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**ON THE POINT LIMIT OF  
THE PAULI–FIERZ MODEL**

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**Abstract.** We continue in this paper the analysis, begun in [NP], of the classical dynamics of the point limit of the Maxwell–Lorentz system in dipole approximation ( the Pauli–Fierz model ). Here, as a first step towards considering the full nonlinear system, we study the case in which a nonlinear external field of force is present. We study the flow of the regularized ( namely with an extended particle ) system, and show that it converges in the appropriate norm, as the radius of the particle tends to zero, to the flow of a closed coupled system of equations, containing the renormalized mass only, which so provides the very definition of the dynamics of the system in the point limit. The Abraham–Lorentz–Dirac equation for the particle position is deduced and turns out to be, in this description, a boundary condition on the vector potential, giving the evolution of its singularity. Moreover, the Hamiltonian structure of the limit system is displayed, and it is shown that the standard Hamiltonian of the Pauli–Fierz model converges to the Hamiltonian of the limit system here given.

**Résumé.** Nous continuons dans ce papier l’analyse, commencée dans le papier [NP], de la dynamique classique du système de Maxwell–Lorentz à la limite ponctuelle dans l’approximation dipolaire (model de Pauli–Fierz). Ici, comme premier pas vers la considération du système complet nonlinéaire, nous étudions le cas dans lequel un champs extérieur de forces est présent. Nous étudions le flot du système régularisé (c’est à dire avec une particule étendue), et montrons qu’il converge dans la norme appropriée, lorsque le rayon de la particule tends vers zéro, au flot d’un système couplé d’equations, contenant seulement la masse renormalisée, qui définit la dynamique même du système dans la limite ponctuelle. Il arrive alors que dans cette description l’equation de Abraham–Lorentz–Dirac pour la position de la particule se présente comme une condition à la frontière pour le potentiel vectoriel, en donnant l’évolution de sa singularité. Enfin, nous donnons la structure Hamiltonienne du système limite, et montrons que l’Hamiltonienne du model de Pauli–Fierz converge à celle du système limite.

## 1. Introduction.

In a recent contribution ( [NP] ) the authors succeeded in defining the dynamics of the system constituted by a *free* charged *point* particle interacting with the electromagnetic field in the dipole ( or linearized ) approximation. As it is well known, this is a highly nontrivial problem due to the fact that the Maxwell–Lorentz system, which of course should be used to describe the dynamics, is, from a rigorous point of view, meaningless when a point particle is considered. The analogous problem shows up in quantum electrodynamics too, where it is at the origin of the divergences which led to the introduction of perturbative renormalization theory. The usual way out consists in trying to define a dynamics by taking a suitable limit ( “point limit” ) on a regularized version of the system itself, typically obtained by attributing a formfactor  $\rho_r$  to the particle. The regularized system so obtained is often called the “Pauli–Fierz model” ( [PF], [Bl], [A] ). In [NP] it is shown how the flow of the regularized system converges to the flow of a well defined dynamical system if and only if mass renormalization is performed, and a suitable constraint ( conserved by the evolution ) between the initial data of vector potential and particle velocity is imposed. More precisely, the limit system for the vector potential  $A$  ( which in the Coulomb gauge is the only relevant variable for the field ) and the particle position  $q$  has the form ( with  $c$  denoting the speed of light )

$$\begin{cases} \frac{1}{c^2}\ddot{A} = -H_m A \\ \dot{q} = Q_A \\ A(0) = A_0, \quad Q_{A_0} = \dot{q}_0, \quad \dot{A}(0) = \dot{A}_0, \quad q(0) = q_0 . \end{cases} \quad (1.1)$$

Here  $H_m$  is a selfadjoint operator, bounded from below and containing the renormalized mass only; such an operator turns out to be of the class of the so called point interactions ( [AGHH] ) and its properties for the case at hand are recalled below ( see theorems 2.1 and 2.2 ).  $Q_A$  is a quantity characterizing the singularity of the generic element  $A$  of the form domain of the operator  $H_m$ , according to the formula

$$Q_A = \frac{3c}{2e} \lim_{r \downarrow 0} r \frac{1}{4\pi r^2} \int_{S_r} A(x) d\mu_r(x) , \quad (1.2)$$

where  $S_r$  denotes the sphere of radius  $r$  about the origin,  $\mu_r$  is the corresponding surface measure, and  $e$  is the particle electric charge.

So the dynamics of the system is completely specified by solving the abstract wave equation for the field given by the first of (1.1), corresponding to initial data in a suitable phase space, and then recovering the time evolution of the particle position from the second equation of the system (1.1) with the aid of formula (1.2).

The main goal would be, of course, the study of the point limit for the complete and relativistic Maxwell–Lorentz system. But this is a problem which, up to now, we are not able to tackle. As a first step, in the present paper, still remaining in the framework of the dipole approximation ( i.e. linearization of the interaction ), we consider the case in which a nonlinearity is introduced through an external nonlinear force field  $F(q)$  acting on the particle. This still constitutes a non trivial generalization because it is not possible to prove convergence by using the same techniques exploited in [NP] which were essentially based on linear methods. In particular a fixed point method, combined with some uniform ( w.r.t. the particle radius  $r$  ) estimates, is needed.

By the way, a further generalization is introduced in as much as it occurs that system (1.1) turns out to describe only situations in which the total linear momentum vanishes; this is due to the fact that the constraint between initial data for particle velocity and vector potential, which in some form is necessary in order to obtain a limit at all, was chosen in [NP] in a form stronger than needed. Such a limitation will be removed here.

The procedure of the present novel approach can be briefly described as follows. Taking inspiration from the Hamiltonian structure of the regularized Pauli–Fierz equations, we write ( see (3.2) ) a first order system equivalent to the original one, introducing as supplementary variables the total linear momentum and the electric field

$$p_r = m_r \dot{q}_r + \frac{e}{c} \langle \rho_r, A_r \rangle, \quad E_r = \frac{1}{4\pi c^2} \dot{A}_r.$$

It turns out that, just as in the free case, a limit system exists if and only if mass is renormalized according to the traditional prescription

$$m_r := m - \frac{8\pi e^2}{3c^2} E(\rho_r), \quad (1.3)$$

where  $E(\rho_r)$  is the electrostatic energy of energy of the distribution  $\rho_r$ ; this energy as it is well known, diverges to  $+\infty$  as  $r \downarrow 0$ , so that, as a consequence, the bare mass  $m_r$  diverges to  $-\infty$  in the same limit. With this mass renormalization, the limit system is

$$\left\{ \begin{array}{l} \dot{A} = 4\pi c^2 E \\ \dot{E} = -\frac{1}{4\pi} H_{m,p} A \\ \dot{q} = Q_A \\ \dot{p} = F(q) \\ A(0) = A_0, \quad E(0) = E_0, \quad q(0) = q_0, \quad p(0) = p_0 \end{array} \right. \quad (1.4)$$

( see theorem 3.4 ). In this set of equations  $H_{m,p}$  is a family of affine operators parametrized by total linear momentum  $p$ , and defined ( see (3.3) and lemma 4.1 ) in terms of the standard Laplacian with one point interaction, namely the operator  $H_m$  already studied in the quoted paper. The system reduces to (1.1) when  $F(q) = 0$ , as it is expected, but *only with the initial condition*  $p_0 = 0$ . Indeed, as already mentioned above, the case treated in [NP] is a particular one corresponding to solutions of the coupled system with vanishing total linear momentum ( see remark 3.6 ).

Of course, due to the its nonlinearity, it is not possible to write down explicitly the flow of system (1.4) as was done for the free case in [NP]. Nevertheless, an alternative and expressive picture of the dynamics is possible for strong solutions of the Cauchy problem (1.4). In fact, exploiting an equivalent definition of the domain of the operator  $H_{m,p}$ , in which the dynamics of the particle appears as a boundary condition on the vector potential ( see lemma 4.1 ), it turns out that the these solutions coincide with the solutions, for the same initial data, of the Cauchy problem for the system

$$\left\{ \begin{array}{l} \frac{1}{c^2} \ddot{A} = \Delta A + \frac{4\pi e}{c} M \dot{q} \delta_0 \\ \ddot{q}(t) = c\sqrt{\lambda_0} \dot{q}(t) + \frac{3c^2}{2e} A_f(t, 0) - \frac{3c^3}{2e^2} p(t) \\ \dot{p} = F(q) \\ A(0) = A_0, \quad \dot{A}(0) = \dot{A}_0, \quad q(0) = q_0, \quad \dot{q}_0 = Q_{A_0}, \quad p(0) = p_0 \end{array} \right. \quad (1.5)$$

( see theorem 4.2 ). Here  $A_f(t, 0)$  is the solution of the free wave equation with the the same initial data for the field as the system (1.4). This is a more familiar problem in which a standard wave equation is coupled with an ordinary differential one. This ordinary differential equation is a low order version of the well known Abraham–Lorentz–Dirac equation ( [D], [L] ),

$$\begin{cases} -m\tau_0 \ddot{q}(t) + m\dot{q}(t) = -\frac{e}{c} \dot{A}_f(t, 0) + F(q(t)) \\ q(0) = q_0, \quad \dot{q}(0) = Q_{A_0}, \quad \ddot{q}(0) = Q_{\dot{A}_0} ; \end{cases}$$

so on one hand this much questioned equation is confirmed in the present more general case, and on the other hand it is settled in a convincing mathematical context.

Finally, we show that the limit system (1.4) is in fact a Hamiltonian system, and that the corresponding Hamiltonian function is the limit in a well defined sense of the classical Hamiltonian of the Pauli–Fierz model ( see theorem 4.5 ). This result, which answers a time honored problem ( we quote only the papers [K] and [VK] ), could be useful to study the many open questions related to electrodynamics of point particles ( e.g. runaway solutions, quantization... ), and could perhaps give valid hints to study the fully non linear system without dipole approximation.

## 2. Definitions and some preliminary results.

We begin by recalling some definitions and results from [NP]. We denote by  $L_*^2(\mathbb{R}^3)$  the Hilbert space of square integrable, divergence-free, vector fields on  $\mathbb{R}^3$ .  $M$  will be the projection from  $L_3^2(\mathbb{R}^3)$ , the Hilbert space of square integrable vector fields on  $\mathbb{R}^3$ , onto  $L_*^2(\mathbb{R}^3)$  and we will use the same symbol  $\langle \cdot, \cdot \rangle$  (  $\| \cdot \|_2$  is the corresponding Hilbert norm ) to denote the scalar products in  $L^2(\mathbb{R}^3)$ ,  $L_3^2(\mathbb{R}^3)$ ,  $L_*^2(\mathbb{R}^3)$  and also to denote the obvious pairing between an element of  $L_3^2(\mathbb{R}^3)$  and one of  $L^2(\mathbb{R}^3)$  ( the result being a vector in  $\mathbb{R}^3$  ). By the same abuse of notation, given two functions  $f$  and  $g$  in  $L^2(\mathbb{R}^3)$ , by  $f \otimes g$  we will denote the operator in  $L_3^2(\mathbb{R}^3)$  defined by  $f \otimes g(A) := f \langle g, A \rangle$ . Moreover  $H^s(\mathbb{R}^3)$ ,  $s \in \mathbb{R}$ , indicates the usual scale of Sobolev–Hilbert spaces, and the meaning of  $H_3^s(\mathbb{R}^3)$  and  $H_*^s(\mathbb{R}^3)$  should now be clear. If  $\gamma$  is a continuous path in  $\mathbb{R}^3$ , defined on the compact time interval  $I(T) := [-T, T]$ ,  $\|\gamma\|_\infty$  denotes the usual supremum norm. With  $\text{Lip}(\mathbb{R}^3; \mathbb{R}^3)$  we denote the space of Lipschitz vector fields. Finally, given a measurable non negative function  $\rho$  we define its energy  $E(\rho)$  as

$$E(\rho) := \frac{1}{4\pi} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{\rho(x)\rho(y)}{|x-y|} dx dy .$$

**Theorem 2.1.** ( [NP, thm. 2.1] ) *As  $r \downarrow 0$ , i.e as  $\rho_r(x) := r^{-3}\rho(r^{-1}x)$ ,  $\rho$  a spherically symmetric probability density with bounded support, weakly converges to  $\delta_0$ , the self-adjoint operator*

$$H_r := -\Delta + \frac{4\pi e^2}{m_r c^2} M \cdot \rho_r \otimes \rho_r ,$$

$$m_r := m - \frac{8\pi e^2}{3c^2} E(\rho_r) = m - \frac{1}{r} \frac{8\pi e^2}{3c^2} E(\rho) ,$$

*converges in norm resolvent sense in  $L_*^2(\mathbb{R}^3)$  to a self-adjoint operator  $H_m$ , where  $H_m$  has the resolvent*

$$(H_m + \lambda)^{-1} = (-\Delta + \lambda)^{-1} + \Gamma_m(\lambda)^{-1} M \cdot G_\lambda \otimes G_\lambda ,$$

and where

$$-\lambda \in \rho(H_m), \quad \lambda > 0, \quad \Gamma_m(\lambda) = -\frac{mc^2}{4\pi e^2} + \frac{\sqrt{\lambda}}{6\pi}, \quad G_\lambda(x) := \frac{1}{4\pi} \frac{e^{-\sqrt{\lambda}|x|}}{|x|}.$$

If  $m_r = \text{const.}$  then  $H_r$  converges in norm resolvent sense in  $L^2(\mathbb{R}^3)$  to  $-\Delta$ . No other definition of the renormalized mass  $m_r$  (up to  $O(r)$  terms) leads to a limit for  $H_r$ .

**Theorem 2.2.** (*[NP, thm. 2.3]*) The vectors  $A$  in the operator domain  $D(H_m)$  of  $H_m$  are of the type

$$A = A_\lambda + \Gamma_m(\lambda)^{-1} M A_\lambda(0) G_\lambda, \quad A_\lambda \in H_*^2(\mathbb{R}^3), \quad -\lambda \in \rho(H_m), \quad \lambda > 0.$$

The above decomposition in a regular part  $A_\lambda$ , and a corresponding singular one, is unique, and with  $A \in D(H_m)$  of this form one has

$$(H_m + \lambda)A = (-\Delta + \lambda)A_\lambda.$$

Let  $F_m$  be the quadratic form corresponding to  $H_m$ . Then the vectors  $A$  in the form domain  $D(F_m)$  are of the type

$$A = A_\lambda + \frac{4\pi e}{c} M Q_A G_\lambda, \quad A_\lambda \in H_*^1(\mathbb{R}^3), \quad Q_A \in \mathbb{R}^3, \quad \lambda > 0.$$

Given  $A \in D(F_m)$ ,  $Q_A$  can be explicitly computed by the formula

$$Q_A = \frac{3c}{2e} \lim_{r \downarrow 0} r \frac{1}{4\pi r^2} \int_{S_r} A(x) d\mu_r(x),$$

where  $S_r$  denotes the sphere of radius  $r$  and  $\mu_r$  is the corresponding surface measure. The above decomposition is unique, and with  $A \in D(F_m)$  of this form one has

$$\begin{aligned} F_m^\lambda(A, A) &:= F_m(A, A) + \lambda \|A\|_2^2 \\ &= \|(-\Delta + \lambda)^{\frac{1}{2}} A_\lambda\|_2^2 + \left(\frac{4\pi e}{c}\right)^2 \Gamma_m(\lambda) |Q_A|^2. \end{aligned}$$

Moreover

$$\sigma_{\text{ess}}(H_m) = \sigma_{\text{ac}}(H_m) = [0, +\infty), \quad \sigma_{\text{sc}}(H_m) = \emptyset,$$

and

$$\sigma_p(H_m) = \left\{ -\left(\frac{3mc^2}{2e^2}\right)^2 \right\} \equiv \{-\lambda_0\},$$

where  $-\lambda_0$  has a threefold degeneration and

$$X_j^0 = 2\sqrt{2\pi m} \frac{c}{e} M e_j G_{\lambda_0},$$

are the corresponding normalized eigenvectors, where  $\{e_j\}_1^3$  is an orthonormal basis.

**Remark 2.3.** This remark would correspond to remark 2.5 in [NP]. However here we proceed in a different way avoiding the use of the fields  $\tilde{X}$ . By functional calculus, and by

$$\frac{1}{\sqrt{y}} = \frac{1}{\pi} \int_0^\infty \frac{1}{y+x} \frac{dx}{\sqrt{x}} \quad y > 0 ,$$

we have, for any  $X \in L_*^2(\mathbb{R}^3)$ ,  $\lambda > \lambda_0$

$$\begin{aligned} & (H_m + \lambda)^{-\frac{1}{2}} X \\ &= (-\Delta + \lambda)^{-\frac{1}{2}} X + \frac{1}{\pi} \int_0^\infty \Gamma_m(x + \lambda)^{-1} M\langle G_{x+\lambda}, X \rangle G_{x+\lambda} \frac{dx}{\sqrt{x}} . \end{aligned} \quad (2.1)$$

Analogously, if  $\lambda > \lambda_0$ , and  $r$  sufficiently small, we have

$$\begin{aligned} & (H_r + \lambda)^{-\frac{1}{2}} X = (-\Delta + \lambda)^{-\frac{1}{2}} X \\ &+ \frac{1}{\pi} \int_0^\infty \Gamma_r(x + \lambda)^{-1} M\langle (-\Delta + x + \lambda)^{-1} \rho_r, X \rangle (-\Delta + x + \lambda)^{-1} \rho_r \frac{dx}{\sqrt{x}} , \end{aligned} \quad (2.2)$$

where

$$\Gamma_r(\lambda) = -\frac{m_r c^2}{4\pi e^2} - \frac{2}{3} \langle (-\Delta + \lambda)^{-1} \rho, \rho \rangle .$$

Moreover, by the definition of  $Q_A$ , we have

$$L_m(X) := Q_{(H_m + \lambda)^{-1/2} X} = \left( \frac{4\pi e}{c} \right)^{-1} \frac{1}{\pi} \int_0^\infty \Gamma_m(x + \lambda)^{-1} \langle G_{x+\lambda}, X \rangle \frac{dx}{\sqrt{x}} . \quad (2.3)$$

**Lemma 2.4.** *If*

$$\lim_{r \downarrow 0} \sup_{|t| \leq T} \|X_r(t) - X(t)\|_2 = 0 ,$$

*then*

$$\lim_{r \downarrow 0} \sup_{|t| \leq T} \left| -\frac{e}{m_r c} \langle \rho_r, (H_r + \lambda)^{-1/2} X_r(t) \rangle - L_m(X(t)) \right| = 0 .$$

*Proof.* By (2.2) we have

$$\begin{aligned} & -\frac{e}{m_r c} \langle \rho_r, (H_r + \lambda)^{-1/2} X_r(t) \rangle = -\frac{e}{c} \frac{\langle \rho_r, (-\Delta + \lambda)^{-\frac{1}{2}} X_r(t) \rangle}{m_r} \\ & - \frac{2e}{3\pi c} \int_0^\infty \Gamma_r(x + \lambda)^{-1} \langle (-\Delta + x + \lambda)^{-1} \rho_r, X_r(t) \rangle \frac{\langle (-\Delta + \lambda)^{-1} \rho_r, \rho_r \rangle}{m_r} \frac{dx}{\sqrt{x}} . \end{aligned}$$

The thesis follows by (2.3),

$$\lim_{r \downarrow 0} \Gamma_r(\lambda) = \Gamma_m(\lambda) ,$$

$$\lim_{r \downarrow 0} \sup_{|t| \leq T} \left| \frac{\langle \rho_r, (-\Delta + \lambda)^{-\frac{1}{2}} X_r(t) \rangle}{m_r} \right| = 0 ,$$

and

$$\lim_{r \downarrow 0} \frac{\langle (-\Delta + \lambda)^{-1} \rho_r, \rho_r \rangle}{m_r} = -\frac{3c^2}{8\pi e^2} .$$

□

Now we recall some results on abstract second order equations ( see [F] for the proof of such results ). Let  $H$  be a bounded from below ( this is a necessary condition ) self-adjoint operator on  $L_*^2(\mathbb{R}^3)$ , and let  $F$  the corresponding quadratic form. Then  $H$  generates a cosine operator function  $C : \mathbb{R} \rightarrow \mathcal{L}(L_*^2(\mathbb{R}^3); L_*^2(\mathbb{R}^3))$ , i.e.  $C$  is a strongly continuous function such that

$$\begin{aligned} C(0) &= 1, & C(s+t)C(s-t) &= 2C(s)C(t) , \\ D(H) &= \{ A \in L_*^2(\mathbb{R}^3) : \lim_{t \rightarrow 0} \frac{2}{t^2}(C(t)A - A) \text{ exists} \} , \\ \forall A \in D(H) & \quad HA &= \lim_{t \rightarrow 0} \frac{2}{t^2}(C(t)A - A) , \\ & \quad C(t)D(H) &\subseteq D(H) . \end{aligned}$$

Moreover  $\forall z \in \mathbb{C}$  such that  $-z^2 \in \rho(H)$  and  $\operatorname{Re} z > |\inf \sigma(H) \wedge 0|^{\frac{1}{2}}$

$$z(H + z^2)^{-1}A = \int_0^\infty e^{-zt}C(t)A dt ,$$

and so, by  $(H + z^2)^{-1}A = z^{-2}A - z^{-2}(H + z^2)HA$ , and by inverse Laplace transform,  $\forall t \geq 0, \forall A \in D(H)$ ,

$$C(t)A = A - \frac{1}{2\pi i} \int_{x-i\infty}^{x+i\infty} e^{zt} \frac{1}{z} (H + z^2)^{-1}HA dz , \quad x > |\inf \sigma(H) \wedge 0|^{\frac{1}{2}} . \quad (2.4)$$

One defines then the sine operator function  $S : \mathbb{R} \rightarrow \mathcal{L}(L_*^2(\mathbb{R}^3); L_*^2(\mathbb{R}^3))$  by

$$S(t)A := \int_0^t C(s)A ds = \frac{1}{2\pi i} \int_{x-i\infty}^{x+i\infty} e^{zt} (H + z^2)^{-1}A dz , \quad x > |\inf \sigma(H) \wedge 0|^{\frac{1}{2}} . \quad (2.5)$$

Obviously, if  $H$  is strictly positive, then, by functional calculus,

$$C(t) = \cos t H^{\frac{1}{2}}, \quad S(t) = H^{-\frac{1}{2}} \sin t H^{\frac{1}{2}} .$$

Given  $A_0, \dot{A}_0 \in L_*^2(\mathbb{R}^3)$ ,  $X \in C(\mathbb{R}; L_*^2(\mathbb{R}^3))$ , let

$$A(t) := C(t)A_0 + S(t)\dot{A}_0 + \int_0^t S(t-s)X(s) ds .$$

Then  $A \in C(\mathbb{R}; L_*^2(\mathbb{R}^3))$ . If  $A_0 \in D(F)$ ,  $\dot{A}_0 \in L_*^2(\mathbb{R}^3)$  then

$$A \in C(\mathbb{R}; D(F)) \cap C^1(\mathbb{R}; L_*^2(\mathbb{R}^3)) .$$

If  $A_0 \in D(H)$ ,  $\dot{A}_0 \in D(F)$ ,  $X \in C^1(\mathbb{R}; L_*^2(\mathbb{R}^3))$ , then

$$A \in C(\mathbb{R}; D(H)) \cap C^1(\mathbb{R}; D(F)) \cap C^2(\mathbb{R}; L_*^2(\mathbb{R}^3)) ,$$

and  $A(t)$  solves the inhomogeneous Cauchy problem

$$\begin{cases} \ddot{A}(t) = -HA(t) + X(t) \\ A(0) = A_0, \quad \dot{A}(0) = \dot{A}_0 . \end{cases}$$

We will denote by  $C_m(t)$ ,  $S_m(t)$ , and by  $C_r(t)$ ,  $S_r(t)$ , the cosine and sine operator functions corresponding to  $c^2 H_m$  and  $c^2 H_r$  respectively.

We conclude this section with the following lemma which is one of the main technical points of the paper.

**Lemma 2.5.** *For any  $\gamma \in C(I(T); \mathbb{R}^3)$  one has*

$$\lim_{r \downarrow 0} \sup_{|t| \leq T} \left\| \frac{4\pi e}{m_r c} (H_r + \lambda)^{-\frac{1}{2}} M\gamma(t) \rho_r + \frac{c}{e} \Gamma_m(\lambda)^{-1} (H_m + \lambda)^{\frac{1}{2}} M\gamma(t) G_\lambda \right\|_2 = 0 .$$

*Proof.* By the inequality

$$\|X + Y\|_2^2 \leq \|X\|_2^2 + \|Y\|_2^2 + 2|\langle X, Z \rangle + \langle Y, Z \rangle| + 2(\|X\|_2^2 + \|Y\|_2^2) \|Y - Z\|_2$$

it follows

$$\begin{aligned} & \left\| \frac{4\pi e}{m_r c} (H_r + \lambda)^{-\frac{1}{2}} M\gamma(t) \rho_r + \frac{c}{e} \Gamma_m(\lambda)^{-1} (H_m + \lambda)^{\frac{1}{2}} M\gamma(t) G_\lambda \right\|_2^2 \\ & \leq \left| \left\| \frac{4\pi e}{m_r c} (H_r + \lambda)^{-\frac{1}{2}} M\gamma(t) \rho_r \right\|_2^2 - \frac{c^2}{e^2} \Gamma_m(\lambda)^{-2} F_m(M\gamma(t) G_\lambda, M\gamma(t) G_\lambda) \right| \\ & + 2 \left| \frac{4\pi e}{m_r c} \langle (H_r + \lambda)^{-\frac{1}{2}} M\gamma(t) \rho_r, \gamma(t) X_j \rangle + \frac{c}{e} \Gamma_m(\lambda)^{-1} \langle (H_m + \lambda)^{\frac{1}{2}} M\gamma(t) G_\lambda, \gamma(t) X_j \rangle \right| \\ & + 2 \left( \left\| \frac{4\pi e}{m_r c} (H_r + \lambda)^{-\frac{1}{2}} M\gamma(t) \rho_r \right\|_2 + \left\| \frac{c}{e} \Gamma_m(\lambda)^{-1} (H_m + \lambda)^{\frac{1}{2}} M\gamma(t) G_\lambda \right\|_2 \right) \times \\ & \times \left\| \frac{c}{e} \Gamma_m(\lambda)^{-1} (H_m + \lambda)^{\frac{1}{2}} M\gamma(t) G_\lambda - M\gamma(t) X_j \right\|_2 \\ & \equiv K_r(t) + 2(B_{r,j}(t) + J_r(t) D_j(t)) , \end{aligned}$$

where  $\{X_j\}_1^\infty \subset L^2(\mathbb{R}^3)$  will be defined below. Since

$$\begin{aligned} F_m(M\gamma(t) G_\lambda, M\gamma(t) G_\lambda) &= \Gamma_m(\lambda) |\gamma(t)|^2 , \\ \langle (H_r + \lambda)^{-1} \rho_r, \rho_r \rangle &= \langle (-\Delta + \lambda)^{-1} \rho_r, \rho_r \rangle + \frac{2}{3} \Gamma_r(\lambda)^{-1} |\langle (-\Delta + \lambda)^{-1} \rho_r, \rho_r \rangle|^2 , \end{aligned} \quad (2.6)$$

and

$$\lim_{r \downarrow 0} \frac{\langle (-\Delta + \lambda)^{-1} \rho_r, \rho_r \rangle}{m_r} = -\frac{3c^2}{8\pi e^2} , \quad (2.7)$$

one has

$$\begin{aligned} & \lim_{r \downarrow 0} \left\| \frac{4\pi e}{m_r c} (H_r + \lambda)^{-\frac{1}{2}} M\gamma(t) \rho_r \right\|_2^2 \\ &= \lim_{r \downarrow 0} \frac{2}{3} \left( \frac{4\pi e}{m_r c} \right)^2 |\gamma(t)|^2 \langle (H_r + \lambda)^{-1} \rho_r, \rho_r \rangle \\ &= \frac{4}{9} \left( \frac{4\pi e}{c} \right)^2 |\gamma(t)|^2 \Gamma_m(\lambda)^{-1} \left( \frac{3c^2}{8\pi e^2} \right)^2 \\ &= \frac{c^2}{e^2} \Gamma_m(\lambda)^{-1} |\gamma(t)|^2 \\ &= \frac{c^2}{e^2} \Gamma_m(\lambda)^{-2} F_m(M\gamma(t) G_\lambda, M\gamma(t) G_\lambda) , \end{aligned}$$

and so

$$\lim_{r \downarrow 0} \sup_{|t| \leq T} K_r(t) = 0 .$$

Now let  $\{f_j\}_{j=1}^\infty \subset L^2(\mathbb{R}^3)$  be a sequence such that

$$\lim_{r \downarrow 0} \|(H_m + \lambda)^{-\frac{1}{2}} f_j - \frac{c}{e} \Gamma_m(\lambda)^{-1} (H_m + \lambda)^{\frac{1}{2}} G_\lambda\|_2 = 0 .$$

Then  $M\gamma(t)(H_m + \lambda)^{-\frac{1}{2}} f_j = (H_m + \lambda)^{-\frac{1}{2}} M\gamma(t)f_j$ , and

$$\lim_{r \downarrow 0} \sup_{|t| \leq T} \|M\gamma(t)(H_m + \lambda)^{-\frac{1}{2}} f_j - \frac{c}{e} \Gamma_m(\lambda)^{-1} (H_m + \lambda)^{\frac{1}{2}} M\gamma(t)G_\lambda\|_2 = 0 .$$

Therefore, if  $X_j := (H_m + \lambda)^{-\frac{1}{2}} f_j$ , one has  $\lim_{j \uparrow \infty} \sup_{|t| \leq T} D_j(t) = 0$ , and

$$\begin{aligned} B_{r,j}(t) \leq & \left| \frac{4\pi e}{m_r c} \langle (H_r + \lambda)^{-1} M\gamma(t)\rho_r, \gamma(t)f_j \rangle + \frac{c}{e} \langle \Gamma_m^{-1} M\gamma(t)G_\lambda, \gamma(t)f_j \rangle \right| \\ & + \left\| \frac{4\pi e}{m_r c} (H_r + \lambda)^{-\frac{1}{2}} M\gamma(t)\rho_r \right\| \left\| (H_r + \lambda)^{-\frac{1}{2}} - (H_m + \lambda)^{-\frac{1}{2}} \right\|_{2,2} |\gamma(t)| \|f_j\|_2 , \end{aligned}$$

so that

$$\forall j \in \mathbb{N} \quad \lim_{r \downarrow 0} \sup_{|t| \leq T} B_{r,j}(t) = 0 .$$

Since

$$\sup_{r > 0} \sup_{|t| \leq T} J_r(t) < +\infty ,$$

in conclusions there follows that, for any  $\epsilon > 0$ , there exists  $j_*$  such that

$$\lim_{r \downarrow 0} \sup_{|t| \leq T} K_r(t) + 2(B_{r,j_*}(t) + J_r(t)D_{j_*}(t)) \leq \epsilon .$$

□

### 3. The point limit of the Maxwell–Lorentz equations with an external force.

Let us consider the regularized Maxwell–Lorentz system in the dipole approximation with an external force  $F$ , i.e. the system of equations on  $L_*^2(\mathbb{R}^3) \times \mathbb{R}^3$  given by

$$\begin{cases} \frac{1}{c^2} \ddot{A}_r = \Delta A_r + \frac{4\pi e}{c} M \dot{q}_r \rho_r \\ m_r \ddot{q}_r = -\frac{e}{c} \langle \rho_r, \dot{A}_r \rangle + \int_{\mathbb{R}^3} \rho_r(x) F(x+q) dx \\ A_r(0) = A_0 \in H_*^1(\mathbb{R}^3), \quad \dot{A}_r(0) = \dot{A}_0 \in L_*^2(\mathbb{R}^3), \quad q_r(0) = q_0, \quad \dot{q}_r(0) = \dot{q}_0 . \end{cases} \quad (3.1)$$

Now, as in [NP], we want to study the point limit ( i.e.  $r \downarrow 0$ , or  $\rho_r \xrightarrow{w} \delta_0$  ) of such a system. At first we rewrite (3.1) as

$$\begin{cases} \dot{A}_r = 4\pi c^2 E_r \\ \dot{E}_r = \frac{1}{4\pi} \Delta A_r - \frac{e^2}{m_r c^2} M \langle \rho_r, A_r \rangle \rho_r + \frac{e}{m_r c} M p_r \rho_r \\ \dot{q}_r = \frac{1}{m_r} p_r - \frac{e}{m_r c} \langle \rho_r, A_r \rangle \\ \dot{p}_r = \langle \rho_r, F_{q_r} \rangle \\ A_r(0) = A_0 \in H_*^1(\mathbb{R}^3), \quad E_r(0) = E_0 \in L_*^2(\mathbb{R}^3), \quad q_r(0) = q_0, \quad p_r(0) = p_0 , \end{cases} \quad (3.2)$$

where  $F_q(x) := F(x+q)$ . Given any path  $p$  in  $\mathbb{R}^3$  we define the time-dependent operator, with domain  $H_*^2(\mathbb{R}^3)$ ,

$$\begin{aligned} H_{r,p(t)}A &:= -\Delta A + \frac{4\pi e^2}{m_r c^2} M \cdot \langle \rho_r, A \rangle \rho_r - \frac{4\pi e}{m_r c} M p(t) \rho_r \\ &\equiv H_r A - \frac{4\pi e}{m_r c} M p(t) \rho_r . \end{aligned}$$

Let us now try to determine the limit, as  $r \downarrow 0$ , of  $H_{r,p}$ . Given  $\lambda > \lambda_0$ , suppose that  $r$  is so small that  $H_r + \lambda$  is strictly positive, and consider the Cauchy problem

$$\begin{cases} \frac{1}{c^2} \ddot{A}_r = - (H_r + \lambda) A_r + \frac{4\pi e}{m_r c} M p \rho_r \\ A(0) = A_0^r \in H_*^1(\mathbb{R}^3), \quad \dot{A}(0) = \dot{A}_0 \in L_*^2(\mathbb{R}^3) . \end{cases}$$

Its ( mild ) solution is given by

$$\begin{aligned} A_r(t) &= \cos(ct(H_r + \lambda)^{\frac{1}{2}}) A_0^r + \sin(ct(H_r + \lambda)^{\frac{1}{2}}) (H_r + \lambda)^{-\frac{1}{2}} \dot{A}_0 \\ &\quad + \frac{4\pi e}{m_r} \int_0^t \sin(c(t-s)(H_r + \lambda)^{\frac{1}{2}}) (H_r + \lambda)^{-\frac{1}{2}} M p(s) \rho_r ds , \end{aligned}$$

which, after an integration by parts, can be rewritten as ( here and below we are supposing that  $p$  is sufficiently regular )

$$\begin{aligned} A_r(t) &= \cos(ct(H_r + \lambda)^{\frac{1}{2}}) A_0^r + \sin(ct(H_r + \lambda)^{\frac{1}{2}}) (H_r + \lambda)^{-\frac{1}{2}} \dot{A}_0 \\ &\quad - \frac{4\pi e}{m_r c} \int_0^t \cos(c(t-s)(H_r + \lambda)^{\frac{1}{2}}) (H_r + \lambda)^{-1} M \dot{p}(s) \rho_r ds \\ &\quad + \frac{4\pi e}{m_r c} (H_r + \lambda)^{-1} M p(t) \rho_r - \frac{4\pi e}{m_r c} \cos(ct(H_r + \lambda)^{\frac{1}{2}}) (H_r + \lambda)^{-1} M p(0) \rho_r . \end{aligned}$$

By thm. 2.1 and by lemma 2.5, if

$$\lim_{r \downarrow 0} \| (H_r + \lambda)^{\frac{1}{2}} A_0^r - (H_m + \lambda)^{\frac{1}{2}} A_0 \|_2 = 0 ,$$

then

$$\lim_{r \downarrow 0} \sup_{|t| \leq T} \| (H_r + \lambda)^{\frac{1}{2}} A_r(t) - (H_m + \lambda)^{\frac{1}{2}} A(t) \|_2 = 0 ,$$

where

$$\begin{aligned} A(t) &= \cos(ct(H_m + \lambda)^{\frac{1}{2}}) A_0 + \sin(ct(H_m + \lambda)^{\frac{1}{2}}) (H_m + \lambda)^{-\frac{1}{2}} \dot{A}_0 \\ &\quad + \frac{c}{e} \Gamma_m(\lambda)^{-1} \int_0^t \cos(c(t-s)(H_m + \lambda)^{\frac{1}{2}}) M \dot{p}(s) G_\lambda ds \\ &\quad - \frac{c}{e} \Gamma_m(\lambda)^{-1} M p(t) G_\lambda - \frac{c}{e} \Gamma_m(\lambda)^{-1} \cos(ct(H_m + \lambda)^{\frac{1}{2}}) M p(0) G_\lambda . \end{aligned}$$

Integrating by parts we can rewrite the above expression as

$$\begin{aligned} A(t) + \frac{c}{e} \Gamma_m(\lambda)^{-1} M p(t) G_\lambda &= \cos(ct(H_m + \lambda)^{\frac{1}{2}}) \left( A_0 + \frac{c}{e} \Gamma_m(\lambda)^{-1} M p(0) G_\lambda \right) \\ &\quad + \sin(ct(H_m + \lambda)^{\frac{1}{2}}) (H_m + \lambda)^{-\frac{1}{2}} \left( \dot{A}_0 + \frac{c}{e} \Gamma_m(\lambda)^{-1} M \dot{p}(0) G_\lambda \right) \\ &\quad + \frac{1}{e} \Gamma_m(\lambda)^{-1} \int_0^t \sin(c(t-s)(H_m + \lambda)^{\frac{1}{2}}) (H_m + \lambda)^{-\frac{1}{2}} M \ddot{p}(s) G_\lambda ds . \end{aligned}$$

This tells us that

$$A_p(t) := A(t) + \frac{c}{e} \Gamma_m(\lambda)^{-1} M p(t) G_\lambda$$

solves the Cauchy problem

$$\begin{cases} \frac{1}{c^2} \ddot{A}_p = - (H_m + \lambda) A_p + \frac{c}{e} \Gamma_m(\lambda)^{-1} M \ddot{p} G_\lambda \\ A_p(0) = A_0 + \frac{c}{e} \Gamma_m(\lambda)^{-1} M p(0) G_\lambda, \quad \dot{A}_p(0) = \dot{A}_0 + \frac{c}{e} \Gamma_m(\lambda)^{-1} M \dot{p}(0) G_\lambda, \end{cases}$$

and so  $A(t)$  solves the Cauchy problem

$$\begin{cases} \frac{1}{c^2} \ddot{A} = - (H_m + \lambda) \left( A + \frac{c}{e} \Gamma_m(\lambda)^{-1} M p G_\lambda \right) \\ A(0) = A_0, \quad \dot{A}(0) = \dot{A}_0. \end{cases}$$

This induces us to define, on the time-dependent domain

$$D(H_{m,p(t)}) := D(H_m) - \frac{c}{e} \Gamma_m(\lambda)^{-1} M p(t) G_\lambda,$$

the time-dependent operator  $H_{m,p(t)}$  given by

$$(H_{m,p(t)} + \lambda) A := (H_m + \lambda) \left( A + \frac{c}{e} \Gamma_m(\lambda)^{-1} M p(t) G_\lambda \right)$$

or, alternatively, by

$$H_{m,p(t)} A := H_m \left( A + \frac{c}{e} \Gamma_m(\lambda)^{-1} M p(t) G_\lambda \right) + \lambda \frac{c}{e} \Gamma_m(\lambda)^{-1} M p(t) G_\lambda. \quad (3.3)$$

In the next two theorems we give the ( local ) existence results for the two dynamical systems defined by equations (3.2) and by their conjectured point limit.

**Theorem 3.1.** *Let  $F \in \text{Lip}(\mathbb{R}^3; \mathbb{R}^3)$ . Then there exists  $T > 0$ , independent of  $r$ , such that the Cauchy problem*

$$\begin{cases} \dot{A}_r = 4\pi c^2 E_r \\ \dot{E}_r = \frac{1}{4\pi} \Delta A_r - \frac{e^2}{m_r c^2} M \langle \rho_r, A_r \rangle \rho_r + \frac{e}{m_r c} M p_r \rho_r \\ \dot{q}_r = \frac{1}{m_r} p_r - \frac{e}{m_r c} \langle \rho_r, A_r \rangle \\ \dot{p}_r = \langle \rho_r, F_{q_r} \rangle \\ A_r(0) = A_0^r \in H_*^1(\mathbb{R}^3), \quad E_r(0) = E_0 \in L_*^2(\mathbb{R}^3), \quad q_r(0) = q_0, \quad p_r(0) = p_0 \end{cases}$$

has an unique mild solution

$$(A_r, E_r, q_r, p_r) \in C(I(T); H_*^1(\mathbb{R}^3)) \times C(I(T); L_*^2(\mathbb{R}^3)) \times C^2(I(T); \mathbb{R}^3) \times C^1(I(T); \mathbb{R}^3).$$

*Proof.* Given  $T > 0$ , let us introduce the map

$$\psi_r : C_{q_0^r}^1(I(T); \mathbb{R}^3) \rightarrow C_{q_0^r}^1(I(T); \mathbb{R}^3) \cap C^1(I(T); \mathbb{R}^3) \quad (3.4)$$

$$\psi_r(\gamma)(t) := \frac{1}{m_r} \tilde{\gamma}(t) - \frac{e}{m_r c} \langle A_r^\gamma(t), \rho_r \rangle ,$$

where

$$C_{\dot{q}_0^r}(I(T); \mathbb{R}^3) := \{\gamma \in C(I(T); \mathbb{R}^3) : \gamma(0) = \dot{q}_0^r\}, \quad \dot{q}_0^r := \frac{1}{m_r} p_0 - \frac{e}{m_r c} \langle \rho_r, A_0^r \rangle ,$$

$$\tilde{\gamma}(t) := p_0 + \int_0^t \int_{\mathbb{R}^3} \rho_r(x) F \left( x + q_0 + \int_0^s \gamma(u) du \right) dx ds ,$$

and where  $A_r^\gamma \in C(I(T); H_*^1(\mathbb{R}^3)) \cap C^1(I(T); L_*^2(\mathbb{R}^3))$  denotes the mild solution of the Cauchy problem

$$\begin{cases} \frac{1}{c^2} \ddot{A}_r^\gamma = -H_{r, \tilde{\gamma}} A_r^\gamma \\ A_r^\gamma(0) = A_0^r, \quad \dot{A}_r^\gamma(0) = \dot{A}_0 \equiv 4\pi c^2 E_0 . \end{cases}$$

If we prove that  $\psi_r$  has an unique fixed point  $\gamma_r^*$ , then

$$\left( A_r^{\gamma_r^*}, \frac{1}{4\pi c^2} \dot{A}_r^{\gamma_r^*}, q_0 + \int_0^t \gamma_r^*(s) ds, \tilde{\gamma}_r^* \right)$$

gives the solution. Let us now show that  $\psi_r$  is a contraction on  $C_{\dot{q}_0^r}(I(T); \mathbb{R}^3)$ . We have

$$|\psi_r(\gamma_1)(t) - \psi_r(\gamma_2)(t)| \leq c_F T^2 \frac{1}{m_r} \|\gamma_1 - \gamma_2\|_\infty + \frac{e}{|m_r|c} |\langle A_r^{\gamma_1}(t) - A_r^{\gamma_2}(t), \rho_r \rangle| ,$$

where  $c_F$  denotes the Lipschitz constant of  $F$ . Since

$$A_r^\gamma(t) = C_r(t) A_0^r + S_r(t) \dot{A}_0 + \frac{4\pi e c}{m_r} \int_0^t S_r(t-s) M \tilde{\gamma}(s) \rho_r ds , \quad (3.5)$$

we have

$$\frac{e}{|m_r|c} |\langle A_r^{\gamma_1}(t) - A_r^{\gamma_2}(t), \rho_r \rangle| \leq c_F T^2 \frac{8\pi e^2}{3m_r^2} \|\gamma_1 - \gamma_2\|_\infty \left| \int_0^t \langle S_r(t-s) \rho_r, \rho_r \rangle ds \right| .$$

Since ( see (2.5) )

$$\langle S_r(t) \rho_r, \rho_r \rangle = \frac{1}{2\pi i} \int_{x-i\infty}^{x+i\infty} e^{zt} \langle (H_r + z^2)^{-1} \rho_r, \rho_r \rangle dz ,$$

by (2.6) and (2.7) it follows that

$$\frac{8\pi e^2}{3m_r^2} \left| \int_0^t \langle S_r(t-s) \rho_r, \rho_r \rangle ds \right|$$

converges, uniformly in  $t$  over compact intervals, and so there exists  $c(T)$ , with  $c(T) \downarrow 0$  as  $T \downarrow 0$ , such that

$$\sup_{|t| \leq T} \sup_{r > 0} \frac{8\pi e^2}{3m_r^2} \left| \int_0^t \langle S_r(t-s) \rho_r, \rho_r \rangle ds \right| \leq c(T) . \quad (3.6)$$

Therefore

$$\|\psi_r(\gamma_1) - \psi_r(\gamma_2)\|_\infty \leq c_F T^2 \left( c(T) + \frac{1}{|m_r|} \right) \|\gamma_1 - \gamma_2\|_\infty .$$

Since  $1/|m_r| \rightarrow 0$ , we can choose, independently of  $r$ , a  $T > 0$  such that

$$c_F T^2 \left( c(T) + \frac{1}{m_r} \right) < 1 ,$$

and the proof is concluded by the contraction mapping principle.  $\square$

**Theorem 3.2.** *Let  $F \in \text{Lip}(\mathbb{R}^3; \mathbb{R}^3)$ . Then there exists  $T > 0$  such that the Cauchy problem*

$$\left\{ \begin{array}{l} \dot{A} = 4\pi c^2 E \\ \dot{E} = -\frac{1}{4\pi} H_{m,p} A \\ \dot{q} = Q_A \\ \dot{p} = F(q) \\ A(0) = A_0 \in D(F_m), \quad E(0) = E_0 \in L_*^2(\mathbb{R}^3), \quad q(0) = q_0, \quad p(0) = p_0 , \end{array} \right.$$

has an unique mild solution

$$(A, E, q, p) \in C(I(T); D(F_m)) \times C(I(T); L_*^2(\mathbb{R}^3)) \times C^1(I(T); \mathbb{R}^3) \times C^1(I(T); \mathbb{R}^3) .$$

*Proof.* Given  $T > 0$ , let us introduce the map

$$\psi : C_{Q_{A_0}}(I(T); \mathbb{R}^3) \rightarrow C_{Q_{A_0}}(I(T); \mathbb{R}^3) \quad (3.7)$$

$$\psi(\gamma)(t) := Q_{A^\gamma(t)} ,$$

where  $C_{Q_{A_0}}(I(T); \mathbb{R}^3) := \{\gamma \in C(I(T); \mathbb{R}^3) : \gamma(0) = Q_{A_0}\}$ , and  $A^\gamma$  denotes the mild solution of the Cauchy problem

$$\left\{ \begin{array}{l} \frac{1}{c^2} \ddot{A}^\gamma = -H_{m, \tilde{\gamma}} A^\gamma \\ A^\gamma(0) = A_0, \quad \dot{A}^\gamma(0) = \dot{A}_0 \equiv 4\pi c^2 E_0 , \end{array} \right. \quad (3.8)$$

with

$$\tilde{\gamma}(t) := p_0 + \int_0^t F \left( q_0 + \int_0^s \gamma(u) du \right) ds .$$

If we prove that  $\psi$  has an unique fixed point  $\gamma^*$  then

$$\left( A^{\gamma^*}, \frac{1}{4\pi c^2} \dot{A}^{\gamma^*}, q_0 + \int_0^t \gamma^*(s) ds, \tilde{\gamma}^* \right)$$

gives the solution. Let us now prove that  $\psi$  is a contraction. Defining

$$\tilde{A} := A^\gamma + \frac{c}{e} \Gamma_m(\lambda)^{-1} M \tilde{\gamma} G_\lambda ,$$

by (3.3),  $A^\gamma$  solves the Cauchy problem (3.8) if and only if  $\tilde{A}$  solves

$$\begin{cases} \frac{1}{c^2} \ddot{\tilde{A}} = -H_m \tilde{A} - \lambda \frac{c}{e} \Gamma_m(\lambda)^{-1} M \tilde{\gamma} G_\lambda + \frac{c}{e} \Gamma_m(\lambda)^{-1} M \ddot{\tilde{\gamma}} G_\lambda \\ \tilde{A}(0) = A_0 + \frac{c}{e} \Gamma_m(\lambda)^{-1} M p_0 G_\lambda, \quad \dot{\tilde{A}}(0) = \dot{A}_0 + \frac{c}{e} \Gamma_m(\lambda)^{-1} M F(q_0) G_\lambda. \end{cases}$$

Therefore the solution of (3.8) is given by

$$\begin{aligned} A^\gamma(t) &= C_m(t) \left( A_0 + \frac{c}{e} \Gamma_m(\lambda)^{-1} M p_0 G_\lambda \right) + S_m(t) \left( \dot{A}_0 + \frac{c}{e} \Gamma_m(\lambda)^{-1} M F(q_0) G_\lambda \right) \\ &\quad + \frac{c}{e} \Gamma_m(\lambda)^{-1} \int_0^t S_m(t-s) (-\lambda M \tilde{\gamma} G_\lambda + M \ddot{\tilde{\gamma}}(s) G_\lambda) ds \\ &\quad - \frac{c}{e} \Gamma_m(\lambda)^{-1} M \tilde{\gamma}(t) G_\lambda, \end{aligned}$$

which, after an integration by parts, can be rewritten as

$$\begin{aligned} A^\gamma(t) &= C_m(t) \left( A_0 + \frac{c}{e} \Gamma_m(\lambda)^{-1} M p_0 G_\lambda \right) + S_m(t) \dot{A}_0 \\ &\quad + \frac{c}{e} \Gamma_m(\lambda)^{-1} \int_0^t C_m(t-s) M \dot{\tilde{\gamma}}(s) G_\lambda ds \\ &\quad + \lambda \frac{c}{e} \Gamma_m(\lambda)^{-1} \int_0^t S_m(t-s) M \tilde{\gamma}(s) G_\lambda ds - \frac{c}{e} \Gamma_m(\lambda)^{-1} M \tilde{\gamma}(t) G_\lambda. \end{aligned}$$

Therefore we have

$$\begin{aligned} &|\psi(\gamma_1)(t) - \psi(\gamma_2)(t)| \\ &\leq \Gamma_m(\lambda)^{-1} \frac{1}{6\pi^2} \int_0^t |\dot{\tilde{\gamma}}_1(s) - \dot{\tilde{\gamma}}_2(s)| |L_m(C_m(t-s)(H_m + \lambda)^{\frac{1}{2}} G_\lambda)| ds \\ &\quad + \lambda \Gamma_m(\lambda)^{-1} \frac{1}{6\pi^2} \int_0^t |\tilde{\gamma}_1(s) - \tilde{\gamma}_2(s)| |L_m(S_m(t-s)(H_m + \lambda)^{\frac{1}{2}} G_\lambda)| ds \\ &\quad + \frac{1}{4\pi} \Gamma_m(\lambda)^{-1} |\tilde{\gamma}_1(t) - \tilde{\gamma}_2(t)| \\ &\leq \Gamma_m(\lambda)^{-1} \frac{c_F T^2}{4\pi} \left( \frac{2}{3\pi} \int_0^t |L_m(C_m(t-s)(H_m + \lambda)^{\frac{1}{2}} G_\lambda)| ds \right. \\ &\quad \left. + \lambda T \frac{2}{3\pi} \int_0^t |L_m(S_m(t-s)(H_m + \lambda)^{\frac{1}{2}} G_\lambda)| ds + 1 \right) \|\gamma_1 - \gamma_2\|_\infty, \end{aligned}$$

and so there exists a  $T$  sufficiently small such that the map  $\psi$  is a contraction. The proof is then concluded by the contraction mapping principle.  $\square$

Before stating our convergence theorem we need the following preliminary result:

**Theorem 3.3.** *For any  $\gamma_r \in C^1(I(T); \mathbb{R}^3)$ ,  $r > 0$ ,  $T > 0$ , let*

$$A_r \in C(I(T); H_*^1(\mathbb{R}^3)) \cap C^1(I(T); L_*^2(\mathbb{R}^3))$$

*be the mild solution of the Cauchy problem*

$$\begin{cases} \frac{1}{c^2} \ddot{A}_r = -H_{r, \gamma_r} A_r \\ A_r(0) = A_0^r \in H_*^1(\mathbb{R}^3), \quad \dot{A}_r(0) = \dot{A}_0 \in L_*^2(\mathbb{R}^3), \end{cases}$$

with  $\lambda > \lambda_0$ . If

$$\lim_{r \downarrow 0} \|\gamma_r - \gamma\|_\infty + \|\dot{\gamma}_r - \dot{\gamma}\|_\infty = 0 ,$$

$\gamma \in C^1(I(T), \mathbb{R}^3)$ , and

$$\lim_{r \downarrow 0} \|(H_r + \lambda)^{\frac{1}{2}} A_0^r - (H_m + \lambda)^{\frac{1}{2}} A_0\|_2 = 0 ,$$

then

$$\lim_{r \downarrow 0} \sup_{|t| \leq T} \|(H_r + \lambda)^{\frac{1}{2}} A_r(t) - (H_m + \lambda)^{\frac{1}{2}} A(t)\|_2 = 0 ,$$

$$\lim_{r \downarrow 0} \sup_{|t| \leq T} \|\dot{A}_r(t) - \dot{A}(t)\|_2 = 0 ,$$

where  $A \in C(I(T); D(F_m)) \cap C^1(\mathbb{R}; L_*^2(\mathbb{R}^3))$  is the mild solution of the Cauchy problem

$$\begin{cases} \frac{1}{c^2} \ddot{A} = -H_{m,\gamma} A \\ A(0) = A_0 \in D(F_m), \quad \dot{A}(0) = \dot{A}_0 \in L_*^2(\mathbb{R}^3) . \end{cases}$$

*Proof.* By proceeding as in the proof of [NP, thm.2.8], and by the definitions of  $H_{r,\gamma_r}$  and  $H_{m,\gamma}$ , it will suffice to prove the analogous statements for the mild solutions of the Cauchy problems ( $\lambda > \lambda_0$ )

$$\begin{cases} \frac{1}{c^2} \ddot{A}_r = - (H_r + \lambda) A_r + \frac{4\pi e}{m_r c} M \gamma_r \rho_r \\ A(0) = A_0^r, \quad \dot{A}(0) = \dot{A}_0 , \end{cases}$$

and

$$\begin{cases} \frac{1}{c^2} \ddot{A} = - (H_{m,\gamma} + \lambda) A \equiv - (H_m + \lambda) (A + \frac{c}{e} \Gamma_m(\lambda)^{-1} M \gamma G_\lambda) \\ A(0) = A_0, \quad \dot{A}(0) = \dot{A}_0 . \end{cases}$$

Proceeding as in the calculations above leading to the definition of  $H_{m,p}$  ( see the beginning of this section ), we have that the solution of the first problem is given by

$$\begin{aligned} A_r(t) &= \cos(ct(H_r + \lambda)^{\frac{1}{2}}) A_0^r + \sin(ct(H_r + \lambda)^{\frac{1}{2}}) (H_r + \lambda)^{-\frac{1}{2}} \dot{A}_0 \\ &\quad - \frac{4\pi e}{m_r c} \int_0^t \cos(c(t-s)(H_r + \lambda)^{\frac{1}{2}}) (H_r + \lambda)^{-1} M \dot{\gamma}_r(s) \rho_r ds \\ &\quad + \frac{4\pi e}{m_r c} (H_r + \lambda)^{-1} M \gamma_r(t) \rho_r , \end{aligned}$$

and the solution of the second one is given by

$$\begin{aligned} A(t) &= \cos(ct(H_m + \lambda)^{\frac{1}{2}}) A_0 + \sin(ct(H_m + \lambda)^{\frac{1}{2}}) (H_m + \lambda)^{-\frac{1}{2}} \dot{A}_0 \\ &\quad + \frac{c}{e} \Gamma_m(\lambda)^{-1} \int_0^t \cos(c(t-s)(H_m + \lambda)^{\frac{1}{2}}) M \dot{\gamma}(s) G_\lambda ds \\ &\quad - \frac{c}{e} \Gamma_m(\lambda)^{-1} M \gamma(t) G_\lambda . \end{aligned}$$

Now, by the known strong resolvent convergence of  $H_r$  to  $H_m$  ( see thm. 2.1 ), and by the  $C^1$  convergence of  $\{\gamma_r\}_{r>0}$  to  $\gamma$ , the proof is concluded by lemma 2.5.  $\square$

We now come to the main result of this paper:

**Theorem 3.4.** *Let  $F \in \text{Lip}(\mathbb{R}^3; \mathbb{R}^3)$ ,  $|F(x)| \leq M_0(1 + |x|)$ ,  $\lambda > \lambda_0$ , and  $E_0 \in L_*^2(\mathbb{R}^3)$ . Let  $A_0^r \in H_*^1(\mathbb{R}^3)$ ,  $A_0 \in D(F_m)$ , such that*

$$\lim_{r \downarrow 0} \|(H_r + \lambda)^{\frac{1}{2}} A_0^r - (H_m + \lambda)^{\frac{1}{2}} A_0\|_2 = 0 . \quad (3.9)$$

Then there exists  $T > 0$  such that, if

$$(A_r, E_r, q_r, p_r) \in C(I(T); H_*^1(\mathbb{R}^3)) \times C(I(T); L_*^2(\mathbb{R}^3)) \times C^2(I(T); \mathbb{R}^3) \times C^1(I(T); \mathbb{R}^3)$$

denotes the mild solution of the Cauchy problem

$$\left\{ \begin{array}{l} \dot{A}_r = 4\pi c^2 E_r \\ \dot{E}_r = \frac{1}{4\pi} \Delta A_r - \frac{e^2}{m_r c^2} M \langle \rho_r, A_r \rangle \rho_r + \frac{e}{m_r c} M p_r \rho_r \\ \dot{q}_r = \frac{1}{m_r} p_r - \frac{e}{m_r c} \langle \rho_r, A_r \rangle \\ \dot{p}_r = \langle \rho_r, F_{q_r} \rangle \\ A_r(0) = A_0^r, \quad E_r(0) = E_0, \quad q_r(0) = q_0, \quad p_r(0) = p_0 , \end{array} \right.$$

then

$$\begin{aligned} \lim_{r \downarrow 0} \sup_{|t| \leq T} \|(H_r + \lambda)^{\frac{1}{2}} A_r(t) - (H_m + \lambda)^{\frac{1}{2}} A(t)\|_2 &= 0, \\ \lim_{r \downarrow 0} \sup_{|t| \leq T} \|E_r(t) - E(t)\|_2 &= 0, \\ \lim_{r \downarrow 0} \sup_{|t| \leq T} |\dot{q}_r(t) - \dot{q}(t)| + \sup_{|t| \leq T} |q_r(t) - q(t)| &= 0, \\ \lim_{r \downarrow 0} \sup_{|t| \leq T} |\dot{p}_r(t) - \dot{p}(t)| + \sup_{|t| \leq T} |p_r(t) - p(t)| &= 0 , \end{aligned}$$

where

$$(A, E, q, p) \in C(I(T); D(F_m)) \times C(I(T); L_*^2(\mathbb{R}^3)) \times C^1(I(T); \mathbb{R}^3) \times C^1(I(T); \mathbb{R}^3)$$

denotes the mild solution of the Cauchy problem

$$\left\{ \begin{array}{l} \dot{A} = 4\pi c^2 E \\ \dot{E} = -\frac{1}{4\pi} H_{m,p} A \\ \dot{q} = Q_A \\ \dot{p} = F(q) \\ A(0) = A_0, \quad E(0) = E_0, \quad q(0) = q_0, \quad p(0) = p_0 . \end{array} \right. \quad (3.10)$$

*Proof.* Let us denote by  $\gamma_r^*$  and  $\gamma^*$  the fixed points of the maps  $\psi_r$  ( see (3.4) ) and  $\psi$  ( see (3.7) ) respectively. If we prove that

$$\lim_{r \downarrow 0} \|\gamma_r^* - \gamma^*\|_\infty = 0 , \quad (3.11)$$

then, obviously,

$$\lim_{r \downarrow 0} \|\tilde{\gamma}_r^* - \tilde{\gamma}^*\|_\infty + \|\dot{\tilde{\gamma}}_r^* - \dot{\tilde{\gamma}}^*\|_\infty = 0 ,$$

and so the proof is concluded by thm. 3.3. Let us consider an arbitrary family  $\{\gamma_r\}_{r>0} \subset C_{q_0^r}(I(T); \mathbb{R}^3)$ , such that  $\|\gamma_r - \gamma\|_\infty \rightarrow 0$ ,  $\gamma \in C_{Q_{A_0}}(I(T); \mathbb{R}^3)$ . Then, writing  $A_r^{\gamma_r} = (H_r + \lambda)^{-\frac{1}{2}} X_r$  and  $A^\gamma = (H_m + \lambda)^{-\frac{1}{2}} X$ , by thm. 3.3 we have

$$\lim_{r \downarrow 0} \sup_{|t| \leq T} \|X_r(t) - X(t)\|_2 = 0 .$$

Therefore, by lemma 2.4, we have

$$\lim_{r \downarrow 0} \left\| \frac{e}{m_r c} \langle A_r^{\gamma_r}, \rho_r \rangle + Q_{A^\gamma} \right\|_\infty = 0 .$$

This gives

$$\lim_{r \downarrow 0} \|\psi_r(\gamma_r) - \psi(\gamma)\|_\infty = 0 ,$$

and so, for any  $n \in \mathbb{N}$ ,

$$\lim_{r \downarrow 0} \|\psi_r^n(\gamma_r) - \psi^n(\gamma)\|_\infty = 0 .$$

Since  $\gamma_r^* = \psi_r(\gamma_r^*)$ , by (3.5) and (3.6), defining  $A_r^0(t) := C_r(t)A_0^r + 4\pi c^2 S_r(t)E_0$ , we have

$$\begin{aligned} |\psi_r(\gamma_r^*)(t)| &\leq \frac{e}{|m_r|c} |\langle A_r^{\gamma_r^*}(t), \rho_r \rangle| + \frac{1}{|m_r|} |\tilde{\gamma}_r^*(t)| \\ &\leq \frac{e}{|m_r|c} \|\langle A_r^0, \rho_r \rangle\|_\infty + \frac{8\pi e^2}{3m_r^2} \|\tilde{\gamma}_r^*\|_\infty \left| \int_0^t \langle S_r(t-s)\rho_r, \rho_r \rangle ds \right| + \frac{1}{|m_r|} \|\tilde{\gamma}_r^*\|_\infty \\ &\leq K_0 + \left( c(T) + \frac{1}{|m_r|} \right) \|\tilde{\gamma}_r^*\|_\infty \\ &\leq K_0 + \left( c(T) + \frac{1}{|m_r|} \right) \left( |p_0| + M_0 T \int_{\mathbb{R}^3} |x| \rho_r(x) dx + M_0 T (1 + |q_0| + T \|\gamma_r^*\|_\infty) \right) . \end{aligned}$$

Therefore, if  $r$  and  $T$  are sufficiently small, then

$$\|\gamma_r^*\|_\infty \leq \left( 1 - K_1 \left( c(T) + \frac{1}{|m_r|} \right) \right)^{-1} K_2 ,$$

and so

$$\sup_{r>0} \|\gamma_r^*\|_\infty < +\infty .$$

These results imply, denoting by  $\sigma < 1$  the common, independent of  $r$ , constant of contractivity of  $\psi_r$  and  $\psi$ ,

$$\begin{aligned} \|\gamma_r^* - \gamma^*\|_\infty &\leq \|\psi_r^n(\gamma_r) - \psi^n(\gamma)\|_\infty + \sup_{r>0} \|\psi_r^n(\gamma_r) - \gamma_r^*\|_\infty + \|\psi^n(\gamma) - \gamma^*\|_\infty \\ &\leq \|\psi_r^n(\gamma_r) - \psi^n(\gamma)\|_\infty + \sigma^n \sup_{r>0} \|\gamma_r - \gamma_r^*\|_\infty + \|\psi^n(\gamma) - \gamma^*\|_\infty . \end{aligned}$$

Therefore (3.11) is proven, and the proof is done.  $\square$

**Remark 3.6.** Here we make the connection with the results obtained, for the case  $F = 0$ , in [NP]. The results there stated are a particular case of the ones given here: indeed in [NP, corollary 2.9] the initial fields  $A_0^r$  and  $A_0$  are not arbitrary elements in  $H_*^1(\mathbb{R}^3)$  and  $D(F_m)$  respectively ( however they belong to  $L^2$ -dense subsets of  $H_*^1(\mathbb{R}^3)$  and  $D(F_m)$  respectively ). Moreover the convergence conditions (3.9) are weaker than the (2.8) used in [NP]. In the present setting, writing  $A_0^r = (H_r + \lambda)^{-\frac{1}{2}} X_0^r$ , these would correspond, besides (3.9), to

$$\lim_{r \downarrow 0} \langle \rho_r, (-\Delta + \lambda)^{-\frac{1}{2}} X_0^r \rangle = \frac{4\pi e}{c} \Gamma_m(\lambda) Q_{A_0} \equiv \frac{4\pi e}{c} \Gamma_m(\lambda) \dot{q}_0 . \quad (3.12)$$

Since the relations (2.8) in [NP] are declared to be necessary and sufficient for the convergence, a contradiction seems to appear. This is not the case. Indeed in [NP] we studied the limit of the Maxwell–Lorentz system when the limit total linear momentum  $p_0$  is vanishing. In fact one has, by (2.2.), (2.3), and (3.9)

$$\begin{aligned} \lim_{r \downarrow 0} p_0^r &= \lim_{r \downarrow 0} m_r \dot{q}_0 + \frac{e}{c} \langle \rho_r, A_0^r \rangle \\ &= \lim_{r \downarrow 0} \left( -\frac{e}{c} \Gamma_r(\lambda) - \frac{2e}{3c} \langle (-\Delta + \lambda)^{-1} \rho_r, \rho_r \rangle \right) \frac{1}{\pi} \int_0^\infty \Gamma_m(x + \lambda)^{-1} \langle G_{x+\lambda}, X_0^r \rangle \frac{dx}{\sqrt{x}} \\ &\quad + \frac{e}{c} \langle \rho_r, (-\Delta + \lambda)^{-\frac{1}{2}} X_0^r \rangle \\ &\quad + \frac{2e}{3c\pi} \int_0^\infty \Gamma_r(x + \lambda)^{-1} \langle (-\Delta + x + \lambda)^{-1} \rho_r, X_0^r \rangle \langle (-\Delta + x + \lambda)^{-1} \rho_r, \rho_r \rangle \frac{dx}{\sqrt{x}} \\ &= \lim_{r \downarrow 0} -\frac{4\pi e^2}{c^2} \Gamma_m(\lambda) \dot{q}_0 + \frac{e}{c} \langle \rho_r, (-\Delta + \lambda)^{-\frac{1}{2}} X_0^r \rangle , \end{aligned}$$

and so, given (3.9), (3.12) is equivalent to  $p_0 = 0$ . This is consistent with the fact that the limit equations (3.10) coincide, in the case  $F = 0$ ,  $p_0 = 0$ , with the ones obtained in [NP]. As a last remark regarding [NP] let us point out that, since, when  $F = 0$ , the linear momentum is conserved, the condition (3.12) is preserved by the flow.

#### 4. The limit dynamics.

In this section we want to give a more detailed description of the solution of the Cauchy problem (3.10). Let us begin with an alternative characterization of the domain of  $H_{m,p}$ . By thm. 2.2 we have that the vectors in  $D(H_{m,p})$  are of the type

$$A = A_\lambda + \Gamma_m(\lambda)^{-1} M \left( A_\lambda(0) - \frac{c}{e} p \right) G_\lambda, \quad A_\lambda \in H_*^2(\mathbb{R}^3), \quad -\lambda \in \rho(H_m), \quad \lambda > 0 .$$

This obviously implies

$$A_\lambda(0) = \left( \frac{4\pi e}{c} \right) \Gamma_m(\lambda) Q_A + \frac{c}{e} p ,$$

and, by the definition of  $\Gamma_m(\lambda)$ , we have

$$A_\lambda(0) - \frac{2}{3} \frac{e}{c} \sqrt{\lambda} Q_A = -\frac{mc}{e} Q_A + \frac{c}{e} p .$$

Since

$$\lim_{r \downarrow 0} \frac{1}{4\pi r^2} \int_{S_r} \left( A - \frac{4\pi e}{c} M Q_A G_0 \right) d\mu_r(x) = A_\lambda(0) - \frac{2}{3} \frac{e}{c} \sqrt{\lambda} Q_A ,$$

in conclusion we have the following

**Lemma 4.1.**  $A(t) \in D(H_{m,p(t)})$  if and only if  $A(t) - \frac{4\pi e}{c} M Q_{A(t)} G_\lambda \in H_*^2(\mathbb{R}^3)$ ,  $-\lambda \in \rho(H_m)$ ,  $\lambda > 0$ , and the following boundary condition holds:

$$\lim_{r \downarrow 0} \frac{1}{4\pi r^2} \int_{S_r} \left( A(t) - \frac{4\pi e}{c} M Q_{A(t)} G_0 \right) d\mu_r(x) = -\frac{mc}{e} Q_{A(t)} + \frac{c}{e} p(t) . \quad (4.1)$$

Moreover

$$H_{m,p(t)} A(t) = -\Delta \left( A(t) - \frac{4\pi e}{c} M Q_{A(t)} G_\lambda \right) - \lambda \frac{4\pi e}{c} M Q_{A(t)} G_\lambda .$$

The next result is analogous to thm. 3.3 in [NP]: it characterizes the solutions of the limit equation (3.10).

**Theorem 4.2.** Given  $F \in \text{Lip}(\mathbb{R}^3; \mathbb{R}^3)$ , let

$$(A, E, q, p) \in C^1([0, T]; D(F_m)) \times C^1([0, T]; L_*^2(\mathbb{R}^3)) \times C^2([0, T]; \mathbb{R}^3) \times C^1([0, T]; \mathbb{R}^3)$$

be the strict solution of the Cauchy problem

$$\left\{ \begin{array}{l} \dot{A} = 4\pi c^2 E \\ \dot{E} = -\frac{1}{4\pi} H_{m,p} A \\ \dot{q} = Q_A \\ \dot{p} = F(q) \\ A(0) = A_0 \in D(H_{m,p_0}), \quad E(0) = E_0 \in D(F_m), \quad q(0) = q_0, \quad p(0) = p_0 , \end{array} \right. \quad (4.2)$$

Then

$$A(t) = A_f(t) + \frac{4\pi e}{c} M A_\delta(t) ,$$

where  $A_f(t)$  is the solution of the free wave equation with initial data  $A_0, E_0$ , and

$$A_\delta(t, x) = \frac{1}{4\pi} \frac{\theta(ct - |x|)}{|x|} Q_{A(t-|x|/c)} ,$$

where  $\theta$  denotes the Heaviside function. Moreover  $Q_A$  satisfies the equation

$$\dot{Q}_{A(t)} = c\sqrt{\lambda_0} Q_{A(t)} + \frac{3c^2}{2e} A_f(t, 0) - \frac{3c^3}{2e^2} p(t) . \quad (4.3)$$

*Proof.* Let

$$A(t) = A_f(t) + \frac{4\pi e}{c} M A_\delta(t) ,$$

where

$$A_\delta(t, x) = \frac{1}{4\pi} \frac{\theta(ct - |x|)}{|x|} Q \left( t - \frac{|x|}{c} \right)$$

denotes the retarded potential of the source  $Q\delta_0$ , and  $A_f(t)$  denotes the solution of the free wave equation with initial data  $A_0$ , and  $E_0$ . Therefore  $A(t)$  satisfies the distributional equation

$$\frac{1}{c^2} \ddot{A} = \Delta A + \frac{4\pi e}{c} MQ\delta_0 .$$

By the Kirchhoff formula one can verify that  $A_f$  gives no contribution to  $Q_A$  ( see [NP] ). Therefore  $Q_{A(t)} = Q(t)$ . Since, by an elementary integration,

$$\begin{aligned} & \lim_{r \downarrow 0} \frac{1}{4\pi r^2} \int_{S_r} \left( A - \frac{4\pi e}{c} MQG_0 \right) d\mu_r(x) \\ &= A_f(t, 0) + \frac{2}{3} \frac{e}{c^2} \lim_{r \downarrow 0} \left( \frac{Q(t - r/c) - Q(t)}{r/c} \right) \\ &= A_f(t, 0) - \frac{2}{3} \frac{e}{c^2} \dot{Q}(t) \end{aligned}$$

we have that  $A$  satisfies the boundary condition (4.1) if and only if (4.3) holds true. By lemma 4.1 ( applied in the case  $p = 0$  ) and by [NP, thm.3.3] it follows that

$$A - \frac{4\pi e}{c} MQG_\lambda \in H_*^2(\mathbb{R}^3) ,$$

and so, by lemma 4.1 again, we have  $A \in D(H_{m,p})$ . The proof is then concluded by the distributional identity

$$\Delta A + \frac{4\pi e}{c} MQ\delta_0 = \Delta \left( A - \frac{4\pi e}{c} MQG_\lambda \right) + \lambda \frac{4\pi e}{c} MQG_\lambda \equiv -H_{m,p} A .$$

□

**Remark 4.3.** Alternatively thm. 4.2 can be rephrased by saying that the strict solution of (4.2) coincides with the solution of the Cauchy problem

$$\left\{ \begin{array}{l} \dot{A} = 4\pi c^2 E \\ \dot{E} = \frac{1}{4\pi} \Delta A + \frac{e}{c} M \dot{q} \delta_0 \\ \ddot{q}(t) = c \sqrt{\lambda_0} \dot{q}(t) + \frac{3c^2}{2e} A_f(t, 0) - \frac{3c^3}{2e^2} p(t) \\ \dot{p} = F(q) \\ A(0) = A_0, \quad \dot{A}(0) = \dot{A}_0, \quad q(0) = q_0, \quad \dot{q}(0) = Q_{A_0}, \quad p(0) = p_0 . \end{array} \right. \quad (4.4)$$

Differentiating ( when possible ) equation (4.4), we obtain the classical Abraham–Lorentz–Dirac equation

$$\left\{ \begin{array}{l} -m\tau_0 \ddot{q}(t) + m\ddot{q}(t) = -\frac{e}{c} \dot{A}_f(t, 0) + F(q(t)) \\ q(0) = q_0, \quad \dot{q}(0) = Q_{A_0}, \quad \ddot{q}(0) = Q_{\dot{A}_0} , \end{array} \right.$$

where  $\tau_0 := \frac{2e^2}{3mc^3}$ .

**Remark 4.4.** By the definition (3.3), and by  $D(H_{m,p}) \subset D(F_m)$ , we have

$$\begin{aligned} \langle H_{m,p} A_1, A_2 \rangle &= F_m \left( A_1 + \frac{c}{e} \Gamma_m(\lambda)^{-1} p G_\lambda, A_2 \right) + \lambda \frac{c}{e} \Gamma_m(\lambda)^{-1} p \cdot \langle G_\lambda, A_2 \rangle \\ &= F_m(A_1, A_2) + 4\pi p \cdot Q_{A_2} . \end{aligned}$$

Therefore, in the case  $F = -\nabla V$ , equations (4.2) are nothing but the Hamilton equations corresponding to the ( degenerate ) Hamiltonian

$$\mathcal{H}_m(A, E, q, p) := 2\pi c^2 \|E\|_2^2 + \frac{1}{8\pi} F_m(A, A) + p \cdot Q_A + V(q) ;$$

this is defined on the symplectic vector space  $(D(F_m) \times L_*^2(\mathbb{R}^3) \times \mathbb{R}^6, \Omega_0)$ , where  $\Omega_0$  denotes the standard symplectic form

$$\Omega_0((A_1, E_1, q_1, p_1), (A_2, E_2, q_2, p_2)) = \langle A_1, E_2 \rangle - \langle A_2, E_1 \rangle + q_1 \cdot p_2 - q_2 \cdot p_1 .$$

Moreover one has the following convergence result:

**Theorem 4.5.** *Let*

$$\mathcal{H}_r(A, E, q, p) = 2\pi c^2 \|E\|_2^2 + \frac{1}{8\pi} \|\nabla A\|_2^2 + \frac{1}{2m_r} \left| p - \frac{e}{c} \langle \rho_r, A \rangle \right|^2 + V(q)$$

*be the Hamiltonian giving the equations (2.2), defined on the symplectic vector space  $(H_*^1(\mathbb{R}^3) \times L_*^2(\mathbb{R}^3) \times \mathbb{R}^6, \Omega_0)$ . Let  $E \in L_*^2(\mathbb{R}^3)$ ,  $(q, p) \in \mathbb{R}^6$ , and let  $A_r \in H_*^1(\mathbb{R}^3)$ ,  $A \in D(F_m)$  satisfy the condition (3.9). Then*

$$\lim_{r \downarrow 0} \mathcal{H}_r(A_r, E, q, p) = \mathcal{H}_m(A, E, q, p) .$$

*Proof.* Let us write  $A_r = (H_r + \lambda)^{-\frac{1}{2}} X_r$ ,  $A = (H_m + \lambda)^{-\frac{1}{2}} X$ ,  $\lambda > \lambda_0$ . Since, by (3.9) and lemma 2.4, one has

$$\lim_{r \downarrow 0} \frac{e}{m_r c} p \cdot \langle \rho_r, A_r \rangle = p \cdot Q_A ,$$

obviously it will suffice to prove that

$$\begin{aligned} \lim_{r \downarrow 0} & \| (-\Delta + \lambda)^{\frac{1}{2}} (H_r + \lambda)^{-\frac{1}{2}} X_r \|_2^2 + \frac{4\pi e^2}{m_r c^2} |\langle \rho_r, (H_r + \lambda)^{-\frac{1}{2}} X_r \rangle|^2 \\ &= \|X\|_2^2 + \left( \frac{4\pi e}{c} \right)^2 |\Gamma_m(\lambda)| L_m(X)|^2 . \end{aligned}$$

By remark 2.3 one obtains

$$\begin{aligned}
& \lim_{r \downarrow 0} \|(-\Delta + \lambda)^{\frac{1}{2}} A_r\|_2^2 + \frac{4\pi e^2}{m_r c^2} |\langle \rho_r, A_r \rangle|^2 = \lim_{r \downarrow 0} \|X_r\|_2^2 \\
& + \frac{2}{\pi} \int_{\mathbb{R}_+} \Gamma_r(x + \lambda)^{-1} \langle (-\Delta + x + \lambda)^{-1} \rho_r, X_r \rangle \cdot \langle (-\Delta + \lambda)^{\frac{1}{2}} (-\Delta + x + \lambda)^{-1} \rho_r, X_r \rangle \frac{dx}{\sqrt{x}} \\
& + \frac{2}{3\pi^2} \int_{\mathbb{R}_+^2} \Gamma_r(x + \lambda)^{-1} \Gamma_r(y + \lambda)^{-1} \langle (-\Delta + x + \lambda)^{-1} \rho_r, X_r \rangle \cdot \langle (-\Delta + y + \lambda)^{-1} \rho_r, X_r \rangle \times \\
& \quad \times \langle (-\Delta + \lambda)(-\Delta + x + \lambda)^{-1} \rho_r, (-\Delta + y + \lambda)^{-1} \rho_r \rangle \frac{dx dy}{\sqrt{xy}} \\
& + \frac{4\pi e^2}{m_r c^2} \left( |\langle \rho_r, (-\Delta + \lambda)^{-\frac{1}{2}} X_r \rangle|^2 \right. \\
& + \frac{4}{3\pi} \int_{\mathbb{R}_+} \Gamma_r(x + \lambda)^{-1} \langle (-\Delta + x + \lambda)^{-1} \rho_r, X_r \rangle \cdot \langle \rho_r, (-\Delta + \lambda)^{-\frac{1}{2}} X_r \rangle \times \\
& \quad \times \langle (-\Delta + x + \lambda)^{-1} \rho_r, \rho_r \rangle \frac{dx}{\sqrt{x}} + \\
& \left. \frac{4}{9\pi^2} \int_{\mathbb{R}_+^2} \Gamma_r(x + \lambda)^{-1} \Gamma_r(y + \lambda)^{-1} \langle (-\Delta + x + \lambda)^{-1} \rho_r, X_r \rangle \cdot \langle (-\Delta + y + \lambda)^{-1} \rho_r, X_r \rangle \times \right. \\
& \quad \left. \times \langle (-\Delta + x + \lambda)^{-1} \rho_r, \rho_r \rangle \langle (-\Delta + y + \lambda)^{-1} \rho_r, \rho_r \rangle \frac{dx dy}{\sqrt{xy}} \right) .
\end{aligned}$$

By (2.3), (2.7), (3.9), and by

$$\begin{aligned}
& \frac{2}{3} \lim_{r \downarrow 0} \frac{8\pi e^2}{3c^2} \frac{\langle (-\Delta + x + \lambda)^{-1} \rho_r, \rho_r \rangle}{m_r} \langle (-\Delta + y + \lambda)^{-1} \rho_r, \rho_r \rangle \\
& \quad + \langle (-\Delta + \lambda)(-\Delta + x + \lambda)^{-1} \rho_r, (-\Delta + y + \lambda)^{-1} \rho_r \rangle \\
& = -\Gamma_m(\lambda) , \\
& \lim_{r \downarrow 0} \frac{8\pi e^2}{3c^2} \frac{\langle (-\Delta + x + \lambda)^{-1} \rho_r, \rho_r \rangle}{m_r} \langle (\rho_r, (-\Delta + \lambda)^{-\frac{1}{2}} X_r) \rangle \\
& \quad + \langle (-\Delta + \lambda)^{\frac{1}{2}} (-\Delta + x + \lambda)^{-1} \rho_r, X_r \rangle = 0 ,
\end{aligned}$$

the proof is done if we prove that

$$\lim_{r \downarrow 0} \frac{|\langle \rho_r, (-\Delta + \lambda)^{-\frac{1}{2}} X_r \rangle|^2}{m_r} = 0 .$$

Obviously this is equivalent to prove that  $\{\sqrt{r}(-\Delta + \lambda)^{-\frac{1}{2}} \rho_r\}_{r>0}$  weakly converges to zero in  $L^2(\mathbb{R}^3)$ . Evidently such a family weakly converges to zero on a dense set, and so we only need to prove that it is bounded in  $L^2(\mathbb{R}^3)$ . This is true since, by standard Sobolev estimates, one has

$$\begin{aligned}
& |\langle \sqrt{r}(-\Delta + \lambda)^{-\frac{1}{2}} \rho_r, f \rangle| \leq \sqrt{r} \|(-\Delta + \lambda)^{-\frac{1}{2}} \rho_r\|_2 \|f\|_2 \\
& \leq C \sqrt{r} \|\rho_r\|_{6/5} \|f\|_2 = C \|\rho\|_{6/5} \|f\|_2 .
\end{aligned}$$

□

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