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COUPLING CONDITIONS FOR THE 3×3 EULER SYSTEM

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Abstract. This paper is devoted to the extension to the full 3×3 Euler system of the basic analytical properties of the equations governing a fluid flowing in a duct with varying section. First, we consider the Cauchy problem for a pipeline consisting of 2 ducts joined at a junction. Then, this result is extended to more complex pipes. A key assumption in these theorems is the boundedness of the total variation of the pipe's section. We provide explicit examples to show that this bound is necessary.

1. Introduction.

We consider Euler equations for the evolution of a fluid flowing in a pipe with varying section $a = a(x)$, see [17, Section 8.1] or [12, 15]:

$$\begin{cases} \partial_t(a\rho) + \partial_x(aq) = 0 \\ \partial_t(aq) + \partial_x\left[aP\left(\rho,q,E\right)\right] = p\left(\rho,e\right)\,\partial_x a \\ \partial_t(aE) + \partial_x\left[aF\left(\rho,q,E\right)\right] = 0 \end{cases} \tag{1.1}$$

where, as usual, ρ is the fluid density, q is the linear momentum density and E is the total energy density. Moreover

$$E(\rho,q,E) = \frac{1}{2}\frac{q^2}{\rho} + \rho e, \quad P(\rho,q,E) = \frac{q^2}{\rho} + p, \quad F(\rho,q,E) = \frac{q}{\rho}(E+p)\,, \tag{1.2}$$

with e being the internal energy density, P the flow of the linear momentum density and F the flow of the energy density. The above equations express the conservation laws for the mass, momentum, and total energy of the fluid through the pipe. Below, we will often refer to the standard case of the ideal gas, characterized by the relations

$$p = (\gamma-1)\rho e, \qquad S = \ln e - (\gamma-1)\ln\rho\,, \tag{1.3}$$

for a suitable $\gamma > 1$. Note however, that this particular equation of state is necessary only in case **(p)** of Proposition 3.1 and has been used in the examples in Section 4. In the rest of this work, the usual hypothesis [16, formula (18.8)], that is $p > 0$, $\partial_\tau p(\tau,S) < 0$ and $\partial^2_{\tau\tau} p(\tau,S) > 0$, are sufficient.

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The case of a sharp discontinuous change in the pipe's section due to a junction sited at, say, $x = 0$, corresponds to $a(x) = a^-$ for $x < 0$ and $a(x) = a^+$ for $x > 0$. Then, the motion of the fluid can be described by

$$\begin{cases} \partial_t \rho + \partial_x q = 0 \\ \partial_t q + \partial_x P(\rho, q, E) = 0 \\ \partial_t E + \partial_x F(\rho, q, E) = 0, \end{cases} \tag{1.4}$$

for $x \neq 0$, together with a *coupling condition* at the junction of the form:

$$\Psi \Big(a^-, (\rho, q, E)(t, 0-); a^+, (\rho, q, E)(t, 0+) \Big) = 0. \tag{1.5}$$

Above, we require the existence of the traces at $x = 0$ of (ρ, q, E). Various choices of the function Ψ are present in the literature, see for instance [1, 5, 8, 9] in the case of the p-system and [10] for the full 3×3 system (1.4). Here, differently from the cited references, we consider the case of a general coupling condition which comprises all the cases found in the literature. Within this more general setting, we prove the well posedness of the Cauchy problem for (1.4)–(1.5). Once this result is obtained, the extension to pipes with several junctions and to pipes with a $\mathbf{W^{1,1}}$ section is achieved by the standard methods considered in the literature. For the analytical techniques to cope with networks having more complex geometry, we refer to [11].

The above statements are global in time and local in the space of the thermodynamic variables (ρ, q, E). Indeed, for any fixed (subsonic) state $(\bar\rho, \bar q, \bar E)$, there exists a bound on the total variation $\mathrm{TV}(a)$ of the pipe's section, such that all sections below this bound give rise to Cauchy problems for (1.4)–(1.5) that are well posed in $\mathbf{L^1}$. We show the necessity of this bound in the conditions found in the current literature. Indeed, we provide explicit examples showing that a wave can be arbitrarily amplified through consecutive interactions with the pipe walls, see Figure 1.

The paper is organized as follows. The next section is divided into three parts, the former one deals with a single junction and two pipes, then we consider n junctions and $n + 1$ pipes, the latter part presents the case of a $\mathbf{W^{1,1}}$ section. Section 3 is devoted to different specific choices of coupling conditions (1.5). In Section 4, an explicit example shows the necessity of the bound on the total variation of the pipe's section. All proofs are gathered in Section 5.

2. **Basic well posedness results.**

Throughout, we let $u = (\rho, q, E)$. We denote by $\mathbb{R}^+$ the real half-line $[0, +\infty[$, while $\mathring{\mathbb{R}}^+ =]0, +\infty[$. Following various results in the literature, such as [1, 2, 5, 8, 9, 10, 13], we limit the analysis in this paper to the *subsonic* region given by $\lambda_1(u) < 0 < \lambda_3(u)$ and $\lambda_2(u) \neq 0$, where λ_i is the $i-$th eigenvalue of (1.4), see (5.1). Without any loss of generality, we further restrict to

$$A_0 = \left\{ u \in \mathring{\mathbb{R}}^+ \times \mathbb{R}^+ \times \mathring{\mathbb{R}}^+ \colon \lambda_1(u) < 0 < \lambda_2(u) \right\}. \tag{2.1}$$

Note that we fix *a priori* the sign of the fluid speed v, since $\lambda_2(u) = q/\rho = v > 0$.

2.1. **A junction and two pipes.**

We now give the definition of weak $\Psi-$solution to the Cauchy Problem for (1.4) equipped with the condition (1.5), extending [5, Definition 2.1] and [9, Definition 2.2] to the 3×3 case (1.4) and comprising the particular case covered in [10, Definition 2.4].

Definition 2.1. Let $\Psi\colon (\mathring{\mathbb{R}}^+ \times A_0)^2 \to \mathbb{R}^3$, $u_o \in \mathbf{BV}(\mathbb{R}; A_0)$ and two positive sections a^-, a^+ be given. A Ψ-solution to (1.4) with initial datum u_o is a map

$$\begin{array}{rcl} u & \in & \mathbf{C^0}\left(\mathbb{R}^+; \mathbf{L^1_{loc}}(\mathbb{R}; A_0)\right) \\ u(t) & \in & \mathbf{BV}(\mathbb{R}; A_0) \quad \text{for a.e. } t \in \mathbb{R}^+ \end{array} \tag{2.2}$$

such that

1.: for $x \neq 0$, u is a weak entropy solution to (1.4);
2.: for a.e. $x \in \mathbb{R}$, $u(0,x) = u_o(x)$;
3.: for a.e. $t \in \mathbb{R}^+$, the coupling condition (1.5) at the junction is met.

Below, extending the 2×2 case of the p-system, see [1, 4, 5, 8, 9], we consider some properties of the coupling condition (1.5), which we rewrite here as

$$\Psi(a^-, u^-; a^+, u^+) = 0\,. \tag{2.3}$$

(Ψ0): Regularity: $\Psi \in \mathbf{C^2}\left((\mathring{\mathbb{R}}^+ \times A_0)^2; \mathbb{R}^3\right)$.

(Ψ1): No-junction case: for all $a > 0$ and all $u^-, u^+ \in A_0$, then

$$\Psi(a, u^-; a, u^+) = 0 \Longleftrightarrow u^- = u^+\,.$$

(Ψ2): Consistency: for all positive a^-, a^0, a^+ and all $u^-, u^0, u^+ \in A_0$,

$$\begin{array}{l} \Psi(a^-, u^-; a^0, u^0) = 0 \\ \Psi(a^0, u^0; a^+, u^+) = 0 \end{array} \Longrightarrow \Psi(a^-, u^-; a^+, u^+) = 0\,.$$

The technique in [6] allows to prove the following well posedness result.

Theorem 2.2. *Assume that Ψ satisfies conditions (Ψ0)-(Ψ2). For every $\bar{a} > 0$ and $\bar{u} \in A_0$ such that*

$$\det\left[D_{u^-}\Psi \cdot r_1(\bar{u}) \quad D_{u^+}\Psi \cdot r_2(\bar{u}) \quad D_{u^+}\Psi \cdot r_3(\bar{u})\right] \neq 0 \tag{2.4}$$

there exist positive δ, L such that for all a^-, a^+ with $\left|a^+ - \bar{a}\right| + \left|a^- - \bar{a}\right| < \delta$ there exists a semigroup $S\colon \mathbb{R}^+ \times \mathcal{D} \to \mathcal{D}$ with the following properties:

1. *$\mathcal{D} \supseteq \left\{u \in \bar{u} + \mathbf{L^1}(\mathbb{R}; A_0) \colon \mathrm{TV}(u) < \delta\right\}$.*
2. *For all $u \in \mathcal{D}$, $S_0 u = u$ and for all $t, s \geq 0$, $S_t S_s u = S_{s+t} u$.*
3. *For all $u, u' \in \mathcal{D}$ and for all $t, t' \geq 0$,*
$$\left\|S_t u - S_{t'} u'\right\|_{\mathbf{L^1}} \leq L \cdot \left(\left\|u - u'\right\|_{\mathbf{L^1}} + \left|t - t'\right|\right)$$
4. *If $u \in \mathcal{D}$ is piecewise constant, then for t small, $S_t u$ is the gluing of solutions to Riemann problems at the points of jump in u and at the junction at $x = 0$.*
5. *For all $u_o \in \mathcal{D}$, the orbit $t \to S_t u_o$ is a Ψ-solution to (1.4) with initial datum u_o.*

The proof is postponed to Section 5. Above $r_i(u)$, with $i = 1, 2, 3$, are the right eigenvectors of $Df(u)$, see (5.1). Moreover, by *solution to the Riemann Problems at the points of jump* we mean the usual Lax solution, see [3, Chapter 5], whereas for the definition of *solution to the Riemann Problems at the junction* we refer to [8, Definition 2.1].

Remark that, by an immediate extension of [9, Lemma 2.1], condition (Ψ0) and (2.4) ensure that (2.3) implicitly defines a map

$$u^+ = T(a^-, a^+; u^-) \tag{2.5}$$

in a neighborhood of any pair of subsonic states u^-, u^+ and sections a^-, a^+ that satisfy $\Psi(a^-, u^-; a^+, u^+) = 0$.

2.2. n **Junctions and** $n+1$ **pipes.** The same procedure used in [9, Paragraph 2.2] allows now to construct the semigroup generated by (1.4) in the case of a pipe with piecewise constant section

$$a = a_0\, \chi_{]-\infty,x_1]} + \sum_{j=1}^{n-1} a_j\, \chi_{[x_j,x_{j+1}[} + a_n\, \chi_{[x_n,+\infty[}$$

with $n \in \mathbb{N}$. In each segment $\left]x_j, x_{j+1}\right[$, the fluid is modeled by (1.4). At each junction x_j, we require condition (1.5), namely

$$\Psi(a_{j-1}, u_j^-; a_j, u_j^+) = 0 \qquad \begin{array}{l}\text{for all } j = 1, \ldots, n, \text{ where} \\ u_j^\pm = \lim_{x \to x_j \pm} u_j(x)\,.\end{array} \tag{2.6}$$

We omit the formal definition of Ψ-solution to (1.4)–(1.5) in the present case, since it is an obvious iteration of Definition 2.1. The natural extension of Theorem 2.2 to the case of (1.4)–(2.6) is the following result.

Theorem 2.3. *Assume that Ψ satisfies conditions (**Ψ0**)-(**Ψ2**). For any $\bar a > 0$ and any $\bar u \in A_0$ satisfying (2.4), there exist positive $M, \Delta, \delta, L, \mathcal{M}$ such that for any pipe's profile satisfying*

$$a \in \mathbf{PC}\left(\mathbb{R}; \left]\bar a - \Delta, \bar a + \Delta\right[\right) \text{ with } \mathrm{TV}(a) < M \tag{2.7}$$

there exists a piecewise constant stationary solution

$$\hat u = \hat u_0 \chi_{]-\infty,x_1[} + \sum_{j=1}^{n-1} \hat u_j \chi_{]x_j,x_{j+1}[} + \hat u_n \chi_{]x_n,+\infty[}$$

to (1.4)–(2.6) satisfying

$$\begin{array}{l} \hat u_j \in A_0 \text{ with } \left|\hat u_j - \bar u\right| < \delta \text{ for } j = 0, \ldots n \\ \Psi\left(a_{j-1}, \hat u_{j-1}; a_j, \hat u_j\right) = 0 \text{ for } j = 1, \ldots, n \\ \mathrm{TV}(\hat u) \le \mathcal{M}\, \mathrm{TV}(a) \end{array} \tag{2.8}$$

and a semigroup $S^a \colon \mathbb{R}^+ \times \mathcal{D}^a \to \mathcal{D}^a$ such that

1. *$\mathcal{D}^a \supseteq \left\{u \in \hat u + \mathbf{L^1}(\mathbb{R}; A_0) \colon \mathrm{TV}(u - \hat u) < \delta\right\}$.*
2. *S_0^a is the identity and for all $t, s \ge 0$, $S_t^a S_s^a = S_{s+t}^a$.*
3. *For all $u, u' \in \mathcal{D}^a$ and for all $t, t' \ge 0$,*
$$\left\|S_t^a u - S_{t'}^a u'\right\|_{\mathbf{L^1}} \le L \cdot \left(\left\|u - u'\right\|_{\mathbf{L^1}} + \left|t - t'\right|\right).$$
4. *If $u \in \mathcal{D}^a$ is piecewise constant, then for t small, $S_t u$ is the gluing of solutions to Riemann problems at the points of jump in u and at each junction x_j.*
5. *For all $u \in \mathcal{D}^a$, the orbit $t \to S_t^a u$ is a weak Ψ-solution to (1.4)–(2.6).*

We omit the proof, since it is based on the natural extension to the present 3×3 case of [9, Theorem 2.4]. Remark that, as in that case, δ and L depend on a only through $\bar a$ and $\mathrm{TV}(a)$. In particular, all the construction above is independent from the number of points of jump in a.

3. **Coupling conditions.** This section is devoted to different specific choices of (2.3).

(S)-Solutions. We consider first the coupling condition inherited from the smooth case. For smooth solutions and pipes' sections, system (1.1) is equivalent to the 3×3 balance law

$$\begin{cases} \partial_t \rho + \partial_x q = -\dfrac{q}{a}\,\partial_x a \\ \partial_t q + \partial_x P(\rho, q, E) = -\dfrac{q^2}{a\rho}\,\partial_x a \\ \partial_t E + \partial_x F(\rho, q, E) = -\dfrac{F}{a}\,\partial_x a. \end{cases} \tag{3.1}$$

The stationary solutions to (1.1) are characterized as solutions to

$$\begin{cases} \partial_x (a(x)\, q) = 0 \\ \partial_x \big(a(x)\, P(\rho, q, E)\big) = p(\rho, e)\, \partial_x a \\ \partial_x \big(a(x)\, F(\rho, q, E)\big) = 0 \end{cases} \quad \text{or} \quad \begin{cases} \partial_x q = -\frac{q}{a}\,\partial_x a \\ \partial_x P\,(\rho, q, E) = -\frac{q^2}{a\rho}\,\partial_x a \\ \partial_x F\,(\rho, q, E) = -\frac{F}{a}\,\partial_x a\,. \end{cases} \tag{3.2}$$

As in the 2×2 case of the p-system, the smoothness of the sections induces a unique choice for condition (2.3), see [9, (2.3) and (2.19)], which reads

$$\textbf{(S)} \quad \Psi = \begin{bmatrix} a^+q^+ - a^-q^- \\ a^+P(u^+) - a^-P(u^-) - \displaystyle\int_{-X}^{X} p\left(\mathcal{R}^a(x), \mathcal{E}^a(x)\right) a'(x)\mathrm{d}x \\ a^+F(u^+) - a^-F(u^-) \end{bmatrix} \tag{3.3}$$

where $a = a(x)$ is a smooth monotone function satisfying $a(-X) = a^-$ and $a(X) = a^+$, for a suitable $X > 0$. $\mathcal{R}^a, \mathcal{E}^a$ are the ρ and e component in the solution to (3.2) with initial datum u^- assigned at $-X$. Note that, by the particular form of (3.3), the function Ψ is independent both from the choice of X and from that of the map a, see [9, 2. in Proposition 2.7].

The definition of **(S)** solution is strictly related to (3.2). As a consequence, it allows to construct the semigroup generated by (1.1) in the case of a pipe whose section satisfies the following condition:

$$\begin{cases} a \in \mathbf{W}^{1,1}\left(\mathbb{R};]\bar a - \Delta, \bar a + \Delta[\right) \text{ for suitable } \Delta > 0,\ \bar a > \Delta \\ \mathrm{TV}(a) < M \text{ for a suitable } M > 0 \\ a'(x) = 0 \text{ for a.e. } x \in \mathbb{R} \setminus [-X, X] \text{ for a suitable } X > 0\,. \end{cases} \tag{3.4}$$

By the same procedure used in [9, Theorem 2.8], thanks to Theorem 2.3, we approximate a with a piecewise constant function a_n. The corresponding problems to (1.4)–(2.6) generate semigroups S_n defined on domains characterized by uniform bounds on the total variation and that are uniformly Lipschitz in time. Here, uniform means also independent from the number of junctions. Therefore, we prove the pointwise convergence of the S_n to a limit semigroup S, along the same lines in [9, Theorem 2.8].

(P)-Solutions. The particular choice of the coupling condition in [10, Section 3] can be recovered in the present setting. Indeed, conditions **(M)**, **(E)** and **(P)** therein amount to the choice

$$\textbf{(P)} \qquad \Psi(a^-, u^-, a^+, u^+) = \begin{bmatrix} a^+q^+ - a^-q^- \\ P(u^+) - P(u^-) \\ a^+F(u^+) - a^-F(u^-) \end{bmatrix}, \tag{3.5}$$

where a^+ and a^- are the pipe's sections. Consider fluid flowing in a horizontal pipe with an elbow or kink, see [14]. Then, it is natural to assume the conservation of the total linear momentum along directions dependent upon the geometry of the elbow, for a general discussion on the relation between the geometry of the junction and condition **(P)**, we refer to [10, Proposition 2.3]. As the angle of the elbow vanishes, one obtains the condition above, see [10, Proposition 2.6].

(L)-Solutions. We can extend the construction in [1, 2, 4] to the 3×3 case (1.4). Indeed, the conservation of the mass and linear momentum in [4] with the conservation of the total energy for the third component lead to the choice

$$\textbf{(L)} \qquad \Psi(a^-, u^-, a^+, u^+) = \begin{bmatrix} a^+ q^+ - a^- q^- \\ a^+ P(u^+) - a^- P(u^-) \\ a^+ F(u^+) - a^- F(u^-) \end{bmatrix}, \tag{3.6}$$

where a^+ and a^- are the pipe's sections. The above is the most immediate extension of the standard definition of Lax solution to the case of the Riemann problem at a junction.

(p)-Solutions. Following [1, 2], motivated by the what happens at the hydrostatic equilibrium, we consider a coupling condition with the conservation of the pressure $p(\rho)$ in the second component of Ψ. Thus

$$\textbf{(p)} \qquad \Psi(a^-, u^-, a^+, u^+) = \begin{bmatrix} a^+ q^+ - a^- q^- \\ p(\rho^+, e^+) - p(\rho^-, e^-) \\ a^+ F(u^+) - a^- F(u^-) \end{bmatrix}, \tag{3.7}$$

where a^+ and a^- are the pipe's sections.

Proposition 3.1. *For every $\bar a > 0$ and $\bar u \in A_0$, each of the coupling conditions Ψ in (3.3), (3.5), (3.6), (3.7) satisfies the requirements* **(Ψ0)**-**(Ψ2)** *and (2.4). In the case of (3.7), we also require that the fluid is perfect, i.e. that (1.3) holds.*

The proof is postponed to Section 5. Thus, Theorem 2.2 applies, yielding the well posedness of (1.4)–(1.5) with each of the particular choices of Ψ in (3.3), (3.5), (3.6), (3.7).

4. **Blow-Up of the total variation.**

Aim of this section is to show the relevance of the bound, required above, on the total variation $\mathrm{TV}(a)$ of the pipe's section. Consider the case in Figure 1.

A wave σ_3^- hits a junction where the pipe's section increases by, say, $\Delta a > 0$. The fastest wave arising from this interaction is σ_3^+, which hits the second junction where the section diminishes by Δa. Up to the leading terms in Δa and in σ_3, we show below that σ_3^{++} is strictly greater than σ_3. Hence, the juxtaposition of a suitable number of situations as that in Figure 1 may lead to the blow up in the total variation of the solution.

Solving the Riemann problem at the first interaction amounts to solve the system

$$L_3\left(L_2\left(T\left(L_1(u;\sigma_1^+)\right);\sigma_2^+\right);\sigma_3^+\right) = T\left(L_3(u;\sigma_3^-)\right), \tag{4.1}$$

where $u \in A_0$, see Figure 2 for the definitions of the waves' strengths σ_i^+ and σ_3^-. Above, T is the map defined in (2.5), which in turn depends from the specific

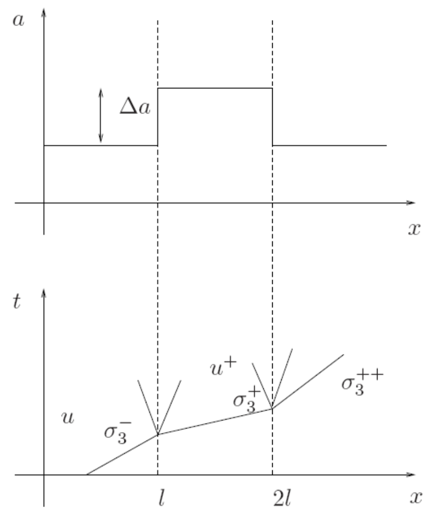

FIGURE 1. A wave σ_3^- hits a junction where the pipe's section increases by Δa. From this interaction, the wave σ_3^+ arises, which hits a second junction, where the pipe section decreases by Δa.

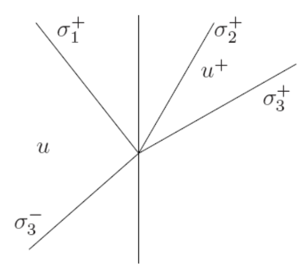

FIGURE 2. Notation used in (4.1) and (4.4).

condition (2.3) chosen. In the expansions below, we use the (ρ, q, e) variables, thus setting $u = (\rho, q, e)$ throughout this section.

Differently from the case of the 2×2 p-system in [9], here we need to consider the second order expansion in $\Delta a = a^+ - a^-$ of the map T; that is

$$T(a, a+\Delta a; u) = u + H(u)\frac{\Delta a}{a} + G(u)\left(\frac{\Delta a}{a}\right)^2 + o\left(\frac{\Delta a}{a}\right)^2 \tag{4.2}$$

The explicit expressions of H and G in (4.2), for each of the coupling conditions (3.3), (3.5), (3.6), (3.7), are in Section 5.2.

Inserting (4.2) in the first order expansions in the wave's sizes of (4.1), with $\tilde{r}_i$ for $i = 1, 2, 3$ as in (5.9), we get a linear system in $\sigma_1^+, \sigma_2^+, \sigma_3^+$. Now, introduce the fluid speed $v = q/\rho$ and the adimensional parameter

$$\vartheta = \left(\frac{v}{c}\right)^2 = \frac{v^2}{\gamma(\gamma-1)e},$$

a sort of *"Mach number"*. Obviously, $\vartheta \in [0, 1]$ for $u \in A_0$. Assuming that the solution is stationary before the interaction of σ_3^- with the junction, we obtain an expression for σ_3^+ of the form

$$\sigma_3^+ = \left(1 + f_1(\vartheta)\,\frac{\Delta a}{a} + f_2(\vartheta)\,\left(\frac{\Delta a}{a}\right)^2\right)\sigma_3^- \,. \tag{4.3}$$

The explicit expressions of f_1 and f_2 in (4.3) are in Section 5.2. Note that obtaining f_1 and f_2 amounts to compute the first order expansion of (4.1) in $\sigma_1^+, \sigma_2^+, \sigma_3^+$, solve the linear system so obtained and expand the solution so obtained at the second order in $(\Delta a)/a$.

Remark that the present situation is different from that of the 2×2 p-system considered in [9]. Indeed, for the p-system $f_2(\vartheta) = f_2(\vartheta^+) = 0$, while here it is necessary to compute the second order term in $(\Delta a)/a$.

Concerning the second junction, similarly, we introduce the parameter $\vartheta^+ = (v^+/c^+)^2$ which corresponds to the state u^+. Recall that u^+ is defined by $u^+ = L_3^-\left(T\left(L_3(u;\sigma_3^-);\sigma_3^+\right)\right)$, see Figure 2. We thus obtain the estimate

$$\sigma_3^{++} = \left(1 - f_1(\vartheta^+)\,\frac{\Delta a}{a} + f_2(\vartheta^+)\,\left(\frac{\Delta a}{a}\right)^2\right)\sigma_3^+, \tag{4.4}$$

where $\vartheta^+ = \vartheta^+\left(\vartheta, \sigma_3^-, (\Delta a)/a\right)$. Now, at the second order in $(\Delta a)/a$ and at the first order in σ_3^-, (4.3) and (4.4) give

$$\begin{aligned}
\sigma_3^{++} &= \left(1 - f_1(\vartheta^+)\,\frac{\Delta a}{a} + f_2(\vartheta^+)\,\left(\frac{\Delta a}{a}\right)^2\right) \\
&\qquad \times \left(1 + f_1(\vartheta)\,\frac{\Delta a}{a} + f_2(\vartheta)\,\left(\frac{\Delta a}{a}\right)^2\right)\sigma_3^- \\
&= \left(1 + \chi(\vartheta)\left(\frac{\Delta a}{a}\right)^2\right)\sigma_3^- \,.
\end{aligned} \tag{4.5}$$

Indeed, computations show that $f_1(\vartheta) - f_1\left(\vartheta^+\right)$ vanishes at the first order in $(\Delta a)/a$, as in the case of the p-system. The explicit expressions of χ are in Section 5.2.

It is now sufficient to compute the sign of χ. If it is positive, then repeating the interaction in Figure 1 a sufficient number of times may lead to an arbitrarily high value of the refracted wave σ_3 and, hence, of the total variation of the solution u. Below, Section 5 is devoted to the computations of χ in the different cases (3.3), (3.5), (3.6) and (3.7). To reduce the formal complexities of the explicit computations below, we consider the standard case of an ideal gas characterized by (1.3) with $\gamma = 5/3$. Remark that obtaining the expressions of χ is immediate once f_1 and f_2 are known.

The results of these computations are in Figure 3. They show that in all the conditions (1.5) considered, there exists a state $u \in A_0$ such that $\chi(\vartheta) > 0$, showing the necessity of condition (2.7). However, in case **(L)**, it turns out that χ is negative on an non trivial interval of values of ϑ. If $\bar{u}$ is chosen in this interval, the wave σ_3 in the construction above is not magnified by the consecutive interactions. The computations leading to the diagrams in Figure 3 are deferred to Section 5.2.

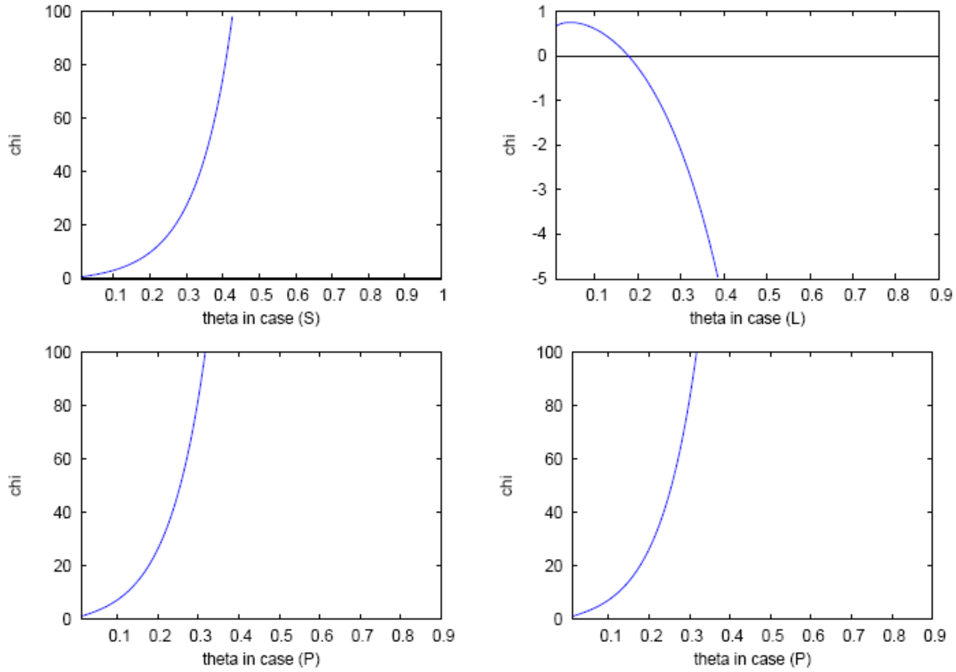

FIGURE 3. Plots of χ as a function of ϑ. Top, left, case **(S)**; right, case **(L)**; bottom, left, case **(P)**; right, case **(p)**. Note that in all four cases, χ attains strictly positive values, showing the necessity of the requirement (2.7).

5. **Technical details.** We recall here basic properties of the Euler equations (1.1), (1.4). The characteristic speeds and the right eigenvectors have the expressions

$$\begin{array}{lll} \lambda_1 = \frac{q}{\rho} - c & \lambda_2 = \frac{q}{\rho} & \lambda_3 = \frac{q}{\rho} + c \\ r_1 = \begin{bmatrix} -\rho \\ \rho c - q \\ qc - E - p \end{bmatrix} & r_2 = \begin{bmatrix} \rho \\ q \\ E + p - \frac{\rho^2 c^2}{\partial_e p} \end{bmatrix} & r_3 = \begin{bmatrix} \rho \\ q + \rho c \\ E + p + qc \end{bmatrix} \end{array} \tag{5.1}$$

whose directions are chosen so that $\nabla \lambda_i \cdot r_i > 0$ for $i = 1, 2, 3$. In the case of an ideal gas, the sound speed $c = \sqrt{\partial_\rho p + \rho^{-2}\, p\, \partial_e p}$ becomes

$$c = \sqrt{\gamma(\gamma - 1)e}\,. \tag{5.2}$$

The shock and rarefaction curves curves of the first and third family are:

$$S_1(u_o, \sigma) \quad = \quad \left\{ \begin{array}{l} \rho = -\sigma + \rho_o \\ v = v_o - \sqrt{-(p - p_o)\left(\frac{1}{\rho} - \frac{1}{\rho_o}\right)} \\ e = e_o - \frac{1}{2}(p + p_o)\left(\frac{1}{\rho} - \frac{1}{\rho_o}\right) \end{array} \right. \quad \text{for} \quad \begin{array}{lcl} \sigma & \leq & 0 \\ \rho & \geq & \rho_o \\ v & \leq & v_o \\ S & \geq & S_o \end{array}$$

$$S_3(u_o,\sigma) = \begin{cases} \rho = \sigma + \rho_o \\ v = v_o - \sqrt{-(p-p_o)\left(\frac{1}{\rho}-\frac{1}{\rho_o}\right)} \\ e = e_o - \frac{1}{2}(p+p_o)\left(\frac{1}{\rho}-\frac{1}{\rho_o}\right) \end{cases} \text{for } \begin{array}{rcl} \sigma & \le & 0 \\ \rho & \le & \rho_o \\ v & \le & v_o \\ S & \le & S_o \end{array}$$

$$R_1(u_o,\sigma) = \begin{cases} \rho = -\sigma + \rho_o \\ v = v_o - \int_{p_o}^{p} [(\rho\, c)(p, S_o)]^{-1}\, dp \\ S(\rho, e) = S(\rho_o, e_o) \end{cases} \text{for } \begin{array}{rcl} \sigma & \ge & 0 \\ \rho & \le & \rho_o \\ v & \ge & v_o \\ e & \le & e_o \end{array}$$

$$R_3(u_o,\sigma) = \begin{cases} \rho = \sigma + \rho_o \\ v = v_o + \int_{p_o}^{p} [(\rho\, c)(p, S_o)]^{-1}\, dp \\ S(\rho, e) = S(\rho_o, e_o) \end{cases} \text{for } \begin{array}{rcl} \sigma & \ge & 0 \\ \rho & \ge & \rho_o \\ v & \ge & v_o \\ e & \ge & e_o \end{array}$$

The forward and backward Lax curves have the expressions

$$\begin{array}{rclrcl} L_1(\sigma;u_o) & = & \begin{cases} S_1(\sigma;u_o) & \sigma < 0 \\ R_1(\sigma;u_o) & \sigma \ge 0 \end{cases} & L_1^-(\sigma;u_o) & = & \begin{cases} S_1^-(\sigma;u_o) & \sigma < 0 \\ R_1^-(\sigma;u_o) & \sigma \ge 0 \end{cases} \\ L_2(\sigma;u_o) & = & \begin{cases} \rho = \sigma + \rho_o \\ v = v_o \\ p = p_o \end{cases} & L_2^-(\sigma;u_o) & = & \begin{cases} \rho = \sigma + \rho_o \\ v = v_o \\ p = p_o \end{cases} \\ L_3(\sigma;u_o) & = & \begin{cases} S_3(\rho;u_o) & \sigma < 0 \\ R_3(\sigma;u_o) & \sigma \ge 0 \end{cases} & L_3^-(\sigma;u_o) & = & \begin{cases} S_3^-(\rho;u_o) & \sigma < 0 \\ R_3^-(\sigma;u_o) & \sigma \ge 0 \end{cases} \end{array} \tag{5.3}$$

5.1. **Proofs of Section 2.** The following result will be of use in the proof of Theorem 2.2.

Lemma 5.1. *With reference to (5.3), the following equalities hold:*

$$\frac{\partial L_1}{\partial \sigma_1}_{|\sigma_1=0} = \begin{bmatrix} -1 \\ -\lambda_1(u_o) \\ -\frac{E_o+p_o}{\rho_o} + \frac{q_o}{\rho_o} c_o \end{bmatrix}, \quad \frac{\partial L_2}{\partial \sigma_2}_{|\sigma_2=0} = \begin{bmatrix} 1 \\ \lambda_2(u_o) \\ \frac{E_o+p_o}{\rho_o} - \frac{\rho_o\, c_o^2}{\partial_e p_o} \end{bmatrix},$$

$$\frac{\partial L_3}{\partial \sigma_3}_{|\sigma_3=0} = \begin{bmatrix} 1 \\ \lambda_3(u_o) \\ \frac{E_o+p_o}{\rho_o} + \frac{q_o}{\rho_o} c_o \end{bmatrix}, \quad \frac{\partial L_i^-}{\partial \sigma_i}_{|\sigma_i=0} = -\frac{\partial L_i}{\partial \sigma_i}_{|\sigma_i=0} \quad i = 1,2,3,$$

$$\frac{\partial L_i}{\partial \rho_o}_{|\sigma_i=0} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \frac{\partial L_i}{\partial q_o}_{|\sigma_i=0} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \frac{\partial L_i}{\partial E_o}_{|\sigma_i=0} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},$$

$$\frac{\partial L_i^-}{\partial \rho_o}_{|\sigma_i=0} = \frac{\partial L_i}{\partial \rho_o}_{|\sigma_i=0}, \quad \frac{\partial L_i^-}{\partial q_o}_{|\sigma_i=0} = \frac{\partial L_i}{\partial q_o}_{|\sigma_i=0}, \quad \frac{\partial L_i^-}{\partial E_o}_{|\sigma_i=0} = \frac{\partial L_i}{\partial E_o}_{|\sigma_i=0}.$$

The proof is immediate and, hence, omitted.

Proof of Theorem 2.2. Following [7, Proposition 4.2], the 3×3 system (1.4) defined for $x \in \mathbb{R}$ can be rewritten as the following 6×6 system defined for $x \in \mathbb{R}^+$:

$$\left\{ \begin{array}{lrcl} \partial_t U + \partial_x \mathcal{F}(U) = 0 & (t,x) & \in & \mathbb{R}^+ \times \mathbb{R}^+ \\ b\left(U(t,0+)\right) = 0 & t & \in & \mathbb{R}^+ \end{array} \right. \tag{5.4}$$

the relations between U and $u = (\rho, q, E)$, between $\mathcal{F}$ and the flow in (1.4) being

$$U(t,x) = \begin{bmatrix} \rho(t,x/\mu) \\ q(t,x/\mu) \\ E(t,x/\mu) \\ \rho(t,x) \\ q(t,x) \\ E(t,x) \end{bmatrix} \quad \text{and} \quad \mathcal{F}(U) = \begin{bmatrix} \mu\, U_2 \\ \mu\, P(U_1,U_2,U_3) \\ \mu\, F(U_1,U_2,U_3) \\ U_5 \\ P(U_4,U_5,U_6) \\ F(U_4,U_5,U_6) \end{bmatrix} \tag{5.5}$$

with $x \in \mathbb{R}^+$ and E, P, F defined in (1.2). Above, we choose

$$\mu < \inf_{u \in \bar{B}(\bar{u},\bar{\delta})} \min \left\{ \frac{\lambda_3(u)}{\lambda_1(u)}, \frac{\lambda_1(u)}{\lambda_2(u)} \right\} \tag{5.6}$$

where $\bar{B}(\bar{u}, r)$ is the closed ball centered at $\bar{u}$ with radius $\bar{\delta}$. In particular $\mu < 0$. The boundary condition in (5.4) is related to (1.5) by

$$b(U) = \Psi\left(a^-, (U_1,U_2,U_3); a^+, (U_4,U_5,U_6)\right)$$

for fixed sections a^- and a^+.

The thesis now follows applying [6, Theorem 2.2] to (5.4), with $n = 6$ and $l = 3$. Indeed, the assumptions **(**γ**)**, **(b)** and **(f)** therein are here satisfied. More precisely, condition **(f)** is the strictly hyperbolicity of (5.4), which follows from the choice of μ in (5.6). Indeed, the eigenvalues of $D\mathcal{F}$ are

$$\begin{array}{lclclclclcl} \Lambda_1 & = & \mu\lambda_3 & \Lambda_2 & = & \mu\lambda_2 & \Lambda_3 & = & \lambda_1 \\ \Lambda_4 & = & \mu\lambda_1 & \Lambda_5 & = & \lambda_2 & \Lambda_6 & = & \lambda_3 \,. \end{array}$$

At the state $\bar{U} \equiv (\bar{u}, \bar{u})$, by the choice (2.1), we have

$$\Lambda_1(\bar{U}) < \Lambda_2(\bar{U}) < \Lambda_3(\bar{U}) < 0 < \Lambda_4(\bar{U}) < \Lambda_5(\bar{U}) < \Lambda_6(\bar{U})\,. \tag{5.7}$$

To verify condition **(b)**, call $\mathcal{R}_1, \ldots, \mathcal{R}_6$ the right eigenvectors of $D\mathcal{F}$, as $\mathcal{F}$ defined in (5.5). Simple computation show that the determinant in **(b)** takes the form

$$\begin{aligned} &\det\left[Db\left(\bar{U}\right)\cdot\mathcal{R}_4\left(\bar{U}\right) \quad Db\left(\bar{U}\right)\cdot\mathcal{R}_5\left(\bar{U}\right) \quad Db\left(\bar{U}\right)\cdot\mathcal{R}_6\left(\bar{U}\right)\right] \\ &= \det\left[D_{u^-}\Psi \cdot \frac{\partial L_1}{\partial \sigma_1}_{|\sigma_1=0} \quad D_{u^+}\Psi \cdot \frac{\partial L_2^-}{\partial \sigma_2}_{|\sigma_2=0} \quad D_{u^+}\Psi \cdot \frac{\partial L_2^-}{\partial u^+}_{|\sigma_2=0} \cdot \frac{\partial L_3^-}{\partial \sigma_3}_{|\sigma_3=0}\right] \\ &= \det\left[D_{u^-}\Psi \cdot r_1(\bar{u}) \quad -D_{u^+}\Psi \cdot r_2(\bar{u}) \quad -D_{u^+}\Psi \cdot r_3(\bar{u})\right] \\ &= \det\left[D_{u^-}\Psi \cdot r_1(\bar{u}) \quad D_{u^+}\Psi \cdot r_2(\bar{u}) \quad D_{u^+}\Psi \cdot r_3(\bar{u})\right], \\ &\neq 0 \end{aligned}$$

by (2.4), above all functions are computed at $\bar{a}$, $\bar{u}$. Finally, condition **(**γ**)** follows directly from (5.7).

Hence, [6, Theorem 2.2] applies, proving all the statements 1.–5., see also [7, Proposition 4.5]. □

Proof of Proposition 3.1. It is immediate to check that each of the coupling conditions (3.3), (3.5), (3.6), (3.7) satisfies the requirements (**Ψ0**) and (**Ψ1**).

To prove that (**Ψ2**) is satisfied, we use an *ad hoc* argument for condition **(S)**. In all the other cases, the function Ψ admits the representation $\Psi(a^-, u^-; a^+, u^+) = \psi(a^-, u^-) - \psi(a^+, u^+)$. Therefore, (**Ψ2**) trivially holds.

We prove below (2.4) in each case separately. Note however that for any of the considered choices of Ψ,

$$D_{u^+}\Psi(a^-, u^-, a^+, u^+)_{|u=\bar u,\, a=\bar a} = -D_{u^-}\Psi(a^-, u^-, a^+, u^+)_{|u=\bar u,\, a=\bar a} \tag{5.8}$$

so that (2.4) reduces to

$$\begin{aligned} & \det\left[D_{u^-}\Psi \cdot r_1(\bar u) \quad D_{u^+}\Psi \cdot r_2(\bar u) \quad D_{u^+}\Psi \cdot r_3(\bar u)\right] \\ = \quad & -\det D_{u^+}\Psi \cdot \det\left[r_1(\bar u) \quad r_2(\bar u) \quad r_3(\bar u)\right] . \end{aligned}$$

Thus, it is sufficient to prove that $\det D_{u^+}\Psi(a^-, u^-, a^+, u^+)_{|u=\bar u,\, a=\bar a} \neq 0$.

(S)-solutions. To prove that the coupling condition (3.3) satisfies (**Ψ2**), simply use the additivity of the integral and the uniqueness of the solution to the Cauchy problem for the ordinary differential equation (3.2).

Next, we have

$$D_u \left(\int_{-X}^{X} p\left(\mathcal{R}^a(x), \mathcal{E}^a(x)\right)\, a'(x)\, \mathrm{d}x \right)_{|u=\bar u,\, a=\bar a} = 0\,,$$

since $a'(x) = 0$ for all x, because $a^- = a^+ = \bar a$. Thus, Ψ in (3.3) satisfies

$$\begin{aligned} & D_{u^+}\Psi(a^-, u^-, a^+, u^+)_{|u=\bar u,\, a=\bar a} \\ & = \bar a^3 \det \begin{bmatrix} 0 & 1 & 0 \\ -\frac{\bar q^2}{\bar\rho^2} + \partial_\rho \bar p + \frac{\partial_e \bar p}{\bar\rho}\left(\frac{\bar q^2}{\bar\rho^2} - \frac{\bar E}{\bar\rho}\right) & \frac{\bar q}{\bar\rho}\left(2 - \frac{\partial_e \bar p}{\bar\rho}\right) & \frac{\partial_e \bar p}{\bar\rho} \\ -\frac{\bar q}{\bar\rho}\left(\partial_\rho \bar p + \frac{\partial_e \bar p}{\bar\rho}\left(\frac{\bar q^2}{\bar\rho^2} - \frac{\bar E}{\bar\rho}\right) - \frac{\bar E + \bar p}{\bar\rho}\right) & \frac{\bar E + \bar p}{\bar\rho} - \frac{\partial_e \bar p}{\bar\rho}\frac{\bar q^2}{\bar\rho^2} & -\frac{\bar q}{\bar\rho}\left(\frac{\partial_e \bar p}{\bar\rho} + 1\right) \end{bmatrix} \\ & = -\bar a^3\, \lambda_1(\bar u)\, \lambda_2(\bar u)\, \lambda_3(\bar u), \end{aligned}$$

which is non zero if $\bar u \in A_0$.

(P)-solutions. Concerning condition (3.5), we have

$$D_{u^+}\Psi(a^-, u^-, a^+, u^+)_{|u=\bar u,\, a=\bar a} = -\bar a^2 \lambda_1(\bar u)\lambda_2(\bar u)\lambda_3(\bar u),$$

which is non zero if $\bar u \in A_0$, as in [10], with $n = 2$.

(L)-solution. For condition (3.6) the computations are very similar to the above case:

$$D_{u^+}\Psi(a^-, u^-, a^+, u^+)_{|u=\bar u,\, a=\bar a} = -\bar a^3 \lambda_1(\bar u)\lambda_2(\bar u)\lambda_3(\bar u),$$

which is non zero if $\bar u \in A_0$.

(p)-solution. Finally, concerning condition (3.7),

$$\begin{aligned} & D_{u^+}\Psi(a^-, u^-, a^+, u^+)_{|u=\bar u,\, a=\bar a} \\ & = \det \begin{bmatrix} 0 & \bar a & 0 \\ \partial_\rho \bar p + \frac{\partial_e \bar p}{\bar\rho}\left(\frac{\bar q^2}{\bar\rho^2} - \frac{\bar E}{\bar\rho}\right) & -\frac{\bar q}{\bar\rho^2}\partial_e \bar p & \frac{\partial_e \bar p}{\bar\rho} \\ -\bar a\frac{\bar q}{\bar\rho}\left(\partial_\rho \bar p + \frac{\partial_e \bar p}{\bar\rho}\left(\frac{\bar q^2}{\bar\rho^2} - \frac{\bar E}{\bar\rho}\right) - \frac{\bar E + \bar p}{\bar\rho}\right) & \bar a\frac{\bar E + \bar p}{\bar\rho} - \bar a\frac{\partial_e \bar p}{\bar\rho}\frac{\bar q^2}{\bar\rho^2} & -\bar a\frac{\bar q}{\bar\rho}\left(\frac{\partial_e \bar p}{\bar\rho} + 1\right) \end{bmatrix} \\ & = \bar a^2 \lambda_2(\bar u)\left(c^2 + \lambda_2^2(\bar u)\frac{\partial_e \bar p}{\bar\rho}\right), \end{aligned}$$

which is non zero if $\bar u \in A_0$ and if the fluid is perfect, i.e. (1.3) holds. □

5.2. **Computation of χ in (4.5).** We use below the tangent vectors to the curves $\sigma \to L_i(\sigma; \rho, q, e)$ at $\sigma = 0$ in the (ρ, q, e) coordinates. They are:

$$\tilde r_1 = \begin{bmatrix} -1 \\ -\frac{q}{\rho} - \sqrt{\gamma(\gamma-1)e} \\ -(\gamma-1)\frac{e}{\rho} \end{bmatrix}, \quad \tilde r_2 = \begin{bmatrix} 1 \\ \frac{q}{\rho} \\ -\frac{e}{\rho} \end{bmatrix}, \quad \tilde r_3 = \begin{bmatrix} 1 \\ \frac{q}{\rho} - \sqrt{\gamma(\gamma-1)e} \\ (\gamma-1)\frac{e}{\rho} \end{bmatrix}. \tag{5.9}$$

The Case of Condition (S). Let Ψ be defined in (3.3) and set

$$\Sigma(a^-, a^+, u) = \int_{-X}^{X} p\left(\mathcal{R}^a(x), \mathcal{E}^a(x)\right) a'(x)dx$$

where the functions $\mathcal{R}^a, \mathcal{E}^a$ have the same meaning as in (3.3). A perturbative method allows to compute the solution to (3.2) with a second order accuracy in $(\Delta a)/a$. Indeed, (3.2) can be rewritten in normal form as

$$\begin{cases} \partial_x \rho &= -\frac{1}{a_o+x}\,\frac{1}{1-1/\vartheta}\,\rho \\ \partial_x q &= -\frac{1}{a_o+x}\,q \\ \partial_x e &= -\frac{1}{a_o+x}\,\frac{\gamma-1}{1-1/\vartheta}\,e\,, \end{cases}$$

where we assumed that $a(x) = a_o + x$, which is possible by [9, (2) in Proposition 2.7]. Writing the above system as $\partial_x u = \varphi(x,u)$, we use below the following relation to compute H and G:

$$u(x) = u(0) + \varphi(0,u_o)\,x + \frac{1}{2}\left(\partial_x\varphi(0,u_o) + \partial_u\varphi(0,u_o)\varphi(0,u_o)\right)x^2 + \ldots$$

obtaining

$$\begin{aligned} \frac{\rho(\Delta a)}{\rho_o} &= 1 + \frac{\vartheta}{1-\vartheta}\,\frac{\Delta a}{a} - \frac{1+\gamma\vartheta}{(1-\vartheta)^3}\vartheta\left(\frac{\Delta a}{a}\right)^2 + \ldots \\ \frac{q(\Delta a)}{q_o} &= 1 - \frac{\Delta a}{a} + \left(\frac{\Delta a}{a}\right)^2 + \ldots \\ \frac{e(\Delta a)}{e_o} &= 1 + \frac{(\gamma-1)\vartheta}{1-\vartheta}\,\frac{\Delta a}{a} - \frac{(\gamma-1)}{2}\,\frac{3-4\vartheta+(\gamma+2)\vartheta^2}{(1-\vartheta)^3}\vartheta\left(\frac{\Delta a}{a}\right)^2 + \ldots \end{aligned}$$

and we get explicitly the terms H and G in (4.2) of the second order expansion of T:

$$H(\rho,q,e) = \begin{bmatrix} \frac{\vartheta}{1-\vartheta}\rho \\ -q \\ \frac{(\gamma-1)\vartheta}{1-\vartheta}e \end{bmatrix} \qquad G(\rho,q,e) = \begin{bmatrix} -\frac{1+\gamma\vartheta}{(1-\vartheta)^3}\vartheta\rho \\ q \\ -\frac{(\gamma-1)}{2}\,\frac{3-4\vartheta+(\gamma+2)\vartheta^2}{(1-\vartheta)^3}\vartheta e \end{bmatrix}.$$

Solving (4.1) at the second order in $(\Delta a)/a$ and assuming that the system is at equilibrium before the interaction of σ_3^- with the junction, we obtain the coefficients f_1, f_2 in (4.3) and χ in (4.5)

$$\begin{aligned} f_1(\vartheta) &= \frac{\vartheta^{3/2} - 3\vartheta - 3\vartheta^{1/2} - 3}{6\,(1-\vartheta^{1/2})^2\,(1+\vartheta^{1/2})} \\ f_2(\vartheta) &= \Big(187\vartheta^7 + 396\vartheta^{13/2} - 904\vartheta^6 - 1872\vartheta^{11/2} + 1827\vartheta^5 + 3636\vartheta^{9/2} \\ &\quad -1908\vartheta^4 - 3724\vartheta^{7/2} + 985\vartheta^3 + 2560\vartheta^{5/2} + 1049\vartheta^2 + 132\vartheta^{3/2} \\ &\quad +447\vartheta + 216\vartheta^{1/2} + 45\Big) \Big/ \Big(180\,(1-\vartheta)^4\Big) \end{aligned}$$

$$\begin{aligned}\chi(\vartheta) &= \Big(374\vartheta^7 + 792\vartheta^{13/2} - 1808\vartheta^6 - 3744\vartheta^{11/2} + 3654\vartheta^5 + 7272\vartheta^{9/2} \\ &\quad -3821\vartheta^4 - 7408\vartheta^{7/2} + 2090\vartheta^3 + 5360\vartheta^{5/2} + 2308\vartheta^2 + 624\vartheta^{3/2} \\ &\quad +1254\vartheta + 432\vartheta^{1/2} + 45\Big) / \Big(180\,(1-\vartheta)^4\Big)\,.\end{aligned}$$

The Case of Condition **(P)**. Let Ψ be defined in (3.5). With reference to (4.2), we show below explicitly the terms H and G in (4.2) of the second order expansion of T,

$$\begin{aligned} H(\rho,q,e) &= \left[\begin{array}{l} \dfrac{8\left(-\vartheta^3+2\,\vartheta^2-\vartheta\right)}{3\,(\vartheta^3-3\,\vartheta^2+3\,\vartheta-1)}\,\rho \\ -q \\ -\dfrac{2\left(5\,\vartheta^4-7\,\vartheta^3-\vartheta^2+3\,\vartheta\right)}{9\,(\vartheta^3-3\,\vartheta^2+3\,\vartheta-1)}\,e \end{array}\right] \\ G(\rho,q,e) &= \left[\begin{array}{l} \dfrac{64\left(\vartheta^3+3\,\vartheta^2\right)}{27\,(\vartheta^3-3\,\vartheta^2+3\,\vartheta-1)}\,\rho \\ q \\ -\dfrac{565\,\vartheta^4-1599\,\vartheta^3+927\,\vartheta^2-405\,\vartheta}{81\,(\vartheta^3-3\,\vartheta^2+3\,\vartheta-1)}\,e \end{array}\right]. \end{aligned}$$

Moreover, the coefficients f_1, f_2 in (4.3) read

$$\begin{aligned} f_1(\vartheta) &= \frac{\sqrt{\vartheta}\,\left(9\,\vartheta^2+2\,\vartheta-27\right)+3\,\vartheta^2-42\,\vartheta-9}{18\sqrt{\vartheta}\,(\vartheta-1)-18\,(\vartheta-1)} \\ f_2(\vartheta) &= \frac{\sqrt{\vartheta}\,\left(154\,\vartheta^5+931\,\vartheta^4-4416\,\vartheta^3+6570\,\vartheta^2+990\,\vartheta+891\right)}{324\left(\sqrt{\vartheta}\,(\vartheta^3-3\,\vartheta^2+3\,\vartheta-1)-\vartheta^3+3\,\vartheta^2-3\,\vartheta+1\right)} \\ &\quad + \frac{86\,\vartheta^5-311\,\vartheta^4-752\,\vartheta^3+7038\,\vartheta^2+1026\,\vartheta+81}{324\left(\sqrt{\vartheta}\,(\vartheta^3-3\,\vartheta^2+3\,\vartheta-1)-\vartheta^3+3\,\vartheta^2-3\,\vartheta+1\right)}. \end{aligned}$$

Next, χ is given by

$$\begin{aligned} \chi &= \frac{\sqrt{\vartheta}\,\left(407\,\vartheta^5+1931\,\vartheta^4-7858\,\vartheta^3+14766\,\vartheta^2+1179\,\vartheta+1863\right)}{324\left(\sqrt{\vartheta}\,(\vartheta^3-3\,\vartheta^2+3\,\vartheta-1)-\vartheta^3+3\,\vartheta^2-3\,\vartheta+1\right)} \\ &\quad + \frac{-23\,\vartheta^5+141\,\vartheta^4+2002\,\vartheta^3+15714\,\vartheta^2+2565\,\vartheta+81}{324\left(\sqrt{\vartheta}\,(\vartheta^3-3\,\vartheta^2+3\,\vartheta-1)-\vartheta^3+3\,\vartheta^2-3\,\vartheta+1\right)}\,. \end{aligned}$$

The Case of Condition **(L)**. Let Ψ be defined in (3.6). Then,

$$H(\rho,q,e) = \left[\begin{array}{l} -\rho \\ -q \\ 0 \end{array}\right] \qquad \text{and} \qquad G(\rho,q,e) = \left[\begin{array}{l} -\dfrac{4\,\vartheta}{3\,(\vartheta-1)}\,\rho \\ q \\ -\dfrac{35\,\vartheta^2-9\,(4\,\vartheta-1)}{9\,(\vartheta-1)}\,e \end{array}\right].$$

The coefficients f_1, f_2 in (4.3) read

$$\begin{aligned} f_1(\vartheta) &= 0 \\ f_2(\vartheta) &= \frac{\sqrt{\vartheta}\,\left(63\,\vartheta^2-106\,\vartheta+27\right)+21\,\vartheta^2-78\,\vartheta+9}{36\left(\sqrt{\vartheta}\,(\vartheta-1)-\vartheta+1\right)}, \end{aligned}$$

so that χ is

$$\chi = \frac{\sqrt{\vartheta}\,\left(63\,\vartheta^2 - 106\,\vartheta + 27\right) + 21\,\vartheta^2 - 78\,\vartheta + 9}{18\left(\sqrt{\vartheta}\,(\vartheta-1) - \vartheta + 1\right)}\,.$$

The Case of Condition **(p)**. Let Ψ be defined in (3.7). With reference to (4.2),

$$\begin{aligned} H(\rho,q,e) &= \begin{bmatrix} -\dfrac{2\left(4\,\vartheta^3+12\,\vartheta^2+9\,\vartheta\right)}{4\,(2\,\vartheta^3+9\,\vartheta^2)+27\,(2\,\vartheta+1)}\,\rho \\ -q \\ \dfrac{2\left(4\,\vartheta^3+12\,\vartheta^2+9\,\vartheta\right)}{4\,(2\,\vartheta^3+9\,\vartheta^2)+27\,(2\,\vartheta+1)}\,e \end{bmatrix} \\ G(\rho,q,e) &= \begin{bmatrix} -\dfrac{4\left(\vartheta^3+3\,\vartheta^2\right)}{4\,(2\,\vartheta^3+9\,\vartheta^2)+27\,(2\,\vartheta+1)}\,\rho \\ q \\ \dfrac{12\left(\vartheta^3+2\,\vartheta^2\right)}{4\,(2\,\vartheta^3+9\,\vartheta^2)+27\,(2\,\vartheta+1)}\,e \end{bmatrix}, \end{aligned}$$

with f_1 and f_2 given by

$$\begin{aligned} f_1(\vartheta) &= \frac{-2\,\vartheta^2+4\,\vartheta^{\frac{3}{2}}+3\,\vartheta-9}{2\,(4\,\vartheta^2+12\,\vartheta+9)} \\ f_2(\vartheta) &= \frac{32\,\vartheta^4+8\,\sqrt{\vartheta}\,\left(4\,\vartheta^3+9\,\vartheta^2-9\,\vartheta\right)+316\,\vartheta^3+558\,\vartheta^2+216\,\vartheta+81}{6\,(16\,\vartheta^4+96\,\vartheta^3+216\,\vartheta^2+216\,\vartheta+81)}, \end{aligned}$$

so that

$$\chi = \frac{60\,\vartheta^4+96\,\sqrt{\vartheta}\,\left(\vartheta^3+\vartheta^2-3\,\vartheta\right)+700\,\vartheta^3+1107\,\vartheta^2-54\,\vartheta+81}{6\,(16\,\vartheta^4+96\,\vartheta^3+216\,\vartheta^2+216\,\vartheta+81)}.$$

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