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A 2-PHASE TRAFFIC MODEL BASED ON A SPEED BOUND*

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Abstract. We extend the classical Lighthill–Whitham–Richards (LWR) traffic model allowing different maximal speeds for different vehicles. Then we add a uniform bound on the traffic speed. The result, presented in this paper, is a new macroscopic model displaying two phases based on a nonsmooth 2×2 system of conservation laws. This model is compared with other models of the same type in the current literature, as well as with a kinetic one. Moreover, we establish a rigorous connection between a *microscopic follow-the-leader* model based on ordinary differential equations and this *macroscopic continuum* model.

Key words. continuum traffic models, 2-phase traffic models, second order traffic models

AMS subject classifications. 35L65, 90B20

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1. Introduction. Several observations of traffic flow result in underlining two different behaviors, sometimes called *phases*; see [11, 15, 17, 28]. At low density and high speed, the flow appears to be reasonably described by a function of the (mean) traffic density. On the contrary, at high density and low speed, flow is not a single-valued function of the density. This paper presents a model providing an explanation to this phenomenon, its two key features being the following:

1. At a given density, different drivers may choose different velocities.
2. There exists a uniform bound on the speed.

By *bound* we do not necessarily mean an official speed limit. On the contrary, we assume that different drivers may have different speeds at the same traffic density. Nevertheless, there exists a speed $V_{\max}$ that no driver exceeds. As a result of this postulate, we obtain a fundamental diagram very similar to those usually observed; see [30, Figure 1], [28, Figure 1], and Figure 1.1, left. In addition, the evolution prescribed by the model so obtained is reasonable and coherent with that of other traffic models in the literature. In particular, we verify that the minimal requirements stated in [4, 14] are satisfied.

Recall the classical Lighthill–Whitham [34] and Richards [37] (LWR) model, first with a general speed $V = V(t, x, \rho)$:

$$\partial_t \rho + \partial_x (\rho V) = 0 \,, \tag{1.1}$$

where ρ is the traffic density. We want to extend this model to comprise the case where the speed V is not the same for all drivers. More precisely, different drivers differ in their *maximal* speed $w = w(t, x)$, so that now $V = w\,\psi(\rho)$, with $w \in [\check{w}, \hat{w}]$, $\check{w} > 0$, being transported along the road at the mean traffic speed V, and where ψ is

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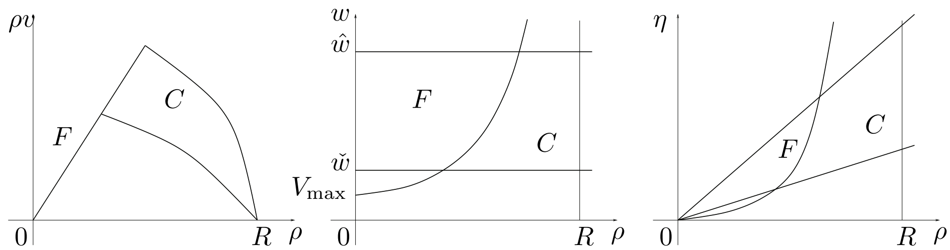

FIG. 1.1. *The free phase* F *and the congested phase* C *resulting from* (1.4) *in the coordinates, from left to right,* $(\rho, \rho v)$, (ρ, w), *and* (ρ, η).

a $\mathbf{C^2}$ function representing the attitude of drivers to adjust their speed to the local traffic density. We identify the different behaviors of the different drivers by means of their maximal speeds; see also [7, 8]. One is thus led to study the equations

$$\left\{\begin{array}{l}\partial_t\rho + \partial_x(\rho v) = 0\\ \partial_t w + v\,\partial_x w = 0\end{array}\right. \qquad \text{with} \qquad v = w\,\psi(\rho)\,. \tag{1.2}$$

Here, the role of the second equation is to let the maximal velocity w be propagated with the traffic speed. Indeed, w is a specific feature of every single driver, in other words a Lagrangian marker. Therefore this model falls into the class of models introduced in [4] and later on extended in [32]; see also [6, formula (1.2)].

Introducing a uniform bound $V_{\max}$ on the speed, we obtain the model

$$\left\{\begin{array}{l}\partial_t\rho + \partial_x(\rho v) = 0\\ \partial_t w + v\,\partial_x w = 0\end{array}\right. \qquad \text{with} \qquad v = \min\left\{V_{\max},\ w\,\psi(\rho)\right\}\ . \tag{1.3}$$

We choose to reformulate the above quasi-linear system in conservation form, similarly to [29, formula (1)], [5, formula (2.2)], [32, formula (1)] (see also [39]) as follows:

$$\left\{\begin{array}{l}\partial_t\rho + \partial_x\left(\rho\, v(\rho,\eta)\right) = 0\\ \partial_t\eta + \partial_x\left(\eta\, v(\rho,\eta)\right) = 0\end{array}\right. \qquad \text{with} \qquad v(\rho,\eta) = \min\left\{V_{\max},\ \frac{\eta}{\rho}\,\psi(\rho)\right\}; \tag{1.4}$$

see Remark 5.3 for further comments on this choice. This model consists of a 2×2 system of conservation laws with a $\mathbf{C^{0,1}}$ but not $\mathbf{C^1}$ flow. Note in fact that $\eta/\rho = w \in [\check{w}, \hat{w}]$. A 2×2 system of conservation laws with a flow having a similar $\mathbf{C^{0,1}}$ regularity is presented in [22] and studied in [1].

From the traffic point of view, we remark that, under mild reasonable assumptions on the function ψ, the flow in (1.4) may vanish if and only if $\rho = 0$, i.e., the road is empty, or $\rho = R$, i.e., the road is fully congested. It is also worth noting the agreement between experimental fundamental diagrams often found in the literature and the one related to (1.4); see Figure 1.1, left. Further comparisons between the present model and those already presented in the literature are in section 3. There we deal with the LWR model [34, 37], the Aw–Rascle (AR) model [4], the 2-phase model [11], and the kinetic description [7]. In particular, we compare the number of required parameters and the basic qualitative features.

Moreover, we rigorously prove that the present model is obtained as the limit of a follow-the-leader model in section 4. Indeed, we consider a system of n ordinary differential equations describing the movement of n vehicles according to the follow-the-leader rule (4.2). As $n \to +\infty$, the solutions to (1.4) are reobtained.

Note that our assumptions also impose a *unique* maximal speed $V_{\max}$ for *all* drivers, which might be unrealistic where, for instance, trucks and cars share the same road. In this case, the same technique used in [7] allows us to extend (1.4) to different populations, each having a different maximal speed.

From the modeling point of view, we stress that, differently from the other 2-phase models in the literature, such as [11, 17], the distinction between the 2 phases here is not the result of an a priori construction. On the contrary, here we assume the presence of a uniform speed bound and, as a consequence, obtain the very definition of the two phases; see (2.1) and (2.2). Indeed, we rigorously identify the phases in the plane $(\rho, \rho\, v)$ of the fundamental diagram: the free phase is where the uniform speed limit is attained, and the congested phase is where traffic flows at a speed lower than $V_{\max}$.

From the analytical point of view, we can extend the present treatment to the more general case of a maximal speed $V_{\max}$ that depends on ρ, i.e., $V_{\max} = V_{\max}(\rho)$. However, we prefer to highlight the main features of the model (1.4) in its simplest analytical framework.

As we already said, the model studied here, inspired from [11], falls into the class of AR models. So we could use the approach and the theoretical results of [3], which should apply here with minor modifications.

However, our approach is different: here, in contrast to the above reference, we establish *directly* a connection between the follow-the-leader model in section 4 and the macroscopic system (1.4) *without* viewing both systems as issued from a same fully discrete system (Godunov scheme) with different limits, and *without* passing in Lagrangian coordinates. For related works that also consider a vacuum, see also [2, 18, 19].

The present paper is arranged in the following way: In the next section we study the Riemann problem for (1.4) and present the qualitative properties of this model from the point of view of traffic. In section 3 we compare the present model with others in the current literature, and in section 4 we establish the connection with a microscopic follow-the-leader model based on ordinary differential equations. All proofs are gathered in the last section.

2. Notation and main results. We assume the following hypotheses throughout:

a. $R, \check{w}, \hat{w}, V_{\max}$ are positive constants, with $\check{w} < \hat{w}$.
b. $\psi \in \mathbf{C^2}\big([0,R];[0,1]\big)$ is such that

$$\begin{aligned} &\psi(0) = 1, && \psi(R) = 0, \\ &\psi'(\rho) \leq 0, && \frac{d^2}{d\rho^2}\big(\rho\,\psi(\rho)\big) \leq 0 \quad \text{for all } \rho \in [0,R]. \end{aligned}$$

c. $\check{w} > V_{\max}$.

Here, R is the maximal possible density, typically $R = 1$ if ρ is normalized as in section 4; $\check{w}$, respectively, $\hat{w}$, is the minimum, respectively, maximum, of the maximal speeds of each vehicle; $V_{\max}$ is the overall uniform upper bound on the traffic speed. In assumption b, the first three assumptions on ψ are the classical conditions usually assumed on speed laws, while the fourth one is technically necessary in the proof of Theorem 2.1. The latter condition means that all drivers do feel the presence of the speed limit.

Moreover, we introduce the notation

$$F = \left\{(\rho, w) \in [0,R] \times [\check{w}, \hat{w}] \colon v(\rho, \rho w) = V_{\max}\right\}, \tag{2.1}$$

$$C = \left\{(\rho, w) \in [0,R] \times [\check{w}, \hat{w}] \colon v(\rho, \rho w) = w\,\psi(\rho)\right\} \tag{2.2}$$

to denote the *free* and the *congested* phases. Note that F and C are closed sets and $F \cap C \neq \emptyset$. Note also that F is one-dimensional in the $(\rho, \rho v)$ plane of the fundamental diagram, while it is two-dimensional in the (ρ, w) and (ρ, η) coordinates; see Figure 1.1. See also Figure 2.1 for a view in three dimensions.

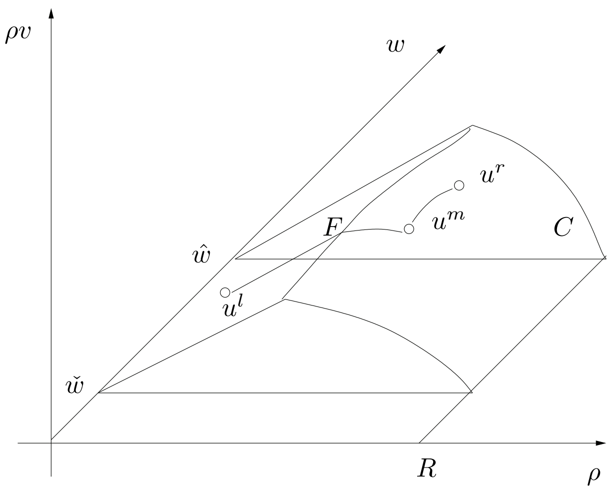

FIG. 2.1. *The phases F and C in the coordinates $(\rho, \rho v, w)$. Note that F is contained in a plane. This figure shows an example of the Riemann problem when $u^l = (\rho^l, \rho^l v^l, w^l) \in F$ and $u^r = (\rho^r, \rho^r v^r, w^r) \in C$.*

Let ρ_* be the maximum of the points of maximum of the flow, i.e., $\rho_* = \max\{\rho \in [0,R] \colon \rho\,\psi(\rho) = \max_{r\in[0,R]} r\,\psi(r)\}$. Then the condition

$$\hat{w}\psi(\rho_*) \geq V_{\max} \tag{2.3}$$

is a further reasonable assumption. Indeed, it means that the maximum flow is attained in the free phase. This is coherent with the *capacity drop* phenomenon, a widely studied feature in traffic engineering according to which the flow along a road is below the expected values when the traffic density is high; see, for instance, [10, 23, 24, 36]. However, (2.3) is not necessary in the following results.

Our next goal is to study the Riemann problem for (1.4).

THEOREM 2.1. *Under assumptions* a, b, *and* c, *for all states $(\rho^l, \eta^l), (\rho^r, \eta^r) \in F \cup C$, the Riemann problem consisting of* (1.4) *with initial data*

$$\rho(0,x) = \begin{cases} \rho^l & \text{if } x < 0, \\ \rho^r & \text{if } x > 0, \end{cases} \qquad \eta(0,x) = \begin{cases} \eta^l & \text{if } x < 0, \\ \eta^r & \text{if } x > 0 \end{cases} \tag{2.4}$$

admits a unique self-similar weak solution $(\rho, \eta) = (\rho, \eta)(t,x)$ constructed as follows:
(1) *If $(\rho^l, \eta^l), (\rho^r, \eta^r) \in F$, then*

$$(\rho, \eta)(t,x) = \begin{cases} (\rho^l, \eta^l) & \text{if } x < V_{\max} t, \\ (\rho^r, \eta^r) & \text{if } x > V_{\max} t. \end{cases} \tag{2.5}$$

(2) *If* $(\rho^l,\eta^l),(\rho^r,\eta^r)\in C$, *then* (ρ,η) *consists of a* 1*-Lax wave (shock or rarefaction) between* (ρ^l,η^l) *and* (ρ^m,η^m), *followed by a* 2*-contact discontinuity between* (ρ^m,η^m) *and* (ρ^r,η^r). *The middle state* (ρ^m,η^m) *is in* C *and is uniquely characterized by the two conditions* $\eta^m/\rho^m=\eta^l/\rho^l$ *and* $v(\rho^m,\eta^m)=v(\rho^r,\eta^r)$.

(3) *If* $(\rho^l,\eta^l)\in C$ *and* $(\rho^r,\eta^r)\in F$, *then the solution* (ρ,η) *consists of a rarefaction wave separating* (ρ^r,η^r) *from a state* (ρ^m,η^m) *and by a linear wave separating* (ρ^m,η^m) *from* (ρ^l,η^l). *The middle state* (ρ^m,η^m) *is in* $F\cap C$ *and is uniquely characterized by the two conditions* $\eta^m/\rho^m=\eta^r/\rho^r$ *and* $v(\rho^m,\eta^m)=V_{\max}$.

(4) *If* $(\rho^l,\eta^l)\in F$ *and* $(\rho^r,\eta^r)\in C$, *then* (ρ,η) *consists of a shock between* (ρ^l,η^l) *and* (ρ^m,η^m), *followed by a contact discontinuity between* (ρ^m,η^m) *and* (ρ^r,η^r). *The middle state* (ρ^m,η^m) *is in* C *and is uniquely characterized by the two conditions* $\eta^m/\rho^m=\eta^l/\rho^l$ *and* $v(\rho^m,\eta^m)=v(\rho^r,\eta^r)$.

(If $\frac{d^2}{d\rho^2}\big(\rho\,\psi(\rho)\big)$ vanishes, then the words "shock" and "rarefaction" above have to be understood as "contact discontinuities.")

We now pass from the solution to single Riemann problems to the properties of the *Riemann solver*, i.e., of the map $\mathcal{R}\colon (F\cup C)^2\to \mathbf{BV}(\mathbb{R};F\cup C)$ such that $\mathcal{R}((\rho^l,\eta^l),(\rho^r,\eta^r))$ is the solution to (1.4)–(2.4) computed at time, say, $t=1$.

To this aim, recall the following definition (see [11]).

DEFINITION 2.2. *A Riemann solver* $\mathcal{R}$ *is* consistent *if the following two conditions hold for all* $(\rho^l,\eta^l),(\rho^m,\eta^m),(\rho^r,\eta^r)\in F\cup C$, *and* $\bar{x}\in\mathbb{R}$:

(C1) *If* $\mathcal{R}((\rho^l,\eta^l),(\rho^m,\eta^m))(\bar{x})=(\rho^m,\eta^m)$ *and* $\mathcal{R}\left((\rho^m,\eta^m),(\rho^r,\eta^r)\right)(\bar{x})=(\rho^m,\eta^m)$, *then*

$$\mathcal{R}\left((\rho^l,\eta^l),(\rho^r,\eta^r)\right)=\begin{cases}\mathcal{R}\left((\rho^l,\eta^l),(\rho^m,\eta^m)\right) & \text{if } x<\bar{x}\,,\\ \mathcal{R}\left((\rho^m,\eta^m),(\rho^r,\eta^r)\right) & \text{if } x\geq\bar{x}.\end{cases}$$

(C2) *If* $\mathcal{R}((\rho^l,\eta^l),(\rho^r,\eta^r))(\bar{x})=(\rho^m,\eta^m)$, *then*

$$\mathcal{R}\left((\rho^l,\eta^l),(\rho^m,\eta^m)\right)=\begin{cases}\mathcal{R}\left((\rho^l,\eta^l),(\rho^r,\eta^r)\right) & \text{if } x\leq\bar{x}\,,\\ (\rho^m,\eta^m) & \text{if } x>\bar{x}\,,\end{cases}$$

$$\mathcal{R}\Big((\rho^m,\eta^m)(\rho^r,\eta^r)\Big)=\begin{cases}(\rho^m,\eta^m) & \text{if } x<\bar{x}\,,\\ \mathcal{R}\left((\rho^l,\eta^l),(\rho^r,\eta^r)\right) & \text{if } x\geq\bar{x}\,.\end{cases}$$

Essentially, (C1) states that whenever two solutions to two Riemann problems can be placed side by side, then their juxtaposition is again a solution to a Riemann problem. Condition (C2) is the opposite; see Figure 2.2.

The next result characterizes the Riemann solver defined above.

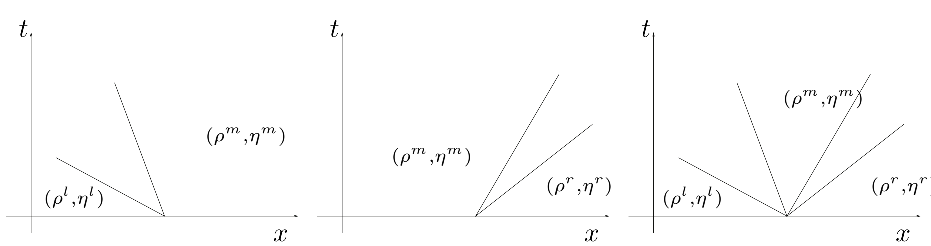

FIG. 2.2. *Conditions* (C1) *and* (C2).

PROPOSITION 2.3. *Let assumptions* a*,* b*, and* c *hold. The Riemann solver* $\mathcal{R}$ *defined in Theorem* 2.1 *enjoys the following three conditions:*

1. *It is consistent in the sense of Definition* 2.2*.*
2. *If* $(\rho^l,\eta^l),(\rho^r,\eta^r) \in F$*, then* $\mathcal{R}((\rho^l,\eta^l),(\rho^r,\eta^r))$ *is the standard solution to the linear system*

$$\begin{cases} \partial_t \rho + \partial_x (\rho V_{\max}) = 0, \\ \partial_t \eta + \partial_x (\eta V_{\max}) = 0. \end{cases} \tag{2.6}$$

3. *If* $(\rho^l,\eta^l) \in F \cup C$ *and* $(\rho^r,\eta^r) \in C$*, then* $\mathcal{R}((\rho^l,\eta^l),(\rho^r,\eta^r))$ *is the standard Lax solution to*

$$\begin{cases} \partial_t \rho + \partial_x \big(\eta\, \psi(\rho)\big) = 0, \\ \partial_t \eta + \partial_x \Big(\frac{\eta^2}{\rho}\, \psi(\rho)\Big) = 0\,. \end{cases} \tag{2.7}$$

Moreover, conditions (C1)*,* 2*, and* 3 *uniquely characterize the Riemann solver* $\mathcal{R}$*.*

The above properties are of use, for instance, in using model (1.4) on traffic networks, according to the techniques described in [16].

The next result presents the relevant qualitative properties of the Riemann solver defined in Theorem 2.1 from the point of view of traffic.

PROPOSITION 2.4. *Let assumptions* a*,* b*, and* c *hold. Then the Riemann solver* $\mathcal{R}$ *enjoys the following properties:*

1. *If the initial datum attains values in* F*,* C*, or* $F \cup C$*; then, respectively, the solution attains values in* F*,* C*, or* $F \cup C$*.*
2. *Traffic density and speed are uniformly bounded.*
3. *Traffic speed vanishes if and only if traffic density is maximal.*
4. *No wave in the solution to* (1.4)–(2.4) *may travel faster than traffic speed; i.e., information may not propagate forward faster than vehicles.*

3. Comparison with other macroscopic models. This section is devoted to a comparison between the present model (1.4) and a sample of models from the literature. In particular, we consider differences in the number of free parameters and functions, in the fundamental diagram, and in the qualitative structures of the solutions. Recall that the evolution described by model (1.4) and the corresponding invariant domain depends on the function ψ and on the parameters $V_{\max}$, R, $\check{w}$, and $\hat{w}$. The fundamental diagram of (1.4) is shown in Figure 1.1, left.

3.1. The LWR model. In the LWR model (1.1), a suitable speed law has to be selected, analogous to the choice of ψ in (1.4). In addition, in (1.4) we also have to set $V_{\max}$, R, and the two geometric positive parameters $\check{w}$ and $\hat{w}$.

The fundamental diagram of (1.4) seems to better agree with experimental data than that of (1.1), shown in Figure 3.1, left. Indeed, compare Figure 1.1, left, with the measurements in [30, Figure 1] and [28, Figure 1].

As long as the data are in F, the solutions to (1.4) are essentially the same as those of (1.1). In the congested phase, the solutions to Riemann problems for (1.4) obviously present a richer structure, for they generically contain two waves instead of one. In particular, the LWR model (1.1) with a strictly concave $V = V(\rho)$ may not describe the "homogeneous-in-speed" solutions, i.e., a type of synchronized flow (see [28, section 2.2] and [25, 40]) which is depicted by the two waves in (1.4).

Finally, note that if in (1.4) the two geometric parameters $\check{w}$ and $\hat{w}$ coincide, then we recover the LWR model (1.1) with $V(\rho) = \min\{V_{\max}, \hat{w}\, \psi(\rho)\}$.

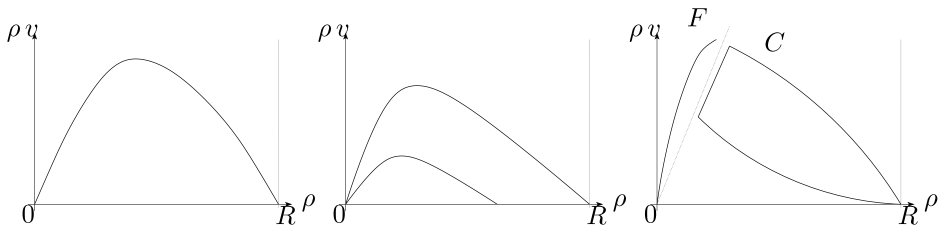

FIG. 3.1. *Fundamental diagrams, from left to right, of the LWR model* (1.1)*, the AR model* (3.1)*, and the* 2*-phase model* (3.2)*.*

3.2. The AR model. Consider now the AR model

$$\left\{ \begin{array}{l} \partial_t \rho + \partial_x \left[\rho\, v(\rho, y)\right] = 0, \\ \partial_t y + \partial_x \left[y\, v(\rho, y)\right] = 0, \end{array} \right. \qquad v(\rho, y) = \frac{y}{\rho} - p(\rho), \tag{3.1}$$

introduced in [4] and successively refined in several papers; see, for instance, [3, 6, 17, 19, 20, 21, 26, 35, 38] and the references therein. Here $v + p\,(\rho)$ plays a role analogous to that of w in (1.4).

In the AR model, R and the *pressure* function need to be selected, similarly to R and ψ in (1.4). No other parameter appears in (3.1), but the definition of an invariant domain requires two parameters, with a role similar to that of $\check{w}$ and $\hat{w}$. Indeed, an invariant domain for (3.1) is

$$\Big\{ (\rho, y) \colon \rho \in [0, R] \text{ and } y \in \Big[\rho\left(v_- + p(\rho)\right), \rho\left(v_- + p(\rho)\right)\Big] \Big\}$$

(see Figure 3.1, center) and depends on the speeds v_- and v_+. More recent versions of (3.1) contain also a suitable relaxation source term on the right-hand side of the second equation; in this case one more arbitrary function needs to be selected. The original AR model does not distinguish between a free and a congested phase. However, it was extended to describe two different phases in [17]. Further comments on (3.1) can be found in [31].

Concerning the analytical properties of the solutions, the Riemann solver for the AR model suffers lack of continuous dependence at vacuum; see [4, section 4]. However, existence of solutions attaining also the vacuum state was proved in [19], while the 2-phase construction in [17] also displays continuous dependence.

A qualitative difference between the AR model and the present one is property 3 in Proposition 2.4. Indeed, solutions to (3.1) may well have zero speed while being at a density strictly lower than the maximal one.

3.3. The hyperbolic 2-phase model. Recall the model presented in [11], with notation similar to the present one:

$$\begin{array}{ll} \text{free flow: } (\rho, q) \in F, & \text{congested flow: } (\rho, q) \in C, \\ \partial_t \rho + \partial_x \left[\rho \cdot v_F(\rho)\right] = 0, & \left\{ \begin{array}{l} \partial_t \rho + \partial_x \left[\rho \cdot v_C(\rho, q)\right] = 0, \\ \partial_t q + \partial_x \left[(q - q_*) \cdot v_C(\rho, q)\right] = 0, \end{array} \right. \\ v_F(\rho) = \left(1 - \dfrac{\rho}{R}\right) \cdot V, & v_C(\rho, q) = \left(1 - \dfrac{\rho}{R}\right) \cdot \dfrac{q}{\rho}, \end{array} \tag{3.2}$$

the phases being defined as

$$F = \{(\rho, q) \in [0, R] \times \mathbb{R}^+ \colon v_f(\rho) \geq V_f,\, q = \rho \cdot V\},$$
$$C = \left\{(\rho, q) \in [0, R] \times \mathbb{R}^+ \colon v_c(\rho . q) \leq V_c,\, \frac{q - q_*}{\rho} \in \left[\frac{Q_1 - q_*}{R}, \frac{Q_2 - q_*}{R}\right]\right\}.$$

In (3.2) no function can be selected; on the other hand the evolution depends on the parameters V, R, and q_*, while the invariant domains F and C depend on V_f, V_c, Q_1, and Q_2. A geometric construction of the solutions to (3.2) in the congested phase is shown in [33].

The main difference between the fundamental diagrams of (3.2) (see Figure 3.1, right) and that of (1.4) is that (3.2) requires the two phases to be *disconnected*: there is a *gap* between the free phase and the congested phase. This restriction is necessary for the well-posedness of the Riemann problem for (3.2) and can hardly be justified on the basis of experimental data. More recently, the global well-posedness of the model (3.2) was proved in [12].

Note that in both models, as well as in that presented in [17], the free phase is one-dimensional, while the congested phase is bidimensional.

The model (3.2) allows for the description of *wide jams*, i.e., of persistent waves in the congested phase moving at a speed *different* from that of traffic. Here, as long as $\frac{d^2}{d\rho^2}\big(\rho\,\psi(\rho)\big) < 0$, persistent phenomena can be described only through waves of the second family, which move at the mean traffic speed. We refer to [31] for further discussions on (3.2) and comparisons with other macroscopic models.

3.4. A kinetic model. Recall, with notation adapted to the present case, the kinetic model introduced in [8, section 1]:

$$\partial_t r(t, x; w) + \partial_x \left[w\, r(t, x; w)\, \psi\left(\int_{\check{w}}^{\hat{w}} r(t, x; w')\, dw'\right)\right] = 0\,. \tag{3.3}$$

The function ψ and the speed w play the same role as here. The unknown $r = r(t, x; w)$ is the probability density of vehicles having maximal speed w that at time t are at point x.

In (3.3) there is one function to be specified, ψ, as in (1.4). The parameters are R (which is normalized to 1 in [8]), $\check{w}$, and $\hat{w}$, similarly to (1.4). Since no speed limit is defined there, no parameter in (3.3) has the same role as $V_{\max}$ here.

Being kinetic in nature, there is no real equivalent to a fundamental diagram for (3.3).

From the analytical point of view, the existence of solutions to (3.3) has not been proved yet. The main result in [8] states only that (3.3) can be rigorously obtained as the limit of systems of $n \times n$ conservation laws describing n populations of vehicles, each characterized by their maximal speed.

Let the measure r solve (3.3) and be such that for suitable functions ρ and w

$$r(t, x; \cdot) = \rho(t, x)\, \delta_{w(t,x)}, \tag{3.4}$$

where δ is the usual Dirac measure. Then, formally, (ρ, w) solves (1.4). Indeed, for the first equation simply substitute (3.4) in (3.3) and integrate; for the second equation substitute (3.4) in (3.3), multiply by w, and integrate over $[\check{w}, \hat{w}]$.

Remark that (3.4) suggests a further interpretation of the quantity ρ in (1.4). Indeed, in the present model, at (t, x) vehicles of only one species are present, namely, those with maximal speed $w(t, x)$.

4. Connections with a follow-the-leader model. Within the framework of (1.3), a single driver starting from $\tilde{p}$ at time $t = 0$ follows the *particle path* $p = p(t)$ that solves the Cauchy problem

$$(4.1)\qquad \begin{cases} \dot{p} = v\left(\rho\left(t, p(t)\right), w\left(t, p(t)\right)\right), \\ p(0) = \tilde{p}, \end{cases} \qquad v(\rho, w) = \min\left\{V_{\max},\, w\,\psi(\rho)\right\};$$

refer to [13] for the well-posedness of the particle path for the LWR model (see also [3]). Recall now that w is a specific feature of every single driver, i.e., $w\left(t, p(t)\right) = w(0, \tilde{p})$ for all $\tilde{p}$. On the other hand, from a microscopic point of view, if n drivers are distributed along the road, then ρ is approximated by $l/(p_{i+1} - p_i)$, where l is a standard length of a car.

We fix $L > 0$ and assume that n drivers are distributed along $[-L, L]$. Then the natural microscopic counterpart to (1.4) is therefore the *follow-the-leader* model defined by the Cauchy problem

$$(4.2)\qquad \begin{cases} \dot{p}_i = v\left(\frac{l}{p_{i+1} - p_i}, w_i\right), & i = 1, \ldots, n, \\ \dot{p}_{n+1} = V_{\max}, & \\ p_i(0) = \tilde{p}_i, & i = 1, \ldots, n+1, \end{cases}$$

where $\tilde{p}_1 = -L$ and $\tilde{p}_{n+1} = L - l$. Proposition 4.1 shows that (4.2) admits a unique global solution defined for every $t \geq 0$ and such that $p_{i+1} - p_i \geq l$ for all $t \geq 0$.

PROPOSITION 4.1. *Let assumptions* a, b, *and* c *hold. Fix $L > 0$. For any $n \in \mathbb{N}$, with $n \geq 2$, choose initial data $\tilde{p}_i^n$ for $i = 1, \ldots, n$ satisfying $\tilde{p}_{i+1}^n - \tilde{p}_i^n \geq l$. Then the Cauchy problem* (4.2) *admits a unique solution $p_i^n = p_i^n(t)$ for $i = 1, \ldots, n+1$, defined for all $t \geq 0$ and satisfying $p_{i+1}^n(t) - p_i^n(t) \geq l$ for all $t \geq 0$ and for $i = 1, \ldots, n$.*

The proof is postponed until section 5.

Our next aim is to rigorously show that in the limit $n \to +\infty$ with $n\,l =$ constant > 0, the microscopic model in (4.2) yields the macroscopic one in (1.4). Given the position p^i of every single vehicle and its maximal speed w_i for $i = 1, \ldots, n+1$, the macroscopic variables ρ, w are given by

$$\rho(x) = \sum_{i=1}^{n} \frac{l}{p_{i+1}^n - p_i^n}\, \chi_{[p_i^n, p_{i+1}^n[}(x) \quad \text{and} \quad w(x) = \sum_{i=1}^{n} w_i^n\, \chi_{[p_i^n, p_{i+1}^n[}(x)\,.$$

Note that necessarily $p_{i+1}^n - p_i^n \geq l$.

On the contrary, given $(\rho, w) \in (\mathbf{L^1} \cap \mathbf{BV})(\mathbb{R}; [0,1] \times [\check{w}, \hat{w}])$, with $\operatorname{supp}\rho, \operatorname{supp} w \subseteq [-L, L]$, we reconstruct a microscopic description defining $l = (\int_{\mathbb{R}} \rho(x)\, dx)/n$ and

$$\begin{aligned} p_{n+1}^n &= L - l, \\ p_i^n &= \max\left\{p \in [-L, L] \colon \int_p^{p_{i+1}} \rho(x)\, dx = l\right\} \qquad \text{for } i = 1, \ldots, n, \\ w_i^n &= w(p_i^n +) \qquad \text{for } i = 1, \ldots, n+1\,. \end{aligned}$$

Note that $\int_{\mathbb{R}} \rho(x)\, dx = nl > 0$. Now we are able to rigorously show that as the number of vehicles increases to infinity, the microscopic model in (4.2) yields the macroscopic one in (1.4).

PROPOSITION 4.2. *Let assumptions* a*,* b*, and* c *hold. Fix* $T > 0$*. Choose* $(\tilde{\rho}, \tilde{w}) \in (\mathbf{L^1} \cap \mathbf{BV})(\mathbb{R}; [0,1] \times [\check{w}, \hat{w}])$ *with* $\operatorname{supp} \tilde{\rho}, \operatorname{supp} \tilde{w} \subseteq [-L, L]$*. Construct the initial data for the microscopic model setting* $l = (\int_{\mathbb{R}} \tilde{\rho}(x)\, dx)/n$ *and*

$$\begin{aligned} \tilde{p}^n_{n+1} &= L - l, \\ \tilde{p}^n_i &= \max\left\{ p \in [-L, L] \colon \int_p^{\tilde{p}_{i+1}} \tilde{\rho}(x)\, dx = l \right\} \qquad \text{for } i = 1, \ldots, n, \\ \tilde{w}^n_i &= \tilde{w}(p^n_i +) \qquad \text{for } i = 1, \ldots, n+1\,. \end{aligned}$$

Let $p^n_i(t)$ *for* $i = 1, \ldots, n$ *be the corresponding solution to* (4.2)*. Define*

$$\rho^n(t,x) = \sum_{i=1}^n \frac{l}{p^n_{i+1}(t) - p^n_i(t)}\, \chi_{[p^n_i(t), p^n_{i+1}(t)[}(x), \tag{4.3}$$

$$w^n(t,x) = \sum_{i=1}^n \tilde{w}^n_i\, \chi_{[p^n_i(t), p^n_{i+1}(t)[}(x)\,. \tag{4.4}$$

If there exists a pair $(\rho, w) \in \mathbf{L}^\infty([0,T]; \mathbf{L^1}(\mathbb{R}; [0,1] \times [\check{w}, \hat{w}])$ *such that*

$$\lim_{n \to +\infty} (\rho^n, w^n)(t,x) = (\rho, w)(t,x) \qquad a.e.,$$

then the pair $(\rho, \rho w)$ *is a weak solution to* (1.4) *with initial datum* $(\tilde{\rho}, \tilde{\rho}\tilde{w})$*.*

The proof is postponed until section 5.

5. Technical details.

We first prove an elementary consequence of our assumption b.

LEMMA 5.1. *Let* ψ *satisfy assumption* b*. Then*

$$\exists\, \bar{\rho} \in [0, R[\quad \text{such that} \quad \begin{cases} \psi \text{ is constant on } [0, \bar{\rho}], \\ \psi \text{ is strictly decreasing on } [\bar{\rho}, R]. \end{cases}$$

Proof. Call $q(\rho) = \rho\, \psi(\rho)$. If ψ is strictly monotone, then $\bar{\rho} = 0$ and the proof is completed. Otherwise, assume that $\psi(\rho_1) = \psi(\rho_2) = c$ for suitable $\rho_1, \rho_2 \in]0, R]$ and $\rho_1 \neq \rho_2$. Then, by assumption b, for all $\rho \in [\rho_1, \rho_2]$ we have $\psi(\rho) = c$ and $q(\rho) = c\rho$. If $\psi(0) = c$, then the proof is complete. Otherwise, note that $q'(0) = \psi(0) > c$ contradicts the concavity of q. □

COROLLARY 5.2. *Assume that assumptions* b *and* c *are satisfied. Then*

$$\bar{\rho} < \min\left\{ \rho \in [0, R] \colon \exists w \in [\check{w}, \hat{w}] \text{ such that } (\rho, w) \in C \right\}\,.$$

The proof is immediate and, hence, omitted.

In what follows, for the basic definitions concerning the standard theory of conservation laws, we refer the reader to [9].

Proof of Theorem 2.1. We consider different cases, depending on the phase of the data (2.4).

Case 1. $(\rho^l, \eta^l), (\rho^r, \eta^r) \in F$.

In this case, (1.4) reduces to the degenerate linear system (2.6) so that the problem (1.4)–(2.4) is solved by (2.5). Note, for later use, that the characteristic speed is $\lambda^F = V_{\max}$.

Case 2. $(\rho^l,\eta^l),(\rho^r,\eta^r) \in C$.

Now, $v(\rho,\eta) = \eta\,\psi(\rho)/\rho$. We show that C is invariant with respect to the 2×2 system of conservation laws (2.7). To this aim, we compute the eigenvalues, right eigenvectors, and Lax curves $\eta = \mathcal{L}_i(\rho;\rho_o,\eta_o)$ in C:

$$\begin{aligned}
\lambda_1(\rho,\eta) &= \eta\,\psi'(\rho) + v(\rho,\eta), & \lambda_2(\rho,\eta) &= v(\rho,\eta),\\
r_1(\rho,\eta) &= \left[\begin{array}{c} -\rho \\ -\eta \end{array}\right], & r_2(\rho,\eta) &= \left[\begin{array}{c} 1 \\ \eta\left(\frac{1}{\rho} - \frac{\psi'(\rho)}{\psi(\rho)}\right) \end{array}\right],\\
\nabla\lambda_1\cdot r_1 &= -\frac{d^2}{d\rho^2}\left[\rho\,\psi(\rho)\right], & \nabla\lambda_2\cdot r_2 &= 0,\\
\mathcal{L}_1(\rho;\rho_o,\eta_o) &= \eta_o\frac{\rho}{\rho_o}, & \mathcal{L}_2(\rho;\rho_o,\eta_o) &= \frac{\rho\, v(\rho_o,\eta_o)}{\psi(\rho)},\ \rho_o < R.
\end{aligned}$$

When $\rho_o = R$, the 2-Lax curve through (ρ_o,η_o) is the segment $\rho = R$, $\eta \in [R\check{w}, R\hat{w}]$.

Shock and rarefaction curves of the first characteristic family coincide by [5, Lemma 2.1]; see also [9, Problem 1, Chapter 5]. The second characteristic field is linearly degenerate. Hence, (2.7) is a Temple system and C is invariant, since its boundary consists of Lax curves; see [27, Theorem 3.2].

Thus, the solution to (1.4) is as described in (2) and attains values in C.

Case 3. $(\rho^l,\eta^l) \in C$, $(\rho^r,\eta^r) \in F$.

Let ρ^m satisfy $\psi(\rho^m) = V_{\max}\rho^r/\eta^r$. Note that such ρ^m exists in $]0,1[$ by assumptions b and c, and it is unique by Corollary 5.2. Define $\eta^m = (\rho^m/\rho^r)\eta^r$ and note that (ρ^l,η^l), (ρ^m,η^m) are connected by a 1-rarefaction wave of (2.7) having maximal speed of propagation $\lambda_1(\rho^m,\eta^m) < V_{\max}$. Hence, a linear wave solution to (2.6) can be juxtaposed connecting (ρ^m,η^m) to (ρ^l,η^l), and the solution to (1.4) is as described in (3).

Case 4. $(\rho^l,\eta^l) \in F$, $(\rho^r,\eta^r) \in C$ (see Figure 2.1).

Note that system (2.7) can be considered on the whole of $F \cup C$. Also this set is invariant for (2.7) by [27, Theorem 3.2]. Then, in this case, we let (ρ,η) be the standard Lax solution to (2.7), as described in (4). □

Proof of Proposition 2.3. We consider different cases depending on the phase of the data (2.4).

If $(\rho^l,\eta^l),(\rho^r,\eta^r) \in F$, then $\mathcal{R}(\rho^l,\eta^l),(\rho^r,\eta^r))$ coincides with the Riemann solver of a linear system, which satisfies (C1). Condition (C2) is immediate since no nontrivial middle state is available.

Similarly, if $(\rho^l,\eta^l),(\rho^r,\eta^r) \in C$, then $\mathcal{R}((\rho^l,\eta^l),(\rho^r,\eta^r))$ coincides with the standard Riemann solver of a 2×2 system, which is consistent. The consistency of $\mathcal{R}$ then follows by the invariance of C, by Case 2 in the proof of Theorem 2.1.

By the same argument, the case $(\rho^l,\eta^l) \in F$ and $(\rho^r,\eta^r) \in C$ is also proved. Indeed, in (C2), note that the only possible nontrivial middles states are in C.

Finally, if $(\rho^l,\eta^l) \in C$ and $(\rho^r,\eta^r) \in F$, then $\mathcal{R}((\rho^l,\eta^l),(\rho^r,\eta^r))$ takes values in $F \cup C$ and is the juxtaposition of two consistent Riemann problems, and hence (C1) holds. Concerning (C2), note that the only possible nontrivial middles states are in C, and (C2) follows by the consistency of the standard Riemann solver for (2.7).

Thus condition 1 of the proposition is proved. Assertions 2 and 3 are immediate consequences of the construction of Theorem 2.1.

Assume now that $\mathcal{R}$ satisfies conditions 2 and 3. Then all Riemann problems with data $(\rho^l,\eta^l),(\rho^r,\eta^r) \in F$, $(\rho^l,\eta^l) \in F$, $(\rho^r,\eta^r) \in C$, and $(\rho^l,\eta^l),(\rho^r,\eta^r) \in C$ are uniquely solved. The solution to Riemann problems with $(\rho^l, \eta^l) \in C$ and $(\rho^r, \eta^r) \in F$ is then uniquely constructed through (C1). □

Proof of Proposition 2.4. Consider the different statements separately.

1. The invariance of F, C, and $F \cup C$ is shown in the proof of Theorem 2.1.
2. By the invariance of $F \cup C$, it is sufficient to observe that on the compact set $F \cup C$, the density ρ, respectively, the speed v, is uniformly bounded by R, respectively, $V_{\max}$.
3. The proof of this condition is immediate; see, for instance, Figure 1.1, left.
4. In phase C we have

$$\lambda_1(\rho, \eta) = \eta\, \psi'(\rho) + v(\rho, \eta) \le v(\rho, \eta) \quad \text{and} \quad \lambda_2(\rho, \eta) \le v(\rho, \eta).$$

In the free phase the wave speed is $V_{\max} = v(\rho, \eta)$. The only case left is that of a phase boundary with left state in F and right state, say (ρ^r, η^r), in C. Then the speed Λ of the phase boundary clearly satisfies $\Lambda \le \lambda_1(\rho^r, \eta^r) < v(\rho^r, \eta^r)$. □

Proof of Proposition 4.1. Note first that the functions $\rho \mapsto v(\rho, w_i)$ in (4.2) are uniformly bounded and Lipschitz continuous for $i = 1, \ldots, n$. We extend them to functions with the same properties and defined on $[0, +\infty[$ setting

$$u_i(\rho) = \begin{cases} V_{\max} & \text{if } \rho < 0, \\ v(\rho, w_i) & \text{if } \rho \in [0,1], \\ 0 & \text{if } \rho > 1. \end{cases} \tag{5.1}$$

We also note that, for $i = 1, \ldots, n$, the composite applications $\delta \mapsto u_i(l/\delta)$ can be extended to uniformly bounded and Lipschitz continuous functions on $[0, +\infty[$. Now we consider the Cauchy problem

$$\begin{cases} \dot{p}_i^n = u_i\left(\frac{l}{p_{i+1}^n - p_i^n}\right), & i = 1, \ldots, n, \\ \dot{p}_{n+1}^n = V_{\max}, & \\ p_i^n(0) = \tilde{p}_i, & i = 1, \ldots, n+1\,. \end{cases} \tag{5.2}$$

Note that $\tilde{p}_i^n$ for $i = 1, \ldots, n+1$ are defined in Proposition 4.2 and satisfy the condition $\tilde{p}_{i+1}^n - \tilde{p}_i^n \ge l > 0$ for every $i = 1, \ldots, n$.

By the standard ordinary differential equation theory, there exists a $\mathbf{C^1}$ solution p_i^n defined as long as $p_{i+1}^n - p_i^n > 0$ for all $i = 1, \ldots, n$. We now prove that in fact $p_{i+1}^n(t) - p_i^n(t) \ge l$ for every $t \ge 0$. To this aim we assume by contradiction that there exist positive $\bar{t}$ and t^*, with $\bar{t} < t^*$, such that $p_{i+1}^n(\bar{t}) - p_i^n(\bar{t}) = l$ and $p_{i+1}^n(t) - p_i^n(t) < l$ for every $t \in]\bar{t}, t^*]$. Then

$$p_i^n(t) = p_i^n(\bar{t}) + \int_{\bar{t}}^{t} \dot{p}_i^n(s)\, ds = p_i^n(\bar{t}) + \int_{\bar{t}}^{t} u_i\left(\frac{l}{p_{i+1}^n(s) - p_i^n(s)}\right) ds = p_i^n(\bar{t}).$$

Then, combining (5.1) and (5.2), we find $\dot{p}_i^n(t) = 0$ for $t \in]\bar{t}, t^*]$, and so $p_i^n(t) = p_i^n(\bar{t})$. This yields a contradiction, since for every $t \in]\bar{t}, t^*]$

$$p_{i+1}^n(t) - p_i^n(t) \ge p_{i+1}^n(\bar{t}) - p_i^n(\bar{t}) = l\,,$$

completing the proof. □

Proof of Proposition 4.2. Recall first the definition of weak solution to (1.4): For all $\varphi \in \mathbf{C_c^\infty}$, setting $v(\rho, w) = \min\{V_{\max}, w\,\psi(\rho)\}$,

$$\int_0^T \int_{\mathbb{R}} \left(\begin{bmatrix} \rho \\ \rho\, w \end{bmatrix} \partial_t \varphi + \begin{bmatrix} \rho\, v(\rho, w) \\ \rho\, w\, v(\rho, w) \end{bmatrix} \partial_x \varphi \right) dx\, dt + \int_{\mathbb{R}} \begin{bmatrix} \tilde\rho \\ \tilde\rho\, \tilde w \end{bmatrix} \varphi(0, x)\, dx = 0,$$

and consider the two components separately.

First insert (4.3) into the above equality and obtain

$$\begin{aligned} I^n &:= \int_0^T \int_{\mathbb{R}} \left(\rho^n \partial_t \varphi + \rho^n\, v(\rho^n, w^n)\, \partial_x \varphi \right)\, dx\, dt + \int_{\mathbb{R}} \tilde\rho\, \varphi(0, x)\, dx \\ &= \sum_{i=1}^n \int_0^T \frac{l}{p_{i+1}^n(t) - p_i^n(t)} \int_{p_i^n(t)}^{p_{i+1}^n(t)} \left[\partial_t \varphi + v\left(\frac{l}{p_{i+1}^n(t) - p_i^n(t)}, w_i^n \right) \partial_x \varphi \right] dt \\ &\qquad + \int_{\mathbb{R}} \rho^n(0, x)\, \varphi(0, x)\, dx + \int_{\mathbb{R}} \left(\tilde\rho - \rho^n(0, x) \right)\, \varphi(0, x)\, dx \\ &= \sum_{i=1}^n \int_0^T \frac{l}{p_{i+1}^n(t) - p_i^n(t)} \int_{p_i^n(t)}^{p_{i+1}^n(t)} \left(\partial_t \varphi(t, x) + \dot p_i^n(t) \partial_x \varphi(t, x) \right) dx\, dt \\ &\qquad + \sum_{i=1}^n \frac{l}{\tilde p_{i+1}^n - \tilde p_i^n} \int_{\tilde p_i}^{\tilde p_{i+1}} \varphi(0, x) dx + \int_{\mathbb{R}} \left(\tilde\rho - \rho^n(0, x) \right) \varphi(0, x)\, dx. \end{aligned}$$

Now approximate $\varphi(t, x)$ with $\varphi\left(t, p_i^n(t)\right)$ for every x in $[p_i^n(t), p_{i+1}^n(t)]$ and introduce the quantity $\|\varphi\|_{\mathbf{C^2}}$ that uniformly bounds from above the modulus of φ and all its derivatives up to second order. Then, for $x \in [p_i^n(t), p_{i+1}^n(t)]$,

$$\left| \partial_t \varphi(t, x) + \dot p_i^n(t) \partial_x \varphi(t, x) - \frac{d}{dt} \varphi\left(t, p_i^n(t)\right) \right| \le \|\varphi\|_{\mathbf{C^2}} \cdot \left| p_{i+1}^n(t) - p_i^n(t) \right|,$$

and, equivalently,

$$\partial_t \varphi(t, x) + \dot p_i^n(t) \partial_x \varphi(t, x) = \frac{d}{dt} \varphi\left(t, p_i^n(t)\right) + \mathcal{O}(1) \cdot \left(p_{i+1}^n(t) - p_i^n(t) \right),$$

where, in particular, $\mathcal{O}(1)$ depends neither on $p_i^n(t)$, nor on i, nor on n. Using the latter estimate in the expression for I^n above,

$$\begin{aligned} I^n &= \sum_{i=1}^n \int_0^T \frac{l}{p_{i+1}^n(t) - p_i^n(t)} \int_{p_i^n(t)}^{p_{i+1}^n(t)} \frac{d}{dt} \varphi\left(t, p_i^n(t)\right)\, dx\, dt \\ &\qquad + \sum_{i=1}^n \int_0^T \frac{l}{p_{i+1}^n(t) - p_i^n(t)} \int_{p_i^n(t)}^{p_{i+1}^n(t)} \mathcal{O}(1) \left(p_{i+1}^n(t) - p_i^n(t) \right)\, dx\, dt \\ &\qquad + \sum_{i=1}^n \frac{l}{\tilde p_{i+1}^n - \tilde p_i^n} \int_{\tilde p_i}^{\tilde p_{i+1}} \varphi(0, x)\, dx + \int_{\mathbb{R}} \left(\tilde\rho - \rho^n(0, x) \right) \varphi(0, x)\, dx \\ &= l \sum_{i=1}^n \int_0^T \frac{d}{dt} \varphi\left(t, p_i^n(t)\right) dt + l \sum_{i=1}^n \int_0^T \mathcal{O}(1) \left(p_{i+1}^n(t) - p_i^n(t) \right) dt \\ &\qquad + \sum_{i=1}^n \frac{l}{\tilde p_{i+1}^n - \tilde p_i^n} \int_{\tilde p_i}^{\tilde p_{i+1}} \varphi(0, x) dx + \int_{\mathbb{R}} \left(\tilde\rho - \rho^n(0, x) \right) \varphi(0, x)\, dx \end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=1}^{n} \frac{l}{\tilde{p}_{i+1}^n - \tilde{p}_i^n} \int_{\tilde{p}_i}^{\tilde{p}_{i+1}} \left[\varphi(0,x) - \varphi(0,\tilde{p}_i^n)\right] \, dx \\
&\qquad + \mathcal{O}(1)\, l \left(p_{n+1}^n(T) - p_1^n(T)\right) + \int_{\mathbb{R}} \left(\tilde{\rho} - \rho^n(0,x)\right) \, \varphi(0,x) \, dx \\
&= \mathcal{O}(1)\, l \,(2L + V_{\max} T) + \int_{\mathbb{R}} \left(\tilde{\rho} - \rho^n(0,x)\right) \, \varphi(0,x) \, dx,
\end{aligned}
$$

and both terms in the latter quantity clearly vanish as $n \to +\infty$.

The computations related to the other component are entirely similar, since w is constant along any set of the form

$$
\left\{ (t,x) \in [0,T] \times \mathbb{R} \colon x \in \left[p_i^n(t), p_{i+1}^n(t)\right[\right\},
$$

and the proof is complete. ∎

REMARK 5.3. *System* (1.2) *is not in conservation form. As far as smooth solutions are concerned, it is equivalent to infinitely many* 2×2 *systems of conservation laws. Indeed, introduce a strictly monotone function* $f \in \mathbf{C}^2 \left([\check{w}, \hat{w}];]0, +\infty[\right)$. *Then elementary computations show that, as long as smooth solutions are concerned, system* (1.2) *is equivalent to*

$$
\left\{
\begin{array}{l}
\partial_t \rho + \partial_x \left(\rho \, \psi(\rho) \, g(\eta/\rho)\right) = 0, \\
\partial_t \eta + \partial_x \left(\eta \, \psi(\rho) \, g(\eta/\rho)\right) = 0,
\end{array}
\right.
\qquad \text{where} \qquad
\begin{array}{l}
\eta = \rho \, f(w) \ \text{and} \\
g\left(f(w)\right) = w.
\end{array}
\tag{5.3}
$$

Clearly, different choices of f *yield different weak solutions to* (5.3), *but they are all equivalent when written in terms of* ρ *and* w.

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