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# THE WAVE EQUATION WITH ONE POINT INTERACTION AND THE ( LINEARIZED ) CLASSICAL ELECTRODYNAMICS OF A POINT PARTICLE

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**Abstract.** *We study the point limit of the linearized Maxwell–Lorentz equations describing the interaction, in the dipole approximation, of an extended charged particle with the electromagnetic field. We find that this problem perfectly fits into the framework of singular perturbations of the Laplacian; indeed we prove that the solutions of the Maxwell–Lorentz equations converge – after an infinite mass renormalization which is necessary in order to obtain a non trivial limit dynamics – to the solutions of the abstract wave equation defined by the self–adjoint operator describing the Laplacian with a singular perturbation at one point. The elements in the corresponding form domain have a natural decomposition into a regular part and a singular one, the singular subspace being three–dimensional. We obtain that this three–dimensional subspace is nothing but the velocity particle space, the particle dynamics being therefore completely determined – in an explicit way – by the behaviour of the singular component of the field. Moreover we show that the vector coefficient giving the singular part of the field evolves according to the Abraham–Lorentz–Dirac equation.*

**Résumé.** *Nous étudions la limite ponctuelle des équations linéarisée de Maxwell–Lorentz, qui décrivent l'interaction d'une particule étendue avec le champ électromagnétique, en approximation de dipôle. Nous démontrons que le problème s'encadre dans la théorie des perturbations singuliére du Laplacien. En effet nous prouvons que la solution des équations de Maxwell–Lorentz converge – à la suite d'une rénormalisation de masse infinie qui est nécessaire pour que la dynamique limite ne soit pas triviale – á un solution d'une équation des ondes abstraite qui est définie par l'opérateur autoadjoint décrivant le Laplacien par le moyen d'une perturbation singulière dans un point. On peut décomposer d'une facon naturelle les éléments du correspondent domaine de forme en une partie régulière et une partie singulière qui appartient à un espace de dimension trois coïncident avec l'espace de la vitesse de la particule. La dynamique de la particule est donc complètement déterminée par le comportement de la composante singulière du champ. En outre nous démontrons que le coefficient qui correspond à la partie singulière du champ est solution de l'équation de Abraham–Lorentz–Dirac.*

## 1. Introduction

One of the most important, and difficult, problems in classical and quantum field theory is the passage to the so called "local limit", i.e. the limit in which the interaction takes place at one point.

The most meaningful theory in which this problem shows up is electrodynamics of point particles, both in its classical and its quantum version ( [D],[Be],[CN] ). The goal of this paper is to show that it is possible to construct a complete and mathematically consistent theory of the dynamical system constituted by the electromagnetic field interacting with a charged point particle, at least in the linear ( also called dipole ) approximation.

Such a program goes back to the early studies of Kramers ( [Kr] ) in renormalization theory, the most comprehensive report on which is perhaps the Ph.D. thesis of Kramers' pupil N.G. VanKampen ( [VK] ).

But these efforts did not reach the goal, and, up to now, a system of equations already renormalized and describing the dynamics of the coupled system was lacking. The model here studied is the point limit of a regularization of the Maxwell–Lorentz system, also called, in this particular case, the Pauli–Fierz model ( [PF],[A],[BG],[Bl] ): it consists of an extended charged particle, described by a spherically symmetric form factor $\rho$, interacting with the electromagnetic field and in linear approximation. In Coulomb gauge this corresponds, for the vector potential $A$ and for the particle position $q$, to the equations

$$\begin{cases} \dfrac{1}{c^2}\ddot{A} =\Delta A + \dfrac{4\pi e}{c} M\dot{q}\rho \\ m_0\ddot{q} = -\dfrac{e}{c}\displaystyle\int_{\mathbb{R}^3} \rho(x)\dot{A}(x)\ dx \\ A(0) =A_0, \quad \dot{A}(0) = \dot{A}_0, \quad q(0) = q_0, \quad \dot{q}(0) = \dot{q}_0\ ; \end{cases} \tag{1.1}$$

here $M$ is the projection onto the divergenceless part, $e$ is the electric charge, $c$ is the light velocity, and $m_0$ is the bare, or mechanical, mass, a parameter to be suitably redefined to get the equations in the point limit, i.e. when $\rho$ weakly converges to $\delta_0$, the Dirac mass at zero. Notice that in the Coulomb gauge the scalar potential $\phi$ plays no role, in that it is purely static, being the solution of $-\Delta\phi = 4\pi e\rho$, and for a spherically symmetric form factor $\rho$ it gives no contribution to the Lorentz force on the particle ( see e.g. [H] ).

As it stands the system is meaningless in the point limit, due to the fact that the solution for the field is not sufficiently regular to be evaluated at the origin, as would be required in the right hand side of the particle equation in (1.1), so that the Lorentz force on the particle appears to be ill defined.

The traditional way of dealing with this problem consists of decoupling the system, firstly eliminating the degrees of freedom of the field by integrating the field equation in (1.1) and then substituting into the particle equation. This procedure leads, through heuristic manipulations, to the classical Abraham–Lorentz–Dirac equation for the particle, namely

$$-m\tau_0\dddot{q}(t) + m\ddot{q}(t) = F_{ext}(t, q(t))\ , \tag{1.2}$$

( [D],[L] ), where $\tau_0 = \frac{2e^2}{3mc^3}$, while $m$ is the renormalized ( or phenomenological ) mass. A first rigorous analysis of the behaviour of the particle in the point limit appears in [BN] and more thoroughly in [B].

But the dynamics of the field is usually not even dealt with. To the authors' knowledge the only previous papers in which the field dynamics plays some role are [IW] and [K] ( both inspired by [EIH] ).

The procedure here adopted is more functional analytic minded, but equally simple in spirit. We too decouple the system, in the reverse way however, obtaining an equation for the field variable alone as follows. Integrate the particle equation in (1.1) and substitute into the field equation, neglecting for the moment the contribution of the initial data $\dot{q}_0, A_0$ – a contribution that, as a matter of fact, will turn out to disappear in the limit ( see lemma 2.6 for this non completely trivial fact ). The result is

$$\frac{1}{c^2}\ddot{A} = \Delta A - \frac{4\pi e^2}{m_0 c^2} M \int_{\mathbb{R}^3} \rho(x) A(x) dx \ \rho \ .$$

Now the right hand side is a finite rank perturbation of the Laplacian, which, as we will show, turns out to converge in the point limit, in norm resolvent sense, to a well defined self–adjoint operator $-H_m$, provided mass is renormalized according to the classical prescription adopted also in the quantum context ( [Be] )

$$m_0 = m - m_{em} = m - \frac{8\pi e^2}{3c^2} E(\rho) \ ,$$

where $E(\rho)$ is the electrostatic energy of the form factor, and $m$ is the phenomenological mass which appears here as an arbitrary parameter not contained in the original system (1.1) ( see thm. 2.1 ). Instead, if such a renormalization is not performed, the limit dynamics turns out to be trivial in a sense to be explained in thm. 2.11.

The limit equation for the field is then

$$\frac{1}{c^2}\ddot{A} = -H_m A \ , \tag{1.3}$$

an abstract wave equation where $H_m$ is an operator that is self–adjoint, bounded from below, containing the renormalized mass only.

Operators of this kind, when $m$ is replaced by a parameter to be interpreted as the scattering length, are well known in ordinary quantum mechanics, and a wide mathematical literature exists on them ( [AGHH] ). In the quantum mechanics context they are related to the so called "Fermi pseudopotentials", which describe potentials with a singular support – one point in our situation ( see remark 2.4 ). Such an operator $H_m$ represents a rigorous version of the formal writing "$-\Delta + M \cdot \delta_0$", in which $\delta_0$ would act as a multiplication operator.

A main characteristics of the operator $H_m$ is the structure of its form domain, the elements of which being the sum of a regular part and of a singular one with a fixed singularity, of the Coulomb type, at the origin ( see thm. 2.3 ). The remarkable fact is that the solutions of the Maxwell–Lorentz system (1.1) converge to the ones of (1.3) if and only if the coefficient of this singular component is proportional to the velocity $\dot{q}$ of the particle through the relation

$$\dot{q} = \frac{3c}{2e} \lim_{r \downarrow 0} \ r \frac{1}{4\pi r^2} \int_{S_r} A(x) d\mu_r(x) \tag{1.4}$$

( see corollary 2.9 and (2.5) ). So to determine the flow of the complete system it suffice to calculate it for the field alone, i.e. for the equation (1.3). This is easily done ( see thm. 3.1 ), and the particle evolution is then readily and explicitly calculated by (1.4) ( see (3.2) and (3.3) ). Moreover it turns out that the vector coefficient determining the singular part of the solution of (1.3), i.e. $\dot{q}(t)$, evolves according to Abraham–Lorentz–Dirac equation (1.2), where $F_{ext}(t, q(t))$ is now the ( linearized ) Lorentz force due to the free evolution; more precisely it satisfies its integrated version

$$-m\tau_0\ddot{q}(t) + m\dot{q}(t) = -\frac{e}{c}A_f(t, 0) ,$$

where $A_f$ denotes the solution of the free wave equation with the given initial data ( see thm. 3.2 ). With such a $q(t)$ the solution of (1.3) satisfies then the distributional equation

$$\frac{1}{c^2}\ddot{A} = \Delta A + \frac{4\pi e}{c}M\dot{q}\delta_0$$

( see remark 3.4 ). This establishes the link with the traditional description.

We conclude with two remarks. The first one deals with the well–known problem of runaway solutions ( [D],[CN] ). The operator $H_m$ has a negative eigenvalue ( given by (2.6) ), so that equation (1.3), and hence the particle motion too, admits runaways. We do not discuss here this relevant physical problem because we are concerned only with the mathematical description of the system. In any case, on the purely mathematical side, the runaways can be readily eliminated by projection onto the absolutely continuous subspace of $H_m$.

The second remark concerns the possibility of extending the procedure here pursued to the non linear complete Maxwell–Lorentz system. Its point limit resists, up to now, every effort of a mathematical study. Some preliminary work shows, contrarily to the most obvious conjecture, that the solution to this problem is not given, as regards the limit operator, by the Laplacian with a singular perturbation moving along the particle trajectory. This indicates that, in the case of a moving delta interaction, the situation for the wave equation is very different from the one for the heat equation, as studied in [DFT].

## 2. The point limit of the Maxwell–Lorentz equations.

Let us start with some notations. We denote by $L^2_*(\mathbb{R}^3)$ the Hilbert space of square integrable, divergence–free, vector fields on $\mathbb{R}^3$. $M$ will be the projection from $L^2_3(\mathbb{R}^3)$, the Hilbert space of square integrable vector fields on $\mathbb{R}^3$, onto $L^2_*(\mathbb{R}^3)$ and we will use the same symbol $\langle\cdot,\cdot\rangle$ ( $\|\cdot\|_2$ is the corresponding Hilbert norm ) to indicate the scalar products in $L^2(\mathbb{R}^3)$, $L^2_3(\mathbb{R}^3)$, $L^2_*(\mathbb{R}^3)$ and also to indicate the obvious pairing between an element of $L^2_3(\mathbb{R}^3)$ and one of $L^2(\mathbb{R}^3)$ ( the result being a vector in $\mathbb{R}^3$ ). By the same abuse of notation, given two functions $f$ and $g$ in $L^2(\mathbb{R}^3)$, by $f \otimes g$ we will indicate the operator in $L^2_3(\mathbb{R}^3)$ defined by $f \otimes g(A) := f\langle g, A\rangle$. $H^s(\mathbb{R}^3)$, $s \in \mathbb{R}$, indicates the usual scale of Sobolev–Hilbert spaces, and the meaning of $H^s_3(\mathbb{R}^3)$ and $H^s_*(\mathbb{R}^3)$ should now be clear. Finally, given a measurable function $\rho$ we define its energy $E(\rho)$ as

$$E(\rho) := \frac{1}{4\pi}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{\rho(x)\rho(y)}{|x-y|}dx\, dy .$$

On $L^2_*(\mathbb{R}^3)\times\mathbb{R}^3$ ( the correct domain of definition will be specified later ) let us consider the system of equations ( $r>0$ )

$$\begin{cases} \frac{1}{c^2}\ddot{A}_r =\Delta A_r + \frac{4\pi e}{c} M\dot{q}_r\rho_r \\ m_r\ddot{q}_r = -\frac{e}{c}\langle\rho_r,\dot{A}_r\rangle \\ A_r(0) =A_0^r, \quad \dot{A}_r(0)=\dot{A}_0, \quad q_r(0)=q_0, \quad \dot{q}_r(0)=\dot{q}_0 \ , \end{cases} \tag{2.1}$$

where

$$m_r := m - \frac{8\pi e^2}{3c^2}E(\rho_r) \ , \tag{2.2}$$

and where $\rho_r(x):=r^{-3}\rho(r^{-1}x)$, $\rho\in L^2(\mathbb{R}^3)$ a spherically symmetric probability density with bounded support. Therefore

$$m_r = m - \frac{1}{r}\frac{8\pi e^2}{3c^2}E(\rho) \ ,$$

( observe that $E(\rho)$ is finite by Sobolev lemma since $\rho\in\cap_{1\le p\le 2}L^p(\mathbb{R}^3)$ ) and $\rho_r$ weakly converges, as $r\downarrow 0$, to $\delta_0$, the Dirac mass at zero. Integrating the particle equation in (2.1) one obtains

$$\dot{q}_r = -\frac{e}{m_r c}\langle\rho_r,A_r\rangle + \dot{q}_0 + \frac{e}{m_r c}\langle\rho_r,A_0^r\rangle \ .$$

Inserting this expression into the field equation one gets

$$\frac{1}{c^2}\ddot{A}_r = -H_r A_r + \frac{4\pi e}{c}M v_r\rho_r \ ,$$

where

$$H_r := -\Delta + \frac{4\pi e^2}{m_r c^2} M\cdot\rho_r\otimes\rho_r \ ,$$

and

$$v_r := \dot{q}_0 + \frac{e}{m_r c}\langle\rho_r,A_0^r\rangle \ .$$

Since $M\cdot\rho_r\otimes\rho_r$ is a bounded symmetric operator, $H_r$ is a self–adjoint operator on $L^2_*(\mathbb{R}^3)$ with operator domain $H^2_*(\mathbb{R}^3)$ and form domain $H^1_*(\mathbb{R}^3)$.

**Theorem 2.1.** *As $r\downarrow 0$, i.e as $\rho_r$ weakly converges to $\delta_0$, the self–adjoint operator $H_r$ defined above converges in norm resolvent sense in $L^2_*(\mathbb{R}^3)$ to a self–adjoint operator $H_m$, where $H_m$ has the resolvent*

$$(H_m+\lambda)^{-1} = (-\Delta+\lambda)^{-1} + \Gamma_m(\lambda)^{-1}M\cdot G_\lambda\otimes G_\lambda \ ,$$

*and where*

$$-\lambda\in\rho(H_m), \quad \lambda>0, \qquad \Gamma_m(\lambda) = -\frac{mc^2}{4\pi e^2}+\frac{\sqrt{\lambda}}{6\pi}, \qquad G_\lambda(x) := \frac{1}{4\pi}\frac{e^{-\sqrt{\lambda}|x|}}{|x|} \ .$$

*Proof.* The proof proceeds as in [AGHH, §II.1.1]. Since

$$\langle f, Mvg\rangle = \frac{2}{3}v\langle f,g\rangle$$

if $v\in\mathbb{R}^3$, $f,g\in L^2(\mathbb{R}^3)$ spherically symmetric, a straightforward computation gives

$$(H_r+z^2)^{-1} = (-\Delta+z^2)^{-1} + \Gamma_r(z^2)^{-1}M\cdot(-\Delta+z^2)^{-1}\rho_r\otimes(-\Delta+z^2)^{-1}\rho_r\ ,$$

where Im $z\neq 0$, Re $z>0$, and

$$\begin{aligned}\Gamma_r(z^2) &= -\frac{m_r c^2}{4\pi e^2} - \frac{2}{3}\langle(-\Delta+z^2)^{-1}\rho_r,\rho_r\rangle\\ &= -\frac{mc^2}{4\pi e^2} + \frac{2}{3}\frac{1}{r}\left(E(\rho) - \langle(-\Delta+r^2z^2)^{-1}\rho,\rho\rangle\right)\ .\end{aligned} \tag{2.3}$$

Obviously

$$\lim_{r\downarrow 0}\ \Gamma_r(z^2) = \Gamma_m(z^2),\quad \lim_{r\downarrow 0}\ \|(-\Delta+z^2)^{-1}\rho_r - G_{z^2}\|_2 = 0\ ,$$

and so

$$\lim_{r\downarrow 0}\ \|(H_r+z^2)^{-1} - (H_m+z^2)^{-1}\|_{HS} = 0$$

( here HS stands for Hilbert–Schmidt ). In order to prove that $(H_m+z^2)^{-1}$ is the resolvent of a self–adjont operator one needs to prove that it is injective and symmetric. This is done as in [AGHH, page 112]. □

**Remark 2.2.** As clearly indicated by (2.3), $H_r$ has a non trivial, i.e. different from $-\Delta$, limit if and only if $m_r$ diverges according to the classical prescription (2.2).

We summarize the properties of the operator $H_m$ in the following
**Theorem 2.3.** *The vectors $A$ in the operator domain $D(H_m)$ of $H_m$ are of the type*

$$A = A_\lambda + \Gamma_m(\lambda)^{-1}MA_\lambda(0)G_\lambda,\quad A_\lambda\in H^2_*(\mathbb{R}^3),\quad -\lambda\in\rho(H_m),\quad \lambda>0\ .$$

*The above decomposition in a regular part $A_\lambda$, and a corresponding singular one, is unique, and with $A\in D(H_m)$ of this form one has*

$$(H_m+\lambda)A = (-\Delta+\lambda)A_\lambda\ . \tag{2.4}$$

*Let $F_m$ be the quadratic form corresponding to $H_m$. Then the vectors $A$ in the form domain $D(F_m)$ are of the type*

$$A = A_\lambda + \frac{4\pi e}{c}MQ_AG_\lambda,\qquad A_\lambda\in H^1_*(\mathbb{R}^3),\quad Q_A\in\mathbb{R}^3,\quad \lambda>0\ .$$

*Given $A\in D(F_m)$, $Q_A$ can be explicitly computed by the formula*

$$Q_A = \frac{3c}{2e}\lim_{r\downarrow 0} r\ \frac{1}{4\pi r^2}\int_{S_r} A(x)d\mu_r(x)\ , \tag{2.5}$$

*where $S_r$ denotes the sphere of radius $r$ and $\mu_r$ is the corresponding surface measure. The above decomposition is unique, and with $A \in D(F_m)$ of this form one has*

$$\begin{aligned} F_m^\lambda(A,A) :=& F_m(A,A) + \lambda\|A\|_2^2 \\ =& \|(-\Delta+\lambda)^{\frac{1}{2}} A_\lambda\|_2^2 + \left(\frac{4\pi e}{c}\right)^2 \Gamma_m(\lambda)|Q_A|^2 \ . \end{aligned}$$

*Moreover*

$$\sigma_{ess}(H_m) = \sigma_{ac}(H_m) = [0,+\infty), \quad \sigma_{sc}(H_m) = \emptyset \ ,$$

*and*

$$\sigma_p(H_m) = \left\{ -\left(\frac{3mc^2}{2e^2}\right)^2 \right\} \equiv \{-\lambda_0\} \ , \tag{2.6}$$

*where $-\lambda_0$ has a threefold degeneration and, given an orthonormal basis $\{e_j\}_1^3$,*

$$X_j^0 = 2\sqrt{2\pi m}\ \frac{c}{e} M e_j G_{\lambda_0} \ ,$$

*are the corresponding normalized eigenvectors.*

*Proof.* Everything follows proceeding as in [AGHH, §I.1.1, §II.1.1] as regards the operator, and as in [T, §2] as regards the form. However, for the reader convenience, we give the proof. At first observe that, if $\lambda > 0$ and $-\lambda \in \rho(H_m)$,

$$\begin{aligned} D(H_m) =& (-H_m+\lambda)^{-1}\left(L_*^2(\mathbb{R}^3)\right) = (-H_m+\lambda)^{-1}(-\Delta+\lambda)\left(H_*^2(\mathbb{R}^3)\right) \\ =& H_*^2(\mathbb{R}^3) + \Gamma_m^{-1}(\lambda) M \cdot G_\lambda \otimes G_\lambda \cdot (-\Delta+\lambda)\left(H_*^2(\mathbb{R}^3)\right) \ , \end{aligned}$$

and $\langle G_\lambda, (-\Delta+\lambda)A\rangle = A(0)$ if $A \in H_*^2(\mathbb{R}^3)$. This gives the result about $D(H_m)$. As regards the univocity of the representation let $A = 0$. Then $A_\lambda = -\Gamma_m(\lambda)^{-1} M A_\lambda(0) G_\lambda$, and the LHS is continuous if and only if $A_\lambda(0) = 0$. This implies $A_\lambda = 0$ and univocity follows. As regards the singular and absolutely continuous spectrum everything follows from a well known theorem of Weyl. The elements in the point spectrum are the poles of the resolvent, i.e. the zeroes of $\Gamma_m(z^2)$, $-z^2 < 0$; a straightforward calculation gives formula (2.6).

On the dense domain

$$D(F_m) = \left\{ A \in L_*^2(\mathbb{R}^3) \ : \ \exists Q_A \in \mathbb{R}^3 \text{ s.t. } A_\lambda := A - \frac{4\pi e}{c} M Q_A G_\lambda \in H_*^1(\mathbb{R}^3) \right\}$$

let us now define the ( positive, when $\lambda > \lambda_0$ ) symmetric quadratic form

$$F_m^\lambda(A,A) := \|(-\Delta+\lambda)^{\frac{1}{2}} A_\lambda\|_2^2 + \left(\frac{4\pi e}{c}\right)^2 \Gamma_m(\lambda)|Q_A|^2 \ .$$

Now we prove that $F_m^\lambda$ is closed. Take any sequence $\{A_n\}_1^\infty \subset D(F_m)$ converging in $L_*^2(\mathbb{R}^3)$, and such that

$$\lim_{n,m\uparrow\infty} F_m^\lambda(A_m - A_n, A_m - A_n) = 0 \ .$$

Then

$$\lim_{n,m\uparrow\infty}\|(A_m)_\lambda-(A_n)_\lambda\|_{H^1_*(\mathbb{R}^3)}=0\ ,$$

and

$$\lim_{n,m\uparrow\infty}|Q_{A_m}-Q_{A_n}|=0\ .$$

Therefore there exists $A_\lambda\in H^1_*(\mathbb{R}^3)$ and $Q_A\in\mathbb{R}^3$ such that

$$\lim_{n\uparrow\infty}\|(A_n)_\lambda-A_\lambda\|_{H^1_*(\mathbb{R}^3)}=0\ ,$$

and

$$\lim_{n\uparrow\infty}|Q_{A_n}-Q_A|=0\ .$$

The two previous formulas and the uniqueness of the strong limit give then

$$A:=A_\lambda+\frac{4\pi e}{c}MQ_AG_\lambda\in D(F_m)\ ,$$

and

$$\lim_{n\uparrow\infty}F^\lambda_m(A_n-A,A_n-A)=0\ .$$

This gives closedness. Obviously $D(H_m)\subset D(F_m)$, and, if $A\in D(H_m)$, then

$$\frac{4\pi e}{c}\Gamma_m(\lambda)Q_A=A_\lambda(0)\ .$$

By a straightforward calculation one then verifies that

$$\forall A\in D(H_m)\qquad\langle(H_m+\lambda)A,A\rangle=F^\lambda_m(A,A)\ .$$

As regards formula (2.5) observe that $Q_{MQ_AG_\lambda}=(\frac{4\pi e}{c})^{-1}Q_A$ and that the $H^{-1}(\mathbb{R}^3)$–norm of the measure $(4\pi r^2)^{-1}\mu_r$ is of the same order as $r^{-1/2}$. So $Q_{A_\lambda}=0$. □

**Remark 2.4.** By (2.4) we have that for any $A\in H^2_*(\mathbb{R}^3)$ such that $A(0)=0$, $H_mA=-\Delta A$. Therefore $H_m$ is a singular perturbation of $-\Delta$, the perturbation taking place only at $x=0$.

**Remark 2.5.** By the definition of $D(F_m)$ given in thm. 2.3, any $A\in D(F_m)$ can be written as

$$A=(H_m+\lambda)^{-\frac12}X=(-\Delta+\lambda)^{-\frac12}\tilde X+\frac{4\pi e}{c}ML_m(X)G_\lambda,\qquad X,\tilde X\in L^2_*(\mathbb{R}^3)\ ,$$

where $L_m\ :\ L^2_*(\mathbb{R}^3)\to\mathbb{R}^3$ is the bounded linear operator defined by

$$L_m(X):=Q_{(H_m+\lambda)^{-1/2}X}\ .$$

Moreover for any $A\in H^1_*(\mathbb{R}^3)$, $\lambda>\lambda_0$, $r$ sufficiently small so that $-\lambda\in\rho(H_r)$, we can write

$$A=(H_r+\lambda)^{-\frac12}X=(-\Delta+\lambda)^{-\frac12}\tilde X_r+\frac{4\pi e}{c}ML_r(X)(-\Delta+\lambda)^{-1}\rho_r,\qquad X,\tilde X_r\in L^2_*(\mathbb{R}^3),$$

where, since $(H_r+\lambda)^{-\frac{1}{2}}$ converges to $(H_m+\lambda)^{-\frac{1}{2}}$ in norm, $\lim_{r\downarrow 0}\|\tilde X_r-\tilde X\|_2=0$, and $L_r\ :\ L^2_*(\mathbb{R}^3)\to\mathbb{R}^3$ is a bounded linear operator such that

$$\lim_{r\downarrow 0}\|L_r-L_m\|_{L^2_*,\mathbb{R}^3}=0\ .$$

Let us now give some general results on second order linear differential equations in Hilbert spaces which we will need below ( for the proofs of such results see [F, Chaps. II and III],[Ki],[S1,2] ). Let $H$ be a bounded from below ( this is a necessary condition ) self–adjoint operator on $L^2_*(\mathbb{R}^3)$, and let $F$ the corresponding quadratic form. Then $H$ generates a cosine operator function $C\ :\ \mathbb{R}\ \to\ \mathcal{L}(L^2_*(\mathbb{R}^3);L^2_*(\mathbb{R}^3))$, i.e. $C$ is a strongly continuous function such that

$$C(0)=1,\qquad C(s+t)C(s-t)=2C(s)C(t)\ ,$$

$$D(H)=\{\ A\in L^2_*(\mathbb{R}^3)\ :\ \lim_{t\to 0}\frac{2}{t^2}(C(t)A-A)\text{ exists }\}\ ,$$

$$\forall A\in D(H)\qquad HA=\lim_{t\to 0}\frac{2}{t^2}(C(t)A-A)\ ,$$

$$C(t)D(H)\subseteq D(H)\ .$$

Moreover $\forall z\in\mathbb{C}$ such that $-z^2\in\rho(H)$ and Re $z>|\inf\sigma(H)\wedge 0|^{\frac{1}{2}}$

$$z(H+z^2)^{-1}A=\int_0^\infty e^{-zt}C(t)A\ dt\ ,$$

and so, by $(H+z^2)^{-1}A=z^{-2}A-z^{-2}(H+z^2)HA$, and by inverse Laplace transform, $\forall t\ge 0$, $\forall A\in D(H)$,

$$C(t)A=A-\frac{1}{2\pi i}\int_{x-i\infty}^{x+i\infty}e^{zt}\frac{1}{z}(H+z^2)^{-1}HA\ dz\ ,\qquad x>|\inf\sigma(H)\wedge 0|^{\frac{1}{2}}\ .\qquad (2.7)$$

One defines then the sine operator function $S\ :\ \mathbb{R}\ \to\ \mathcal{L}(L^2_*(\mathbb{R}^3);L^2_*(\mathbb{R}^3))$ by

$$S(t)A:=\int_0^t C(s)A\ ds\ .$$

Given $A_0,\dot A_0,X\in L^2_*(\mathbb{R}^3)$, let

$$A(t):=C(t)A_0+S(t)\dot A_0+\int_0^t S(s)X\,ds\ .$$

Then $A\in C(\mathbb{R};L^2_*(\mathbb{R}^3))$,

$$A_0\in D(F),\dot A_0\in L^2_*(\mathbb{R}^3)\quad\Rightarrow\quad A\in C(\mathbb{R};D(F))\cap C^1(\mathbb{R};L^2_*(\mathbb{R}^3))\ ,$$

$$A_0\in D(H),\dot A_0\in D(F)\quad\Rightarrow\quad A\in C(\mathbb{R};D(H))\cap C^1(\mathbb{R};D(F))\cap C^2(\mathbb{R};L^2_*(\mathbb{R}^3))\ ,$$

and $A(t)$ solves the inhomogeneous Cauchy problem

$$\begin{cases} \ddot{A}(t) = -HA(t) + X \\ A(0) = A_0, \quad \dot{A}(0) = \dot{A}_0 \ . \end{cases}$$

Moreover there is a well defined dynamics on the "phase–space" in the following sense: on $D(F) \times L^2_*(\mathbb{R}^3)$ define the Hilbert norm

$$|||(A, \dot{A})|||^2 := \|((H+\lambda)^{\frac{1}{2}} \times 1)(A, \dot{A})\|^2_{L^2_* \times L^2_*} \equiv F(A,A) + \lambda\|A\|^2_2 + \|\dot{A}\|^2_2 \ ,$$

with $\lambda$ s.t. $H+\lambda$ is strictly positive. Then $U(t)$ defined by

$$U(t) := \begin{bmatrix} C(t) & S(t) \\ \dot{C}(t) & C(t) \end{bmatrix} \quad + \quad \begin{bmatrix} \int_0^t S(s)X ds \\ S(t)X \end{bmatrix}$$

is a strongly continuous ( w.r.t. the energy norm $|||\cdot|||$ ) one–parameter group on $D(F) \times L^2_*(\mathbb{R}^3)$ with generator

$$\begin{bmatrix} 0 & 1 \\ -H & 0 \end{bmatrix} \quad + \quad \begin{bmatrix} 0 \\ X \end{bmatrix} ,$$

and $U(t)\,(D(H) \times D(F)) \subseteq D(H) \times D(F)$.

The lemma below is the key ingredient in the proof of the next theorem and hence of the successive corollary. It essentially says that convergence of the solutions of the Cauchy problem (2.1) occurs only and only if some compatibility conditions on the initial data are satisfied and the inhomogeneous term $Mv_r\rho_r$ disappears in the limit.

**Lemma 2.6.** *Given $X, X_0 \in L^2_*(\mathbb{R}^3)$, $\dot{q}_0 \in \mathbb{R}^3$, $\lambda > \lambda_0$, let*

$$A_0^r := (-\Delta+\lambda)^{-\frac{1}{2}}X_0 + \frac{4\pi e}{c} ML_r(X_0)(-\Delta+\lambda)^{-1}\rho_r, \qquad v_r := \dot{q}_0 + \frac{e}{m_r c}\langle \rho_r, A_0^r\rangle \ .$$

*Then*

$$\lim_{r\downarrow 0} \|(H_r+\lambda)^{-\frac{1}{2}}Mv_r\rho_r - (H_m+\lambda)^{-\frac{1}{2}}X\|_2 = 0$$

*if and only if*

$$X = 0, \quad \lim_{r\downarrow 0} L_r(X_0) = \dot{q}_0, \quad \lim_{r\downarrow 0} \langle \rho_r, (-\Delta+\lambda)^{-\frac{1}{2}}X_0\rangle = \frac{4\pi e}{c}\Gamma_m(\lambda)\dot{q}_0 \ . \tag{2.8}$$

**Remark 2.7.** The third condition in (2.8) requires, obviously, some regularity at the origin for $X_0$. Such a limit exists if $X_0 \in H_*^{\frac{1}{2}+\epsilon}(U_0)$, $U_0$ a neighbourhood of the origin. Moreover note that if we define $A_0^r := (H_r+\lambda)^{-1}Y_0$, $Y_0 \in L^2_*(\mathbb{R}^3)$, then one can prove ( by proceeding as in the proof below ) that lemma 2.6 holds true under the condition, beside $X=0$,

$$(-\Delta+\lambda)^{-1}Y_0(0) = \frac{4\pi e}{c}\Gamma_m(\lambda)\dot{q}_0 \ ,$$

which coincides with the relation, holding for an element in $D(H_m)$, between $Q_A$ and $A_\lambda(0)$ that can be obtained by thm. 2.3.

*Proof of lemma 2.6.* By the expression for $(H_r+\lambda)^{-1}$ we have

$$\begin{aligned}\|(H_r+\lambda)^{-\frac{1}{2}}Mv_r\rho_r\|_2^2 =&\frac{2}{3}|v_r|^2\langle(-\Delta+\lambda)^{-1}\rho_r,\rho_r\rangle\\ &+\frac{4}{9}\Gamma_r(\lambda)^{-1}(|v_r|\langle(-\Delta+\lambda)^{-1}\rho_r,\rho_r\rangle)^2 \ .\end{aligned}$$

Since

$$\lim_{r\downarrow 0}\ \Gamma_r(\lambda)=\Gamma_m(\lambda)=\frac{1}{6\pi}(\sqrt{\lambda}-\sqrt{\lambda_0})>0\ ,$$

we have the sum of two positive terms, and so $|v_r|\langle(-\Delta+\lambda)^{-1}\rho_r,\rho_r\rangle$ needs to converge. However $\langle(-\Delta+\lambda)^{-1}\rho_r,\rho_r\rangle$ diverges, and so $|v_r|$,

$$v_r=\dot q_0+\frac{e}{m_rc}\langle\rho_r,(-\Delta+\lambda)^{-\frac{1}{2}}X_0\rangle+\frac{8\pi e^2}{3m_rc^2}L_r(X_0)\langle(-\Delta+\lambda)^{-1}\rho_r,\rho_r\rangle\ ,$$

needs to converge to zero. Since $(-\Delta+\lambda)^{-\frac{1}{2}}X_0\in H^1_*(\mathbb{R}^3)\subset L^6_*(\mathbb{R}^3)$, and $\|\rho_r\|_{6/5}=r^{-1/2}\|\rho\|_{6/5}$, by Hölder inequality we obtain

$$\lim_{r\downarrow 0}\ \frac{\langle\rho_r,(-\Delta+\lambda)^{-\frac{1}{2}}X_0\rangle}{m_r}=0\ ,$$

and so we need

$$\lim_{r\downarrow 0}\ \dot q_0+\frac{8\pi e^2}{3c^2}L_r(X_0)\frac{\langle(-\Delta+\lambda)^{-1}\rho_r,\rho_r\rangle}{m_r}=0\ .$$

Since

$$\lim_{r\downarrow 0}\ \frac{\langle(-\Delta+\lambda)^{-1}\rho_r,\rho_r\rangle}{m_r}=-\frac{3c^2}{8\pi e^2}\ ,$$

we obtain

$$\lim_{r\downarrow 0}\ L_r(X_0)=L_m(X_0)=\dot q_0\ .$$

Therefore

$$\begin{aligned}\lim_{r\downarrow 0}\ v_r\langle(-\Delta+\lambda)^{-1}\rho_r,\rho_r\rangle=\lim_{r\downarrow 0}\ \frac{\langle(-\Delta+\lambda)^{-1}\rho_r,\rho_r\rangle}{m_r}\times\\ \times\left(\dot q_0\left(m_r+\frac{8\pi e^2}{3c^2}\langle(-\Delta+\lambda)^{-1}\rho_r,\rho_r\rangle\right)+\frac{e}{c}\langle\rho_r,(-\Delta+\lambda)^{-\frac{1}{2}}X_0\rangle\right)\ .\end{aligned}$$

As

$$\lim_{r\downarrow 0}\ \frac{\langle(-\Delta+\lambda)^{-1}\rho_r,\rho_r\rangle}{m_r}\left(m_r+\frac{8\pi e^2}{3c^2}\langle(-\Delta+\lambda)^{-1}\rho_r,\rho_r\rangle\right)=\frac{3}{2}\Gamma_m(\lambda)\ ,$$

we have

$$\lim_{r\downarrow 0}\ v_r\langle(-\Delta+\lambda)^{-1}\rho_r,\rho_r\rangle=-\frac{3}{2}\left(\lim_{r\downarrow 0}\ \frac{c}{4\pi e}\langle\rho_r,(-\Delta+\lambda)^{-\frac{1}{2}}X_0\rangle-\Gamma_m(\lambda)\dot q_0\right)\ .$$

Since $H_r$ converges in norm resolvent sense to $H_m$,

$$\lim_{r\downarrow 0}\ \|(H_r+\lambda)^{-\frac{1}{2}}Mv_r\rho_r-(H_m+\lambda)^{-\frac{1}{2}}X\|_2=0$$

if and only if

$$\lim_{r\downarrow 0} \|(H_r+\lambda)^{-\frac{1}{2}}(Mv_r\rho_r - X)\|_2 = 0 \ .$$

Moreover

$$\|(H_r+\lambda)^{-\frac{1}{2}}(Mv_r\rho_r - X)\|_2^2 = \|(-\Delta+\lambda)^{-\frac{1}{2}}(Mv_r\rho_r - X)\|_2^2 + \Gamma_r(\lambda)^{-1}\langle(-\Delta+\lambda)^{-1}\rho_r, Mv_r\rho_r - X\rangle^2 \ ,$$

so we have the sum of two positive terms, and therefore we need

$$\lim_{r\downarrow 0} \|(-\Delta+\lambda)^{-\frac{1}{2}}(Mv_r\rho_r - X)\|_2 = 0 \ .$$

Since $v_r\langle(-\Delta+\lambda)^{-1}\rho_r,\rho_r\rangle$ converges, and $v_r$ converges to zero, we have

$$\lim_{r\downarrow 0} \|(-\Delta+\lambda)^{-\frac{1}{2}}Mv_r\rho_r\|_2 = 0 \ .$$

Therefore

$$\|(-\Delta+\lambda)^{-\frac{1}{2}}X\|_2 = 0,$$

and so $X=0$. This implies that $v_r\langle(-\Delta+\lambda)^{-1}\rho_r,\rho_r\rangle$ needs to converge to zero, and so

$$\lim_{r\downarrow 0} \langle\rho_r, (-\Delta+\lambda)^{-\frac{1}{2}}X_0\rangle = \frac{4\pi e}{c}\Gamma_m(\lambda)\dot{q}_0 \ .$$

□

Now we want to study the convergence of $U_r(t)$, the one–parameter group on $H^1_*(\mathbb{R}^3)\oplus L^2_*(\mathbb{R}^3)$ generated by solving the equation

$$\frac{1}{c^2}\ddot{A}_r = -H_rA_r + \frac{4\pi e}{c}Mv_r\rho_r \ .$$

To avoid the problem given by the fact that the limit operator $H_m$ generates an energy norm different from the one given by $H_r$, we will indeed study the convergence, w.r.t. the $L^2_*(\mathbb{R}^3)\times L^2_*(\mathbb{R}^3)$ norm, of

$$((H_r+\lambda)^{\frac{1}{2}}\times 1)\cdot U_r(t)\cdot((H_r+\lambda)^{-\frac{1}{2}}\times 1) \ .$$

**Theorem 2.8.** *Given $X, X_0 \in L^2_*(\mathbb{R}^3)$, $\dot{q}_0\in\mathbb{R}^3$, $\lambda>\lambda_0$, let*

$$A_0^r := (-\Delta+\lambda)^{-\frac{1}{2}}X_0 + \frac{4\pi e}{c}ML_r(X_0)(-\Delta+\lambda)^{-1}\rho_r, \qquad v_r := \dot{q}_0 + \frac{e}{m_rc}\langle\rho_r, A_0^r\rangle \ .$$

*let $U_r(t)$ be the strongly continuous group on $H^1_*(\mathbb{R}^3)\times L^2_*(\mathbb{R}^3)$ with generator*

$$\begin{bmatrix} 0 & 1 \\ -c^2H_r & 0 \end{bmatrix} \quad + \quad 4\pi ec\begin{bmatrix} 0 \\ Mv_r\rho_r \end{bmatrix} \ ,$$

*and let $U_m(t)$ be the strongly continuous group on $D(F_m)\times L^2_*(\mathbb{R}^3)$ with generator*

$$\begin{bmatrix} 0 & 1 \\ -c^2H_m & 0 \end{bmatrix} \quad + \quad \begin{bmatrix} 0 \\ X \end{bmatrix} \ .$$

*Then*

$$((H_r+\lambda)^{\frac12}\times 1)\cdot U_r(t)\cdot((H_r+\lambda)^{-\frac12}\times 1)$$

*converges, strongly in* $L^2_*(\mathbb{R}^3)\times L^2_*(\mathbb{R}^3)$, *uniformly in* $t$ *over compact intervals, to*

$$((H_m+\lambda)^{\frac12}\times 1)\cdot U_m(t)\cdot((H_m+\lambda)^{-\frac12}\times 1)$$

*if and only if*

$$X=0,\quad \lim_{r\downarrow 0}\,L_r(X_0)=\dot q_0,\quad \lim_{r\downarrow 0}\,\langle\rho_r,(-\Delta+\lambda)^{-\frac12}X_0\rangle=\frac{4\pi e}{c}\Gamma_m(\lambda)\dot q_0\ . \tag{2.8}$$

*Proof.* Let us at first prove that the theorem holds true for $\tilde U_r(t)$ and $\tilde U_m(t)$, the affine, strongly continuous, groups with generators

$$\begin{bmatrix}0 & 1\\ -c^2(H_r+\lambda) & 0\end{bmatrix}\ +\ 4\pi ec\begin{bmatrix}0\\ Mv_r\rho_r\end{bmatrix},$$

$$\begin{bmatrix}0 & 1\\ -c^2(H_m+\lambda) & 0\end{bmatrix}\ +\ \begin{bmatrix}0\\ X\end{bmatrix}.$$

By functional calculus we have

$$\tilde U_r(t)=\hat U_r(t)\quad +\quad 4\pi e\begin{bmatrix}\frac1c(H_r+\lambda)^{-1}(1-\cos ct(H_r+\lambda)^{\frac12})Mv_r\rho_r\\ (H_r+\lambda)^{-\frac12}\sin ct(H_r+\lambda)^{\frac12}Mv_r\rho_r\end{bmatrix},$$

$$\hat U_r(t):=\begin{bmatrix}\cos ct(H_r+\lambda)^{\frac12} & \frac1c(H_r+\lambda)^{-\frac12}\sin ct(H_r+\lambda)^{\frac12}\\ -c(H_r+\lambda)^{\frac12}\sin ct(H_r+\lambda)^{\frac12} & \cos ct(H_r+\lambda)^{\frac12}\end{bmatrix},$$

$$\tilde U_m(t)=\hat U_m(t)\quad +\quad \begin{bmatrix}\frac1{c^2}(H_m+\lambda)^{-1}(1-\cos ct(H_m+\lambda)^{\frac12})X\\ \frac1c(H_m+\lambda)^{-\frac12}\sin ct(H_m+\lambda)^{\frac12}X\end{bmatrix},$$

$$\hat U_m(t):=\begin{bmatrix}\cos ct(H_m+\lambda)^{\frac12} & \frac1c(H_m+\lambda)^{-\frac12}\sin ct(H_m+\lambda)^{\frac12}\\ -c(H_m+\lambda)^{\frac12}\sin ct(H_m+\lambda)^{\frac12} & \cos ct(H_m+\lambda)^{\frac12}\end{bmatrix}.$$

By norm resolvent convergence of $H_r$ to $H_m$ we have that

$$((H_r+\lambda)^{\frac12}\times 1)\cdot\hat U_r(t)\cdot((H_r+\lambda)^{-\frac12}\times 1)$$

converges, strongly in $L^2_*(\mathbb{R}^3)\times L^2_*(\mathbb{R}^3)$, uniformly in $t$ over compact intervals, to

$$((H_m+\lambda)^{\frac12}\times 1)\cdot\hat U_m(t)\cdot((H_m+\lambda)^{-\frac12}\times 1)\ ,$$

and so we only need to study the behaviour of $(H_r+\lambda)^{-\frac12}Mv_r\rho_r$ as $r\downarrow 0$. By the previous lemma we have therefore proved the theorem with $\tilde U_r(t)$ and $\tilde U_m(t)$ in the place of $U_r(t)$ and $U_m(t)$ respectively. Let us now go back to $U_r(t)$ and $U_m(t)$. Let

$$\Lambda:=\begin{bmatrix}0 & 0\\ c^2\lambda & 0\end{bmatrix},$$

be the difference between the generators of $\tilde{U}_r(t)$, $\tilde{U}_m(t)$ and $U_r(t)$, $U_m(t)$. Then define the non autonomous, affine, vector fields

$$\Lambda_r(t) := \hat{U}_r(t)^{-1} \cdot \Lambda \cdot \tilde{U}_r(t) \ ,$$
$$\Lambda_m(t) := \hat{U}_m(t)^{-1} \cdot \Lambda \cdot \tilde{U}_m(t) \ ,$$

and let $P_r(t,s)$ and $P_m(t,s)$ be the corresponding evolution operators. With these notations we have, as can be easily verified,

$$U_r(t) = \tilde{U}_r(t) \cdot P_r(t,0), \qquad U_m(t) = \tilde{U}_m(t) \cdot P_m(t,0) \ .$$

Since

$$((H_r+\lambda)^{\frac{1}{2}} \times 1) \cdot \tilde{U}_r(t) \cdot ((H_r+\lambda)^{-\frac{1}{2}} \times 1)$$

converges, strongly in $L^2_*(\mathbb{R}^3) \times L^2_*(\mathbb{R}^3)$, to

$$((H_m+\lambda)^{\frac{1}{2}} \times 1) \cdot \tilde{U}_m(t) \cdot ((H_m+\lambda)^{-\frac{1}{2}} \times 1)$$

if and only if (2.8) holds, we have that

$$((H_r+\lambda)^{\frac{1}{2}} \times 1) \cdot \Lambda_r(t) \cdot ((H_r+\lambda)^{-\frac{1}{2}} \times 1)$$

converges, strongly in $L^2_*(\mathbb{R}^3) \times L^2_*(\mathbb{R}^3)$, to

$$((H_m+\lambda)^{\frac{1}{2}} \times 1) \cdot \Lambda_m(t) \cdot ((H_m+\lambda)^{-\frac{1}{2}} \times 1)$$

if and only if (2.8) holds. Therefore

$$((H_r+\lambda)^{\frac{1}{2}} \times 1) \cdot P_r(t,s) \cdot ((H_r+\lambda)^{-\frac{1}{2}} \times 1)$$

converges, strongly in $L^2_*(\mathbb{R}^3) \times L^2_*(\mathbb{R}^3)$, to

$$((H_m+\lambda)^{\frac{1}{2}} \times 1) \cdot P_m(t,s) \cdot ((H_m+\lambda)^{-\frac{1}{2}} \times 1)$$

if and only if (2.8) holds. In conclusion

$$((H_r+\lambda)^{\frac{1}{2}} \times 1) \cdot \tilde{U}_r(t) \cdot P_r(t,0) \cdot ((H_r+\lambda)^{-\frac{1}{2}} \times 1)$$

converges, strongly in $L^2_*(\mathbb{R}^3) \times L^2_*(\mathbb{R}^3)$, uniformly in t, to

$$((H_m+\lambda)^{\frac{1}{2}} \times 1) \cdot \tilde{U}_m(t) \cdot P_m(t,0) \cdot ((H_m+\lambda)^{-\frac{1}{2}} \times 1) \ ,$$

and the proof is done. □

We can now state our main result about the convergence of the solutions of the Maxwell–Lorentz system:

**Corollary 2.9.** *Let* $(A_r, q_r) \in C(\mathbb{R}; H^1_*(\mathbb{R}^3) \times \mathbb{R}^3) \cap C^1(\mathbb{R}; L^2_*(\mathbb{R}^3) \times \mathbb{R}^3)$ *be the mild solution of the Cauchy problem*

$$\begin{cases} \frac{1}{c^2}\ddot{A}_r =\Delta A_r + \frac{4\pi e}{c} M\dot{q}_r\rho_r \\ m_r\ddot{q}_r = -\frac{e}{c}\langle \rho_r, \dot{A}_r\rangle \\ A_r(0) =(-\Delta+\lambda)^{-\frac{1}{2}}X_0 + \frac{4\pi e}{c}ML_r(X_0)(-\Delta+\lambda)^{-1}\rho_r \\ \dot{A}_r(0) =\dot{A}_0, \quad q_r(0) = q_0, \quad \dot{q}_r(0) = \dot{q}_0 \ , \end{cases} \tag{2.9}$$

*with* $\lambda > \lambda_0$, $\dot{A}_0 \in L^2_*(\mathbb{R}^3)$, *and with* $X_0 \in L^2_*(\mathbb{R}^3)$ *such that*

$$\lim_{r\downarrow 0} L_r(X_0) = \dot{q}_0, \quad \lim_{r\downarrow 0} \langle \rho_r, (-\Delta+\lambda)^{-\frac{1}{2}}X_0\rangle = \frac{4\pi e}{c}\Gamma_m(\lambda)\dot{q}_0 \ . \tag{2.10}$$

*Then* $\forall T > 0$

$$\begin{aligned} &\lim_{r\downarrow 0}\ \sup_{|t|\le T} \|(H_r+\lambda)^{\frac{1}{2}}A_r(t) - (H_m+\lambda)^{\frac{1}{2}}A(t)\|_2 = 0, \\ &\lim_{r\downarrow 0}\ \sup_{|t|\le T} \|\dot{A}_r(t) - \dot{A}(t)\|_2 = 0, \\ &\lim_{r\downarrow 0}\ \sup_{|t|\le T} |\dot{q}_r(t) - Q_{A(t)}| = 0 \ , \end{aligned} \tag{2.11}$$

*where* $A \in C(\mathbb{R}; D(F_m)) \cap C^1(\mathbb{R}; L^2_*(\mathbb{R}^3))$ *is the mild solution of the Cauchy problem*

$$\begin{cases} \frac{1}{c^2}\ddot{A} = -H_m A \\ A(0) = (-\Delta+\lambda)^{-\frac{1}{2}}X_0 + \frac{4\pi e}{c}M\dot{q}_0 G_\lambda \\ \dot{A}(0) = \dot{A}_0 \ . \end{cases} \tag{2.12}$$

**Remark 2.10.** If in the corollary 2.9 we are concerned with strict solutions of (2.9), i.e. if $A_r(0) = (H_r+\lambda)^{-1}Y_0$, $\dot{A}_r(0) = (H_r+\lambda)^{-\frac{1}{2}}\dot{Y}_0$, $Y_0, \dot{Y}_0 \in L^2_*(\mathbb{R}^3)$, then, by remark 2.7, the (2.11)'s hold under the condition

$$(-\Delta+\lambda)^{-1}Y_0(0) = \frac{4\pi e}{c}\Gamma_m(\lambda)\dot{q}_0 \ ,$$

and $A(t)$ is now the strict solution of (2.12) with initial data

$$A(0) = (-\Delta+\lambda)^{-1}Y_0 + \frac{4\pi e}{c}M\dot{q}_0 G_\lambda, \quad \dot{A}(0) = (H_m+\lambda)^{-\frac{1}{2}}\dot{Y}_0 \ .$$

*Proof of corollary 2.9.* The assertions about the convergence of the $A_r$'s immediately follows from the previous theorem equating the initial data $\dot{q}_0, X_0$ to the elements defining the vector $v_r$. Let us now prove the convergence of the $q_r$'s.

Let $C_r(t),\ S_r(t)$ and $C_m(t),\ S_m(t)$ be the cosine and the sine operator functions of $c^2H_r$ and $c^2H_m$ respectively. Then, writing $A_r(0)=(H_r+\lambda)^{-\frac{1}{2}}Y_0^r$, $A(0)=(H_m+\lambda)^{-\frac{1}{2}}Y_0$,

$$\begin{aligned}A_r(t)&=C_r(t)(H_r+\lambda)^{-\frac{1}{2}}Y_0^r+S_r(t)\dot{A}_0+4\pi ec\int_0^t S_r(s)Mv_r\rho_r ds\\ &\equiv (H_r+\lambda)^{-\frac{1}{2}}X_r(t)\ ,\end{aligned}$$

where

$$X_r(t):=C_r(t)Y_0^r+(H_r+\lambda)^{\frac{1}{2}}\left(S_r(t)\dot{A}_0+4\pi ec\int_0^t S_r(s)Mv_r\rho_r ds\right)\ ,$$

and

$$A(t)=C_m(t)(H_m+\lambda)^{-\frac{1}{2}}Y_0+S_m(t)\dot{A}_0\equiv(H_m+\lambda)^{-\frac{1}{2}}X(t)\ ,$$

where

$$X(t):=C_m(t)Y_0+(H_m+\lambda)^{\frac{1}{2}}S_m(t)\dot{A}_0\ .$$

By the previous theorem, since $\lim_{r\downarrow 0}\ \|Y_0^r-Y_0\|_2=0$,

$$\lim_{r\downarrow 0}\ \sup_{|t|\le T}\|X_r(t)-X(t)\|_2=0\ .$$

Since

$$\dot{q}_r(t)=-\frac{e}{m_rc}\langle\rho_r,A_r(t)\rangle+v_r$$

$$=-\frac{e}{c}\frac{\langle\rho_r,(-\Delta+\lambda)^{-\frac{1}{2}}\tilde{X}_r(t)\rangle}{m_r}-\frac{8\pi e^2}{3c^2}L_r(X_r(t))\frac{\langle(-\Delta+\lambda)^{-1}\rho_r,\rho_r\rangle}{m_r}+v_r\ ,$$

the thesis follows from

$$\lim_{r\downarrow 0}\ v_r=0,$$

$$\lim_{r\downarrow 0}\ \sup_{|t|\le T}\left|\frac{\langle\rho_r,(-\Delta+\lambda)^{-\frac{1}{2}}\tilde{X}_r(t)\rangle}{m_r}\right|=0\ ,$$

$$\lim_{r\downarrow 0}\ \sup_{|t|\le T}|L_r(X_r(t))-L_m(X(t))|=0\ ,$$

$$\lim_{r\downarrow 0}\ \frac{\langle(-\Delta+\lambda)^{-1}\rho_r,\rho_r\rangle}{m_r}=-\frac{3c^2}{8\pi e^2}\ .$$

□

In the next theorem we study the case in which no mass renormalization is done, i.e. $m_r$ =const. By abuse of notation we will denote such a constant by the symbol $m$. In this situation we prove that the limit dynamics is trivial ( As regards the particle dynamics, and for zero initial data for the field, the same result already appeared in [BN] ).

**Theorem 2.11.** *Let $(A_r,q_r)\in C(\mathbb{R};H^2_*(\mathbb{R}^3)\times\mathbb{R}^3)\cap C^2(\mathbb{R};L^2_*(\mathbb{R}^3)\times\mathbb{R}^3)$ be the strict solution of the Cauchy problem*

$$\begin{cases}\frac{1}{c^2}\ddot{A}_r=\Delta A_r+\frac{4\pi e}{c}M\dot{q}_r\rho_r\\ m\ddot{q}_r=-\frac{e}{c}\langle\rho_r,\dot{A}_r\rangle\\ A_r(0)=A_0\in H^2_*(\mathbb{R}^3),\quad \dot{A}_r(0)=\dot{A}_0\in H^1_*(\mathbb{R}^3),\quad q_r(0)=q_0,\quad \dot{q}_r(0)=\dot{q}_0\ .\end{cases}$$

*Then* $\forall T > 0$

$$\lim_{r\downarrow 0}\sup_{|t|\le T}\|A_r(t)-A_f(t)\|_2=0\ ,$$
$$\lim_{r\downarrow 0}\sup_{|t|\le T}|q_r(t)-q_0|=0\ , \tag{2.13}$$

*where* $A_f$ *is the solution of the free wave equation with the same initial data* $A_0, \dot{A}_0$.

*Proof.* By remark 2.2 $H_r$ converges in norm resolvent sense to $-\Delta$. Therefore, proceeding as in thm. 2.8, the first formula in (2.13) follows from

$$\begin{aligned}&\lim_{r\downarrow 0}\|(H_r+\lambda)^{-1}Mv_r\rho_r\|_2\\ =&\lim_{r\downarrow 0}\left(1+\frac{2}{3}\Gamma_r(\lambda)^{-1}\langle(-\Delta+\lambda)^{-1}\rho_r,\rho_r\rangle\right)\|(-\Delta+\lambda)^{-1}Mv_r\rho_r\|_2=0\ .\end{aligned}$$

Since, if $t\ge 0$,

$$\begin{aligned}q_r(t)=&q_0+v_rt-\frac{e}{mc}\int_0^t\langle\rho_r,A_r(s)\rangle ds\\ =&q_0+v_rt-\frac{e}{mc}\left(\frac{1}{2\pi i}\int_{x-i\infty}^{x+i\infty}\frac{1}{c^2}e^{zt}\langle\rho_r,(H_r+(z/c)^2)^{-1}A_0\rangle\,dz\right.\\ &+\frac{1}{2\pi i}\int_{x-i\infty}^{x+i\infty}\frac{1}{c^2}e^{zt}\frac{1}{z}\langle\rho_r,(H_r+(z/c)^2)^{-1}\dot{A}_0\rangle\,dz\\ &\left.+\,4\pi ec\,\frac{1}{2\pi i}\int_{x-i\infty}^{x+i\infty}\frac{1}{c^2}e^{zt}\frac{1}{z^2}\langle\rho_r,(H_r+(z/c)^2)^{-1}Mv_r\rho_r\rangle\,dz\right)\ ,\end{aligned}$$

by

$$\begin{aligned}&\lim_{r\downarrow 0}\langle\rho_r,(H_r+z^2)^{-1}X\rangle\\ =&\lim_{r\downarrow 0}\left(1+\frac{2}{3}\Gamma_r(z^2)^{-1}\langle(-\Delta+z^2)^{-1}\rho_r,\rho_r\rangle\right)\langle\rho_r,(-\Delta+z^2)^{-1}X\rangle=0\ ,\end{aligned}$$

$X\in L^2_*(\mathbb{R}^3)$, $-z^2\in\rho(H_m)$, by

$$\lim_{r\downarrow 0}v_r=\dot{q}_0+\lim_{r\downarrow 0}\frac{e}{mc}\langle\rho_r,A_0\rangle=\dot{q}_0+\frac{e}{mc}A_0(0)\ ,$$

by

$$\lim_{r\downarrow 0}\langle\rho_r,(H_r+z^2)^{-1}Mv_r\rho_r\rangle=\frac{mc^2}{4\pi e^2}\left(\dot{q}_0+\frac{e}{mc}A_0(0)\right)\ ,$$

and by Lebesgue dominated convergence theorem, we obtain

$$\lim_{r\downarrow 0}\sup_{|t|\le T}|q_r(t)-q_0|=0\ ,$$

and the proof is done. □

## 3. The limit dynamics.

The results in corollary 2.9 induce us to declare that the system

$$\begin{cases} \frac{1}{c^2}\ddot{A} = -H_m A \\ \quad \dot{q} = Q_A \\ A(0) = A_0 \in D(F_m),\ Q_{A_0} = \dot{q}_0,\ \dot{A}(0) = \dot{A}_0 \in L^2_*(\mathbb{R}^3),\ q(0) = q_0 \ . \end{cases} \tag{3.1}$$

describes the classical electrodynamics of a point charged particle in the dipole approximation. The next theorem gives us the solution of (3.1).

**Theorem 3.1.** *Let $C_m(t),\ S_m(t)$ the cosine and the sine operator functions of $c^2 H_m$ and let $C_0(t),\ S_0(t)$ the cosine and the sine operator functions of $-c^2\Delta$. Then, $\forall t \geq 0$,*

$$C_m(t) = C_0(t) + M\cdot K_1(t) + c\sqrt{\lambda_0}M\cdot K_2(t)\ , \qquad S_m(t) = S_0(t) + M\cdot K_2(t)\ ,$$

*where*

$$K_1(t)A(x) = \frac{3}{8\pi}\frac{\theta(ct-|x|)}{|x|}\frac{1}{ct-|x|}\int_{S_{ct-|x|}} A(y)\,d\mu_{ct-|x|}(y)\ ,$$

*and $K_2(t)$ is the Hilbert–Schmidt operator with kernel given by*

$$K_2(t;x,y) = \frac{3}{8\pi c}\frac{1}{|x|\,|y|}\theta(ct-|x|-|y|)\exp\sqrt{\lambda_0}(ct-|x|-|y|)\ .$$

*Here $\theta$ denotes the Heaviside function. Moreover $\dot{C}_m(t)$, which is defined on $D(F_m)$, is given by*

$$\dot{C}_m(t) = \dot{C}_0(t) + c\sqrt{\lambda_0}M\cdot K_1(t) + c^2\lambda_0 M\cdot K_2(t) + cM\cdot K_1(t)\cdot\nabla_r + M\cdot K_3(t)\ ,$$

*where $\nabla_r$ denotes the radial derivative and*

$$K_3(t)A(x) = -\frac{c}{ct-|x|}K_1(t)A(x)\ .$$

*Proof.* By the explicit expression for the resolvent of $H_m$ we have $C_m(t) = C_0(t)+M\cdot K(t)$ where $K(t)$ has a Laplace transform given by the integral operator with kernel

$$\begin{aligned}\tilde{K}(z;x,y) =& \frac{z}{c^2}\Gamma_m(z^2/c^2)G_{z^2/c^2}(x)G_{z^2/c^2}(y)\\ =&\frac{3}{8\pi c}\ \frac{2ze^2}{-3mc^3+2e^2 z}\ \frac{1}{|x|\,|y|}\exp-\frac{z}{c}(|x|+|y|)\\ =&\frac{3}{8\pi}\left(\frac{1}{c}+\frac{3mc^2}{2e^2}\ \frac{2e^2}{-3mc^3+2e^2 z}\right)\ \frac{1}{|x|\,|y|}\exp-\frac{z}{c}(|x|+|y|)\ .\end{aligned}$$

Since ( here we use the same trick as in [ABD, §3.1] )

$$\frac{2e^2}{-3mc^3+2e^2 z}\exp-\frac{z}{c}(|x|+|y|) =$$

$$= \frac{\exp-\frac{z}{c}(|x|+|y|)}{z} + \frac{3mc^2}{2e^2}\int_0^\infty \exp\frac{3mc^2}{2e^2}s\ \frac{\exp-\frac{z}{c}(|x|+|y|+s)}{z}ds\ ,$$

since $\delta_{t-s}$ and $\theta(t-s)$ are the inverse Laplace transforms of $e^{-zs}$ and $(1/z)e^{-zs}$ respectively, and since

$$\theta(ct-|x|-|y|)+\frac{3mc^2}{2e^2}\int_0^\infty \exp\frac{3mc^2}{2e^2}s\;\theta(ct-|x|-|y|-s)ds=$$

$$=\theta(ct-|x|-|y|)\exp\frac{3mc^2}{2e^2}(ct-|x|-|y|)\;,$$

the statements about $C_m(t)$ and $\dot{C}_m(t)$ then readily follows. By the definition of $S_m(t)$ the proof is then concluded by straightforward calculations. □

Suppose now that $A_0\in D(F_m)$, $\dot{A}_0\in L^2_*(\mathbb{R}^3)$ are such that

$$Q_{A_0}=\dot{q}_0,\quad A_\lambda(r,\theta,\phi)\equiv A_0(r,\theta,\phi)-\frac{4\pi e}{c}M\dot{q}_0G_\lambda(r)=A^1_\lambda(r)A^2_\lambda(\theta,\phi)\;,$$

and

$$\dot{A}_0(r,\theta,\phi)=\dot{A}^1_0(r)\dot{A}^2_0(\theta,\phi)$$

( here we are using spherical coordinates ). Then, if $\langle\cdot\rangle$ denotes the spherical mean,

$$K_1(t)A_0(x)=\left(\frac{3}{2}(ct-|x|)\langle A^2_\lambda\rangle A^1_\lambda(ct-|x|)+\frac{e}{c}\dot{q}_0\;e^{-\sqrt{\lambda}(ct-|x|)}\right)\frac{\theta(ct-|x|)}{|x|}\;,$$

$$\begin{aligned}K_2(t)A_0(x)=&\left(\frac{3}{2c}\langle A^2_\lambda\rangle e^{\sqrt{\lambda_0}(ct-|x|)}\int_0^{ct-|x|}A^1_\lambda(r)e^{-\sqrt{\lambda_0}r}rdr\right.\\&\left.-\frac{e}{c^2}\dot{q}_0\frac{1}{\sqrt{\lambda_0}+\sqrt{\lambda}}\left(e^{-\sqrt{\lambda}(ct-|x|)}-e^{\sqrt{\lambda_0}(ct-|x|)}\right)\right)\frac{\theta(ct-|x|)}{|x|}\;,\end{aligned}$$

$$K_2(t)\dot{A}_0(x)=\left(\frac{3}{2c}\langle\dot{A}^2_0\rangle e^{\sqrt{\lambda_0}(ct-|x|)}\int_0^{ct-|x|}\dot{A}^1_0(r)e^{-\sqrt{\lambda_0}r}rdr\right)\frac{\theta(ct-|x|)}{|x|}\;.$$

Since $C_0(t)A_\lambda+S_0(t)\dot{A}_0\in H^1_*(\mathbb{R}^3)$, since

$$C_0(t)M\dot{q}_0G_\lambda=M\dot{q}_0\Phi_\lambda(t)\;,$$

where

$$\Phi_\lambda(t,x)=\frac{\lambda}{4\pi}\int_{B_{ct}(x)}\frac{G_\lambda(y)}{|x-y|}dy-\frac{1}{4\pi}\frac{\theta(ct-|x|)}{|x|}+G_\lambda(x)\;,$$

$B_r(x):=\{\;y\;:\;|x-y|<r\;\}$, we have that $C_0(t)A_0+S_0(t)\dot{A}_0$ gives no contribution to $Q_{A(t)}$, and so

$$\begin{aligned}Q_{A(t)}=&\frac{3c^2}{2e}\;t\;\langle A^2_\lambda\rangle A^1_\lambda(ct)+\dot{q}_0e^{-\sqrt{\lambda}ct}\\&+\frac{3c\sqrt{\lambda_0}}{2e}\langle A^2_\lambda\rangle e^{\sqrt{\lambda_0}ct}\int_0^{ct}A^1_\lambda(r)e^{-\sqrt{\lambda_0}r}rdr-\dot{q}_0\frac{\sqrt{\lambda_0}}{\sqrt{\lambda_0}+\sqrt{\lambda}}\left(e^{-\sqrt{\lambda}ct}-e^{\sqrt{\lambda_0}ct}\right)\\&+\frac{3}{2e}\langle\dot{A}^2_0\rangle e^{\sqrt{\lambda_0}ct}\int_0^{ct}\dot{A}^1_0(r)e^{-\sqrt{\lambda_0}r}rdr\;.\end{aligned}\tag{3.2}$$

Let us now consider strict solutions, i.e. we suppose that $A_0 \in D(H_m)$, $\dot{A}_0 \in D(F_m)$. Then

$$A_\lambda(0) = \langle A^2_\lambda\rangle A^1_\lambda(0) = \frac{4\pi e}{c}\Gamma_m(\lambda)\dot{q}_0 = \frac{3c}{2e}(\sqrt{\lambda} - \sqrt{\lambda_0})\dot{q}_0 \ ,$$

$$\dot{A}_\lambda(r,\theta,\phi) \equiv \dot{A}_0(r,\theta,\phi) - \frac{4\pi e}{c} M Q_{\dot{A}_0} G_\lambda(r) = \dot{A}^1_\lambda(r)\dot{A}^2_\lambda(\theta,\phi) \ ,$$

and

$$\begin{aligned} Q_{A(t)} =& \frac{3c^2}{2e}\ t\ \langle A^2_\lambda\rangle A^1_\lambda(ct) + \dot{q}_0 e^{-\sqrt{\lambda}ct} \\ &+ \frac{3c\sqrt{\lambda_0}}{2e}\langle A^2_\lambda\rangle e^{\sqrt{\lambda_0}ct}\int_0^{ct} A^1_\lambda(r)e^{-\sqrt{\lambda_0}r} r dr - \dot{q}_0 \frac{\sqrt{\lambda_0}}{\sqrt{\lambda_0}+\sqrt{\lambda}}\left(e^{-\sqrt{\lambda}ct} - e^{\sqrt{\lambda_0}ct}\right) \\ &+ \frac{3}{2e}\langle \dot{A}^2_\lambda\rangle e^{\sqrt{\lambda_0}ct}\int_0^{ct} \dot{A}^1_\lambda(r)e^{-\sqrt{\lambda_0}r} r dr - \frac{Q_{\dot{A}_0}}{c}\frac{1}{\sqrt{\lambda_0}+\sqrt{\lambda}}\left(e^{-\sqrt{\lambda}ct} - e^{\sqrt{\lambda_0}ct}\right) . \end{aligned} \tag{3.3}$$

Moreover $A \in C^1(\mathbb{R}; D(F_m))$, $Q_A \in C^1(\mathbb{R};\mathbb{R}^3)$, and

$$\begin{aligned} Q_{\dot{A}(t)} = \frac{d}{dt}Q_{A(t)} =& c\sqrt{\lambda_0}Q_{A(t)} - c\sqrt{\lambda}\dot{q}_0 e^{-\sqrt{\lambda}ct} + \frac{3c^2}{2e}\langle A^2_\lambda\rangle A^1_\lambda(ct) \\ &+ \frac{3c^3}{2e}\ t\ \langle A^2_\lambda\rangle \frac{d}{dr}A^1_\lambda(ct) + Q_{\dot{A}_0}e^{-\sqrt{\lambda}ct} + \frac{3c^2}{2e}\langle \dot{A}^2_\lambda\rangle \dot{A}^1_\lambda(ct) \ . \end{aligned} \tag{3.4}$$

These results, and the classical Kirchhoff formula for the solution of the free wave equation, give us the following

**Theorem 3.2.** *Let $A \in C(\mathbb{R}; D(F_m)) \cap C^1(\mathbb{R}; L^2_*(\mathbb{R}^3))$ be the mild solution of the Cauchy problem*

$$\begin{cases} \dfrac{1}{c^2}\ddot{A} = -H_m A \\ A(0) = A_0 \in D(F_m),\ Q_{A_0} = \dot{q}_0, \quad \dot{A}(0) = \dot{A}_0 \in L^2_*(\mathbb{R}^3) \ . \end{cases}$$

*Then*

$$A(t) = A_f(t) + \frac{4\pi e}{c} M A_\delta(t),$$

*where $A_f(t)$ is the solution of the free wave equation with initial data $A_0, \dot{A}_0$, and*

$$A_\delta(t,x) = \frac{1}{4\pi}\frac{\theta(ct-|x|)}{|x|}Q_{A(t-|x|/c)} \ .$$

*Moreover if $A_0 \in D(H_m)$ and $\dot{A}_0 \in D(F_m)$, i.e. if we are dealing with strict solutions, then $Q_A \in C^1(\mathbb{R};\mathbb{R}^3)$, and it solves the Cauchy problem*

$$\begin{cases} \dot{Q}_{A(t)} = c\sqrt{\lambda_0}Q_{A(t)} + \dfrac{3c^2}{2e}A_f(t,0) \\ Q_{A(0)} = \dot{q}_0 \ . \end{cases}$$

**Remark 3.3.** Let us note that in the above Cauchy problem the function $t \mapsto A_f(t,0)$ is well defined for any $t > 0$, and, by (3.3) and the Kirchhoff formula, one has

$$\lim_{t\downarrow 0} A_f(t,0) = \frac{2e}{3c^2}\left(Q_{\dot{A}_0} - c\sqrt{\lambda_0}\dot{q}_0\right) \ .$$

**Remark 3.4.** One can rephrase the previous theorem saying that the strict solution of the Cauchy problem

$$\begin{cases} \frac{1}{c^2}\ddot{A} = -H_m A \\ A(0) = A_0 \in D(H_m),\ Q_{A_0} = \dot{q}_0, \quad \dot{A}(0) = \dot{A}_0 \in D(F_m) \end{cases}$$

coincides with the first component of $(A, Q)$, the solution of

$$\begin{cases} \frac{1}{c^2}\ddot{A} = \Delta A + \frac{4\pi e}{c} MQ\delta_0 \\ \dot{Q}(t) = c\sqrt{\lambda_0}Q(t) + \frac{3c^2}{2e}A_f(t,0) \\ A(0) = A_0 \in D(H_m),\ Q_{A_0} = \dot{q}_0, \quad \dot{A}(0) = \dot{A}_0 \in D(F_m), \quad Q(0) = \dot{q}_0 \ . \end{cases}$$

Moreover if $A_\lambda \in H^3_*(\mathbb{R}^3)$, and $\dot{A}_\lambda \in H^2_*(\mathbb{R}^3)$, then, by (3.4), $Q_A \in C^2(\mathbb{R};\mathbb{R}^3)$ and one obtains, defining

$$q(t) := q_0 + \int_0^t Q_{A(s)} ds \ ,$$

the Abraham–Lorentz–Dirac equation

$$\begin{cases} -m\tau_0 \dddot{q}(t) + m\ddot{q}(t) = -\frac{e}{c}\dot{A}_f(t,0) \\ q(0) = q_0, \quad \dot{q}(0) = \dot{q}_0, \quad \ddot{q}(0) = Q_{\dot{A}_0} \ , \end{cases}$$

where

$$\tau_0 := \frac{2e^2}{3mc^3} \ .$$

Observe that $-\frac{e}{c}\dot{A}_f(t,0)$ is nothing but the ( linearized ) Lorentz force due to the free field evaluated at $x = 0$. Moreover let us point out that the relation $\ddot{q}(0) = Q_{\dot{A}_0}$, as the correct initial acceleration for the Abraham–Lorentz–Dirac equation, already appeared in [B].

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