaining coordinates equal to 0, i.e.

$$\varepsilon^{(k)} = (1, 0, \dots, M-1, 0, \dots, 0).$$

Set

$$\begin{aligned} E_2 &= \left\{\varepsilon \in E \;:\; \varepsilon = j\varepsilon^{(2)},\ j = 0, \dots, M-1\right\} \\ &= \{\varepsilon \in E \;:\; \varepsilon = (j, M-j, 0, \dots, 0),\ j = 0, \dots, M-1\}. \end{aligned}$$

Clearly $|E_2| = M$, where $|\cdot|$ denotes the cardinality. We have

$$\sum_{\varepsilon \in E_2} h(\omega + \vartheta\varepsilon) = \sum_{\varepsilon \in E_2} C_1(\omega + \vartheta\varepsilon) \prod_{k=2}^{N-1} C_k(\omega) D(\omega)$$

and, since $\alpha_1 = \alpha_2$, by Lemma 2 we obtain

$$\begin{aligned} &\sum_{\varepsilon \in E_2} C_1(\omega + \vartheta\varepsilon) \\ &= \sum_{j=0}^{M-1} f(\omega_1 + j\vartheta)^{\alpha_1} f(\omega_2 + (M-j)\vartheta)^{\alpha_2} A_{\alpha_2}(\omega_1 + j\vartheta, \omega_2 + (M-j)\vartheta) \\ &= f(\omega_1 + \omega_2)^{M(\alpha_2 - 1) + 1}. \end{aligned}$$

Using the relation $\alpha_3 = \beta_2 + M(\alpha_2 - 1) + 1$ we can also write

$$\sum_{\varepsilon \in E_2} h(\omega + \vartheta\varepsilon) = f(\omega_1 + \omega_2)^{\alpha_3} f(\omega_3)^{\alpha_3} A_{\alpha_3}(\omega_1 + \omega_2, \omega_3) \prod_{k=3}^{N-1} C_k(\omega) D(\omega).$$

We consider now for every $l = 0, \dots, M-1$ the sets

$$E_2 + l\varepsilon^{(3)} = \{\ \varepsilon = (j + l, M - j, M - l, 0 \dots, 0),\ j = 0, \dots, M-1\}.$$

Then, obviously

$$\sum_{\varepsilon\in E_2+l\varepsilon^{(3)}} h(\omega+\vartheta\varepsilon) = \sum_{\varepsilon\in E_2} h(\omega+l\varepsilon^{(3)}\vartheta+\varepsilon\vartheta)$$

$$= f(\omega_1+\omega_2+l\vartheta)^{\alpha_3} f(\omega_3+(M-l)\vartheta)^{\alpha_3}\times$$

$$\times A_{\alpha_3}(\omega_1+\omega_2+l\vartheta,\omega_3+(M-l)\vartheta)\prod_{k=3}^{N-1} C_k(\omega)D(\omega).$$

We define now the set

$$E_3=\bigcup_{l=0}^{M-1}\left(E_2+l\varepsilon^{(3)}\right)$$

which consists of the elements of E with at most the first three coordinates different from zero and having sum equal to 0. Obviously $|E_3|=M^2$. Then, using again Lemma 2 and the definition of α_4 we get

$$\sum_{\varepsilon\in E_3} h(\omega+\vartheta\varepsilon) = f(\omega_1+\omega_2+\omega_3)^{\alpha_4} f(\omega_4)^{\alpha_4} A_{\alpha_4}(\omega_1+\omega_2+\omega_3,\omega_4)\times$$

$$\times\prod_{k=4}^{N-1} C_k(\omega)D(\omega).$$

At the j–th step we define recursively the set E_j $(j=2,\ldots,N)$ as

$$E_j=\bigcup_{l=0}^{M-1}\left(E_{j-1}+l\varepsilon^{(j)}\right).$$

Notice that $|E_j|=M^{j-1}$ and that E_j consists of the elements of E with at most the first j coordinates different from zero and having sum equal to 0. Furthermore,

$$\sum_{\varepsilon\in E_j} h(\omega+\vartheta\varepsilon) = f(\omega_1+\cdots+\omega_j)^{\alpha_{j+1}} f(\omega_{j+1})^{\alpha_{j+1}} A_{\alpha_{j+1}}(\omega_1+\cdots+\omega_j,\omega_{j+1})\times$$

$$\times\prod_{k=j+1}^{N-1} C_k(\omega)D(\omega).$$

At the last step (using also the definition of α_{N+1}) we obtain

$$\sum_{\varepsilon\in E_N} h(\omega+\vartheta\varepsilon) = f(\omega_1+\cdots+\omega_N)^{2\alpha_{N+1}} B_{\alpha_{N+1}}(\omega_1+\cdots+\omega_N),$$

where E_N is the subgroup of the elements of E whose sum is 0 $\bmod M$. Consider now the elements of E of the form

$$\eta^{(j)} = (j, 0, \ldots, 0) \qquad j = 0, \ldots, M-1.$$

Then E is the disjoint union of the cosets $E_N + \eta^{(j)}$. Hence, it follows Lemma 3 that

$$\begin{aligned}\sum_{\varepsilon \in E} h(\omega + \vartheta\varepsilon) &= \sum_{j=0}^{M-1} \sum_{\varepsilon \in E_N + \eta^{(j)}} h(\omega + \vartheta\varepsilon) \\ &= \sum_{j=0}^{M-1} f(\omega_1 + \cdots + \omega_N + j\vartheta)^{2\alpha_{N+1}} B_{\alpha_{N+1}}(\omega_1 + \cdots + \omega_N + j\vartheta) \\ &= 1.\end{aligned}$$

■

Remark 4. In the case $N = 1$ conditions (41) must simply be replaced by the condition $2\alpha_2 = \alpha_1$.

4 Regularity

In this section we show that, for fixed exponents a_k and b_k in m_0 (38) it is possible to choose α_k and β_k in $\widetilde{m}_0$ (39) in such a way that $\widehat{\widetilde{\varphi}}$ has arbitrarily fast (polynomial) decay. At the same time we will show that (24) can be satisfied. To this end we will use the regularity exponent of a filter, a notion thoroughly investigated in the literature; see e.g., [5], [8], [10], [26] in the case $M = 2$ and [1], [14], [16], [23], [24], [25] in the general case.

Let $p(t)$ be a trigonometric polynomial such that $p(k\vartheta) = \delta_{k,0}$ for $k = 0, \ldots, M-1$. Then, there exists a positive integer L such that

$$p(t) = \left(\frac{1 + e^{-it} + e^{-i2t} + \cdots + e^{-i(M-1)t}}{M} \right)^L P(t)$$

where P is a trigonometric polynomial such that $P(0) = 1$ and $P(k\vartheta) \neq 0$ for $k \neq 0$. Set

$$\mathcal{K}_\ell = \mathcal{K}_\ell(P) = \frac{1}{\ell} \log_M \left(\sup_t \prod_{h=1}^{\ell} \left| P(M^{-h} t) \right| \right),$$

$$\mathcal{K} = \mathcal{K}(P) = \inf_\ell \mathcal{K}_\ell(P) = \lim_{\ell \to \infty} \mathcal{K}_\ell(P).$$

The number $\mathcal{K}(P)$ is called the regularity exponent of P. It is known (see e.g [8, p.82]) that for every ℓ there exists a constant C_ℓ such that for every $n \geq \ell$ one has

$$\prod_{h=1}^{n} \left|P(M^{-h}t)\right| \leq C_\ell (1+|t|)^{\mathcal{K}_\ell}. \tag{42}$$

Therefore we have $\left|\prod_{h=1}^{\infty} p(M^{-h}t)\right| \leq C_\ell (1+|t|)^{-L+\mathcal{K}_\ell}$, so that, depending on the value of $\mathcal{K}$, the infinite product belongs to various functional spaces. We shall use (42) to prove not only the validity of (24) but also the regularity of the dual scaling function.

We start recalling that for

$$\begin{aligned} P_s(t) = \sum_{|j|=M(s-1)+1} & \binom{M(s-1)+1}{j} g\left(\tfrac{j_0}{M(s-1)+1}, \dots, \tfrac{j_{M-1}}{M(s-1)+1}\right) \times \\ & \times \left(\frac{\sin^2\left(\frac{Mt}{2}\right)}{M^2 \sin^2\left(\frac{t}{2}\right)}\right)^{j_0 - s} \prod_{\ell=1}^{M-1} \left(\frac{\sin^2\left(\frac{Mt}{2}\right)}{M^2 \sin^2\left(\frac{t}{2} + \frac{\ell\pi}{M}\right)}\right)^{j_\ell} \end{aligned} \tag{43}$$

the following asymptotic estimates for $\mathcal{K}(P_s)$ are known. If $M = 2$ then, as $s \to \infty$, $\mathcal{K}(P_s) \sim s \log_2 3$ [5], [26]. If $M = 3$, then $\mathcal{K}(P_s) \sim s(-1 + \log_3 16)$, [23, Theorems 7 and 9]. Finally, if $M > 3$ then, for s sufficiently large one has $\mathcal{K}(P_s) < s\left(1 + (M-1)\log_M\left(\frac{M}{M-1}\right)\right)$, [24], [23, Theorem 7].

Thus we set

$$\rho_M = \begin{cases} \log_2 3 & \text{if } M = 2, \\ -1 + \log_3 16 & \text{if } M = 3, \\ 1 + (M-1)\log_M\left(\frac{M}{M-1}\right) & \text{if } M > 3. \end{cases}$$

Note that

$$0 < \rho_M < 2 \quad \text{for all } M. \tag{44}$$

Theorem 3 *Let $\widetilde{m}_0$ be as in (39) with r as in (40). Then, for every $\mu > 0$ there exist a constant $C > 0$ and exponents α_k and β_k satisfying conditions (41) of Theorem 2 such that for every sufficiently large n one has*

$$\prod_{h=1}^{n} \left|r(M^{-h}\omega)\right| \leq C \prod_{k=1}^{N} (1+|\omega_k|)^{\alpha_k - a_k - \mu} \times \prod_{k=2}^{N} (1 + |\omega_1 + \cdots + \omega_k|)^{\beta_k - b_k - \mu}.$$

Proof. Let P_s be as in (43) above. Then, remembering the explicit form of A_k and using Schwartz inequality we have for every $k \leq N-1$

$$\left|A_{\alpha_{k+1}}(\omega_1 + \cdots + \omega_k, \omega_{k+1})\right| \leq P^{1/2}_{\alpha_{k+1}}(\omega_1 + \cdots + \omega_k) P^{1/2}_{\alpha_{k+1}}(\omega_{k+1}).$$

Of course, $|B_{\alpha_{N+1}}(\omega_1+\cdots+\omega_N)| = P_{\alpha_{N+1}}(\omega_1+\cdots+\omega_N)$. Thus

$$\prod_{h=1}^{n} \left|r(M^{-h}\omega)\right| \le \prod_{k=1}^{N}\prod_{h=1}^{n} P_{\alpha_k}^{1/2}(M^{-h}\omega_k) \prod_{k=2}^{N-1}\prod_{h=1}^{n} P_{\alpha_{k+1}}^{1/2}(M^{-h}(\omega_1+\cdots+\omega_k))\times$$
$$\times \prod_{h=1}^{n} P_{\alpha_{N+1}}(M^{-h}(\omega_1+\cdots+\omega_N)).$$

Set $\mathcal{K}(\ell,s) = \mathcal{K}_\ell(P_s)$ and $\mathcal{K}(s) = \mathcal{K}(P_s)$. Then, for every ℓ, if $n \ge \ell$ we get

$$\prod_{h=1}^{n} \left|r(M^{-h}\omega)\right| \le C_\ell \prod_{k=1}^{N} (1+|\omega_k|)^{\mathcal{K}(\ell,\alpha_k)/2}\times$$
$$\times \prod_{k=2}^{N-1} (1+|\omega_1+\cdots+\omega_k|)^{\mathcal{K}(\ell,\alpha_{k+1})/2} \times (1+|\omega_1+\cdots+\omega_N|)^{\mathcal{K}(\ell,\alpha_{N+1})}.$$

Choose ε such that $\rho_M(1+\varepsilon) < 2$ (this is possible by (44)). Then, choose the α_k so large that $\mathcal{K}(\alpha_k) \le \alpha_k\rho_M(1+\varepsilon/2)$. For this choice of the α_k choose ℓ so large that $\mathcal{K}(\ell,\alpha_k) \le \mathcal{K}(\alpha_k) + \varepsilon\alpha_k\rho_M/2$. Then we get from (45)

$$\prod_{h=1}^{n} \left|r(M^{-h}\omega)\right| \le C \prod_{k=1}^{N} (1+|\omega_k|)^{\alpha_k\rho_M(1+\varepsilon)/2}\times$$
$$\times \prod_{k=2}^{N-1} (1+|\omega_1+\cdots+\omega_k|)^{\alpha_{k+1}\rho_M(1+\varepsilon)/2} \times (1+|\omega_1+\cdots+\omega_N|)^{\alpha_{N+1}\rho_M(1+\varepsilon)}. \tag{45}$$

To complete the proof we have to show that there exist β_k and arbitrarily large α_k which satisfy (41) and

$$\begin{aligned}
\alpha_k\rho_M\frac{(1+\varepsilon)}{2} &\le \alpha_k - a_k - \mu\ , \quad && 1 \le k \le N\\
\alpha_{k+1}\rho_M\frac{(1+\varepsilon)}{2} &\le \beta_k - b_k - \mu, && 2 \le k \le N-1\\
2\alpha_{N+1}\rho_M\frac{(1+\varepsilon)}{2} &\le \beta_N - b_N - \mu && k = N.
\end{aligned}$$

Let $\alpha_1 > 0$ be such that $\alpha_1\left(1-\rho_M\frac{(1+\varepsilon)}{2}\right) > \mu + \max_k(a_k)$; this is possible by (44). Then every $\alpha_k \ge \alpha_1$ satisfies the first set of inequalities. By defining β_k by means of (41), the second set of inequalities can be recast as

$$\alpha_{k+1}\left(1-\rho_M\frac{(1+\varepsilon)}{2}\right) \ge M(\alpha_k-1)+1+b_k+\mu$$

which can be recursively satisfied in the range $\alpha_k \geq \alpha_1$. Analogously, we see that the last inequality can be satisfied for large values of α_{N+1}. ■

Finally we get that $\widetilde{\varphi}$ can have Fourier transform with arbitrarily fast decay in the directions of Ξ.

Corollary 1 *Let* $\widetilde{m}_0$ *be as in (39) with* r *as in (40). Moreover let* $\widehat{\widetilde{\varphi}}(\omega) = \prod_{j=1}^{\infty} \widetilde{m}_0(M^{-j}\omega)$. *Then, for every* $\mu > 0$ *there exist a constant* $C > 0$ *and exponents* α_k *and* β_k *satisfying conditions (41) of Theorem 2 such that*

$$\left|\widehat{\widetilde{\varphi}}(\omega)\right| \leq C \prod_{k=1}^{N}(1+|\omega_k|)^{-\mu} \prod_{k=2}^{N}(1+|\omega_1+\cdots+\omega_k|)^{-\mu}.$$

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