



On the stability of a model for the cutting of metal plates by means of laser beams



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ABSTRACT

In a class of systems of balance laws in several space dimensions, we prove the stability of solutions with respect to variations in the flow and in the source. This class comprises a model describing the cutting of metal plates by means of laser beam.

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1. Introduction

Following [1], we consider this system of n balance laws in several space dimensions:

$$\begin{cases} \partial_t u_i + \operatorname{div}_x \varphi_i(t, x, u_i, \vartheta * u) = \Phi_i(t, x, u_i, \vartheta * u) & i = 1, \dots, n. \\ u_i(0, x) = \bar{u}_i(x) \end{cases} \quad (1)$$

Here, $t \in [0, +\infty[$ is time, $x \in \mathbb{R}^N$ is the space coordinate and u_1, \dots, u_n are the unknowns. The function ϑ is a smooth function defined in \mathbb{R}^N attaining values in $\mathbb{R}^{m \times n}$, so that

$$\vartheta \in \mathbf{C}_c^2(\mathbb{R}^N; \mathbb{R}^{m \times n}), \quad (\vartheta * u(t))(x) = \int_{\mathbb{R}^N} \vartheta(x - \xi) u(t, \xi) d\xi, \quad (\vartheta * u(t))(x) \in \mathbb{R}^m.$$

Requirements on the flows φ_i , on the sources Φ_i and on the initial data \bar{u}_i ensuring the well posedness of (1) are provided below.

A key property of system (1) is that the equations are coupled only through the nonlocal convolution term $\vartheta * u$. It is this feature that allows a well posedness and stability theory, although we are dealing with *systems* of balance laws in *several* space dimensions.

The driving example motivating (1) is a new model for the cutting of metal plates by means of a laser beam, presented in [1, Section 3], see also [2,3]. However, (1) also comprises the model [4], see also

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[1, Section 4], devoted to the dynamics on a conveyor belt, as well as several crowd dynamics models, e.g. [5–7]. **Theorem 2.3**, applied to each of these cases, provides the stability of solutions with respect to perturbations of fluxes and sources.

2. Results

Throughout, $\text{grad}_x f$ and $\text{div}_x f$ denote the gradient and the divergence of f with respect to the space variable $x \in \mathbb{R}^N$. Throughout, we fix the non trivial time interval $\widehat{I} = [0, \widehat{T}]$. For any $k > 0$, we also denote $\mathcal{U}_k = [-k, k]$ and $\mathcal{U}_k^m = [-k, k]^m$.

Recall the definition of solution to (1), based on [8, Definition 1], and the well posedness result obtained in [1].

Definition 2.1 ([1, Definition 2.1]). Let $\bar{u} \in \mathbf{L}^\infty(\mathbb{R}^N, \mathbb{R}^n)$. A map $u : \widehat{I} \rightarrow \mathbf{L}^\infty(\mathbb{R}^N, \mathbb{R}^n)$ is a solution on \widehat{I} to (1) with initial datum \bar{u} if, for $i = 1, \dots, n$, setting for all $z \in \mathbb{R}$

$$\widetilde{\varphi}_i(t, x, z) = \varphi_i(t, x, z, (\vartheta * u)(t, x)) \quad \text{and} \quad \widetilde{\Phi}_i(t, x, z) = \Phi_i(t, x, z, (\vartheta * u)(t, x)),$$

the map u_i is a Kruřkov solution [8] to $\begin{cases} \partial_t u_i + \text{div}_x \widetilde{\varphi}_i(t, x, u_i) = \widetilde{\Phi}_i(t, x, u_i), \\ u_i(0, x) = \bar{u}_i(x) \end{cases}$, for $i = 1, \dots, n$.

Theorem 2.2 ([1, Theorem 2.2]). Assume φ, Φ and ϑ satisfy the following conditions, for a given $\lambda \in (\mathbf{C}^0 \cap \mathbf{L}^1)(\widehat{I} \times \mathbb{R}^N \times \mathbb{R}^+; \mathbb{R}^+)$:

(φ) For any $U > 0$, $\varphi \in (\mathbf{C}^2 \cap \mathbf{W}^{2,\infty})(\widehat{I} \times \mathbb{R}^N \times \mathcal{U}_U \times \mathcal{U}_U^m; \mathbb{R}^{n \times N})$ and for all $t \in \widehat{I}$, $x \in \mathbb{R}^N$, $u \in \mathcal{U}_U$, $A \in \mathcal{U}_U^m$

$$\max \left\{ \begin{array}{ll} \|\text{grad}_x \varphi(t, x, u, A)\|, & \|\text{div}_x \varphi(t, x, u, A)\|, \\ \|\text{grad}_x \text{div}_x \varphi(t, x, u, A)\|, & \|\text{grad}_x \text{grad}_A \varphi(t, x, u, A)\|, \\ \|\text{grad}_A \varphi(t, x, u, A)\|, & \|\text{grad}_A^2 \varphi(t, x, u, A)\| \end{array} \right\} \leq \lambda(t, x, U).$$

(Φ) For any $U > 0$, $\Phi \in (\mathbf{C}^1 \cap \mathbf{W}^{1,\infty})(\widehat{I} \times \mathbb{R}^N \times \mathcal{U}_U \times \mathcal{U}_U^m; \mathbb{R}^n)$ and for all $t \in \widehat{I}$, $x \in \mathbb{R}^N$, $u \in \mathcal{U}_U$, $A \in \mathcal{U}_U^m$, $\max \{ \|\Phi(t, x, u, A)\|, \|\text{grad}_x \Phi(t, x, u, A)\| \} \leq \lambda(t, x, U)$.

(ϑ) $\vartheta \in \mathbf{C}_c^2(\mathbb{R}^N; \mathbb{R}^{m \times n})$.

Then, for any $\bar{\mathcal{C}} > 0$ there exists a $T_* \in \widehat{I}$ and positive \mathcal{L}, \mathcal{C} such that for any datum

$$\bar{u} \in (\mathbf{L}^1 \cap \mathbf{L}^\infty \cap \mathbf{BV})(\mathbb{R}^N; \mathbb{R}^n) \quad \text{with} \quad \begin{array}{l} \|\bar{u}_i\|_{\mathbf{L}^1(\mathbb{R}^N; \mathbb{R}^n)} \leq \bar{\mathcal{C}}, \\ \|\bar{u}_i\|_{\mathbf{L}^\infty(\mathbb{R}^N; \mathbb{R}^n)} \leq \bar{\mathcal{C}}, \\ \text{TV}(\bar{u}_i) \leq \bar{\mathcal{C}}, \end{array} \tag{2}$$

problem (1) admits a unique solution $u \in \mathbf{C}^0([0, T_*]; \mathbf{L}^1(\mathbb{R}^N; \mathbb{R}^n))$ in the sense of Definition 2.1, satisfying for all $t \in [0, T_*]$ the bounds $\|u(t)\|_{\mathbf{L}^1(\mathbb{R}^N; \mathbb{R}^n)} \leq \mathcal{C}$, $\|u(t)\|_{\mathbf{L}^\infty(\mathbb{R}^N; \mathbb{R}^n)} \leq \mathcal{C}$ and $\text{TV}(u(t)) \leq \mathcal{C}$. Moreover, if also \bar{w} satisfies (2) and w is the corresponding solution to (1), the following Lipschitz estimate holds: $\|u(t) - w(t)\|_{\mathbf{L}^1(\mathbb{R}^N; \mathbb{R}^n)} \leq \mathcal{L} \|\bar{u} - \bar{w}\|_{\mathbf{L}^1(\mathbb{R}^N; \mathbb{R}^n)}$.

We complete the above statement proving the stability of solutions with respect to Φ_i, φ_i , for $i = 1, \dots, n$, and with respect to ϑ .

Theorem 2.3. Let φ^1, φ^2 and Φ^1, Φ^2 satisfy (φ) and (Φ) in Theorem 2.2, with the same function λ . Let ϑ^1, ϑ^2 satisfy (ϑ). Assume moreover that

$$\int_{\widehat{I}} \int_{\mathbb{R}^N} \sup_{u \in \mathcal{U}_U} \lambda(t, x, u) dx dt < +\infty. \tag{3}$$

Then, the solutions $u^\ell \equiv (u_1^\ell, \dots, u_n^\ell)$ to $\begin{cases} \partial_t u_i^\ell + \operatorname{div}_x \varphi_i^\ell(t, x, u_i, \vartheta^\ell * u) = \Phi_i^\ell(t, x, u_i, \vartheta^\ell * u) \\ u_i(0, x) = \bar{u}_i(x) \end{cases}$, for $\ell = 1, 2$, satisfy (where $\operatorname{grad}_{xA} = [\partial_{x_1}, \dots, \partial_{x_n}, \partial_{A_{11}}, \dots, \partial_{A_{nn}}]$)

$$\begin{aligned} \|u^1(t) - u^2(t)\|_{\mathbf{L}^1(\mathbb{R}^N; \mathbb{R}^n)} &\leq C^* \left(\|\partial_u(\varphi^1 - \varphi^2)\|_{\mathbf{L}^\infty(I \times \mathbb{R}^N \times \mathcal{U}_U \times \mathcal{U}_U^m; \mathbb{R}^n)} \right. \\ &\quad + \|\operatorname{grad}_{xA}(\varphi^1 - \varphi^2)\|_{\mathbf{L}^1(I \times \mathbb{R}^N; \mathbf{L}^\infty(\mathcal{U}_U \times \mathcal{U}_U^m; \mathbb{R}^{(N+m \times n) \times n})} \\ &\quad \left. + \|\Phi^1 - \Phi^2\|_{\mathbf{L}^1(I \times \mathbb{R}^N; \mathbf{L}^\infty(\mathcal{U}_U \times \mathcal{U}_U^m; \mathbb{R}^n))} + \|\vartheta^1 - \vartheta^2\|_{\mathbf{L}^1(I \times \mathbb{R}^N; \mathbb{R}^{m \times n})} \right) \end{aligned} \quad (4)$$

for a constant C^* dependent on the assumptions (φ) , (Φ) and (ϑ) , whose value is estimated in (9), (10) and (11).

Proof of Theorem 2.3. Below, we often use the standard bound

$$\|\vartheta * u\|_{\mathbf{L}^\infty(I \times \mathbb{R}^N; \mathbb{R}^n)} \leq \|\vartheta\|_{\mathbf{L}^\infty(\mathbb{R}^N; \mathbb{R}^{m \times n})} \|u\|_{\mathbf{C}^0(I; \mathbf{L}^1(\mathbb{R}^N; \mathbb{R}))}, \quad (5)$$

that holds for ϑ satisfying (ϑ) and $u \in \mathbf{C}^0(I; \mathbf{L}^1(\mathbb{R}^N; \mathbb{R}^n))$. By (ϑ) , we may assume that $\|\vartheta_j\|_{\mathbf{L}^1(\mathbb{R}^N; \mathbb{R})} \leq 1/n$ for all $j = 1, \dots, m$ and $i = 1, \dots, n$, so that in the estimates below, for $U \in \mathbb{R}^+$, we have

$$u_i(x) \in \mathcal{U}_U \text{ for all } i = 1, \dots, n \text{ and } x \in \mathbb{R}^N \quad \Rightarrow \quad (\vartheta * u)(x) \in \mathcal{U}_U^m \text{ for all } x \in \mathbb{R}^N.$$

As in [1, Formula (5.3)], define

$$A(t, U) = \|\lambda(\cdot, \cdot, U)\|_{\mathbf{L}^1([0, t] \times \mathbb{R}^N; \mathbb{R})}. \quad (6)$$

Fix positive U and R with

$$\|\bar{u}\|_{\mathbf{L}^1(\mathbb{R}^N; \mathbb{R}^n)} < R, \quad \|\bar{u}\|_{\mathbf{L}^\infty(\mathbb{R}^N; \mathbb{R}^n)} < U \quad \text{and} \quad \operatorname{TV}(\bar{u}) < R.$$

Introduce the \mathbf{L}^1 closed ball centered at the initial datum \bar{u} with radius R

$$B_{\mathbf{L}^1}(\bar{u}, R, U) = \left\{ u \in \mathbf{L}^1(\mathbb{R}^N; \mathbb{R}^n) : \|u - \bar{u}\|_{\mathbf{L}^1(\mathbb{R}^N; \mathbb{R}^n)} \leq R \text{ and } \forall x \ u(x) \in \mathcal{U}_U^n \right\}.$$

Throughout, we denote by C a quantity dependent only on λ and on the assumptions (φ) , (Φ) and (ϑ) , but independent of T , R and U . Similarly, C_U is a constant depending only on $\|\varphi\|_{\mathbf{W}^{2, \infty}(I \times \mathbb{R}^N \times \mathcal{U}_U \times \mathcal{U}_U^m; \mathbb{R}^{n \times m})}$ and on $\|\Phi\|_{\mathbf{W}^{1, \infty}(I \times \mathbb{R}^N \times \mathcal{U}_U \times \mathcal{U}_U^m; \mathbb{R}^n)}$.

For any positive $T \in \widehat{I}$, denote $I = [0, T]$ and define the map

$$\begin{aligned} \mathcal{T} : \mathbf{C}^0(I; B_{\mathbf{L}^1}(\bar{u}, R, U)) \times \mathcal{P} &\rightarrow \mathbf{C}^0(I; B_{\mathbf{L}^1}(\bar{u}, R, U)) \\ w, (\varphi, \Phi, \vartheta) &\rightarrow u \end{aligned} \quad (7)$$

where $u \equiv (u_1, \dots, u_n)$ solves $\partial_t u_i + \operatorname{div}_x \varphi_i(t, x, u_i, \vartheta * w) = \Phi_i(t, x, u_i, \vartheta * w)$ as in [1, Formula (5.1)] and the parameter space is $\mathcal{P} = (\mathbf{C}^2 \cap \mathbf{W}^{2, \infty})(\widehat{I} \times \mathbb{R}^N \times \mathcal{U}_{\bar{U}} \times \mathcal{U}_{\bar{U}}^m; \mathbb{R}^{n \times N}) \times (\mathbf{C}^1 \cap \mathbf{W}^{1, \infty})(\widehat{I} \times \mathbb{R}^N \times \mathcal{U}_{\bar{U}} \times \mathcal{U}_{\bar{U}}^m; \mathbb{R}^n) \times \mathbf{C}_c^2(\mathbb{R}^N; \mathbb{R}^{m \times n})$.

The map \mathcal{T} is proved to be a contraction for T small in [1, Theorem 2.2]. Hence, the present proof consists in showing that \mathcal{T} is Lipschitz continuous in φ, Φ, ϑ . We consider the three variables φ, Φ and ϑ separately and apply repeatedly [9, Theorem 2.6], as refined in [10, Theorem 2.5]. The assumptions **(H1*)** and **(H2*)** are verified in [1, Section 5]. We now check **(H3*)**.

In the estimates below we set for simplicity $\bar{u} = 0$.

Step 1: (H3) holds:* Fix an index $i \in \{1, \dots, n\}$, define $\Omega_T^U = I \times \mathbb{R}^n \times \mathcal{U}_U$ and for a fixed $w \in \mathbf{C}^0(I; B_{\mathbf{L}^1}(\bar{u}, R, U))$ denote

$$\begin{aligned} f(t, x, u) &= \varphi_i^1(t, x, u, \vartheta^1 * w(t, x)) & F(t, x, u) &= \Phi_i^1(t, x, u, \vartheta^1 * w(t, x)) \\ g(t, x, u) &= \varphi_i^2(t, x, u, \vartheta^2 * w(t, x)) & G(t, x, u) &= \Phi_i^2(t, x, u, \vartheta^2 * w(t, x)). \end{aligned} \quad (8)$$

Then, we directly have:

$$\partial_u(f - g) \in \mathbf{L}^\infty(\Omega_T^U; \mathbb{R}^N) \text{ holds by } (\varphi) \text{ since } \partial_u(f - g) = \partial_u(\varphi_i^1 - \varphi_i^2).$$

$$\partial_u(F - G) \in \mathbf{L}^\infty(\Omega_T^U; \mathbb{R}^N): \text{ holds by } (\Phi), \text{ since } \partial_u(F - G) = \partial_u(\Phi_i^1 - \Phi_i^2).$$

$\int_0^T \int_{\mathbb{R}^N} \|((F - G)(t, x, \cdot) - (\operatorname{div}_x(f - g))(t, x, \cdot))\|_{\mathbf{L}^\infty([-U, U]; \mathbb{R})} dx dt < +\infty$ holds, due to the inequality $\|((F - G)(t, x, U) - (\operatorname{div}_x(f - g))(t, x, U))\| \leq 4\lambda(t, x, U)$ and (3) applies.

We use below A as a dummy variable for the fourth argument in $\varphi^1, \varphi^2, \Phi^1$ and Φ^2 .

Step 2: Dependence on φ . Assume that $\Phi^1 = \Phi^2$ and $\vartheta^1 = \vartheta^2$ in (8). Then, with reference to the notation in [10, Theorem 2.2] and using also [8, theorem 1], we have

$$\begin{aligned} \kappa_0^* &\leq C C_U (1 + RT) \quad (\text{as in [1, Formula (5.6)])} \\ \kappa^* &\leq \|\partial_u F\|_{\mathbf{L}^\infty(I \times \mathbb{R}^N \times \mathcal{U}_U; \mathbb{R})} + \|\partial_u \operatorname{div}_x(f - g)\|_{\mathbf{L}^\infty(I \times \mathbb{R}^N \times \mathcal{U}_U; \mathbb{R})} \\ &\leq \|\Phi^1\|_{\mathbf{W}^{1, \infty}(I \times \mathbb{R}^N \times \mathcal{U}_U \times \mathcal{U}_U^m; \mathbb{R})} + \|\partial_u \operatorname{div}_x(\varphi^1 - \varphi^2)\|_{\mathbf{L}^\infty(I \times \mathbb{R}^N \times \mathcal{U}_U \times \mathcal{U}_U^m; \mathbb{R})} \\ &\quad + \|\partial_u \operatorname{grad}_A(\varphi^1 - \varphi^2)\|_{\mathbf{L}^\infty(I \times \mathbb{R}^N \times \mathcal{U}_U \times \mathcal{U}_U^m; \mathbb{R}^m)} \|\operatorname{div}_x \vartheta\|_{\mathbf{L}^\infty(\mathbb{R}^N; \mathbb{R}^{m \times n})} \|w\|_{\mathbf{L}^1(I \times \mathbb{R}^N; \mathbb{R}^n)} \\ &\leq C C_U (1 + RT) \\ \frac{e^{\kappa_0^* t} - e^{\kappa^* t}}{\kappa_0^* - \kappa^*} &\leq t e^{\max\{\kappa_0^*, \kappa^*\} t} \quad (\text{by [10, Remark 2.8] and [1, Formula (5.16)])} \\ &\leq t e^{C C_U (1 + RT) t} \end{aligned}$$

and clearly $\|\partial_u(f - g)\|_{\mathbf{L}^\infty(I \times \mathbb{R}^N \times \mathcal{U}_U; \mathbb{R})} = \|\partial_u(\varphi^1 - \varphi^2)\|_{\mathbf{L}^\infty(I \times \mathbb{R}^N \times \mathcal{U}_U \times \mathcal{U}_U^m; \mathbb{R})}$. Moreover,

$$\int_0^T \int_{\mathbb{R}^N} \|\operatorname{grad}_x(F - \operatorname{div}_x f)(t, x, \cdot)\|_{\mathbf{L}^\infty(\mathcal{U}_U; \mathbb{R}^N)} dx dt \leq C_U \Lambda(T, U) (1 + RT + R^2 T^2)$$

the latter expression above is computed in [1, Formula (5.5)]. We also need to estimate

$$\begin{aligned} &\int_0^T \int_{\mathbb{R}^N} \|((F - G) - \operatorname{div}_x(f - g))(t, x, \cdot)\|_{\mathbf{L}^\infty(\mathcal{U}_U; \mathbb{R})} dx dt \\ &= \int_0^T \int_{\mathbb{R}^N} \|\operatorname{div}_x(f - g)(t, x, \cdot)\|_{\mathbf{L}^\infty(\mathcal{U}_U; \mathbb{R})} dx dt \\ &\leq \int_0^T \int_{\mathbb{R}^N} \|\operatorname{div}_x(\varphi^1 - \varphi^2)(t, x, \cdot, (\vartheta * w)(t, x))\|_{\mathbf{L}^\infty(\mathcal{U}_U; \mathbb{R})} dx dt \\ &\quad + \int_0^T \int_{\mathbb{R}^N} \|\operatorname{grad}_A(\varphi^1 - \varphi^2)(t, x, \cdot, (\vartheta * w)(t, x))\|_{\mathbf{L}^\infty(\mathcal{U}_U; \mathbb{R}^{m \times n})} \|\operatorname{div}_x(\vartheta * w)\|_{\mathbf{L}^\infty(I \times \mathbb{R}^N; \mathbb{R}^m)} dx dt \\ &\leq \int_0^T \int_{\mathbb{R}^N} \|\operatorname{div}_x(\varphi^1 - \varphi^2)(t, x, \cdot, \cdot)\|_{\mathbf{L}^\infty(\mathcal{U}_U \times \mathcal{U}_U^m; \mathbb{R})} dx dt \\ &\quad + C_U \int_0^T \int_{\mathbb{R}^N} \|\operatorname{grad}_A(\varphi^1 - \varphi^2)(t, x, \cdot, \cdot)\|_{\mathbf{L}^\infty(\mathcal{U}_U \times \mathcal{U}_U^m; \mathbb{R}^{m \times n})} dx dt \\ &\leq C_U \|\operatorname{grad}_{xA}(\varphi^1 - \varphi^2)\|_{\mathbf{L}^1(I \times \mathbb{R}^N; \mathbf{L}^\infty(\mathcal{U}_U \times \mathcal{U}_U^m; \mathbb{R}^{N+m \times n}))}. \end{aligned}$$

Inserting the estimates above in the one provided by [10, Theorem 2.5], we have:

$$\begin{aligned} &\|u^1(t) - u^2(t)\|_{\mathbf{L}^1(\mathbb{R}^N; \mathbb{R}^n)} \\ &\leq (1 + C C_U \Lambda(T, U) (1 + RT + R^2 T^2)) T \operatorname{TV}(\bar{u}) e^{C C_U (1 + RT) T} \|\partial_u(\varphi^1 - \varphi^2)\|_{\mathbf{L}^\infty(I \times \mathbb{R}^N \times \mathcal{U}_U \times \mathcal{U}_U^m; \mathbb{R})} \\ &\quad + C_U e^{C C_U (1 + RT) T} \|\operatorname{grad}_{xA}(\varphi^1 - \varphi^2)\|_{\mathbf{L}^1(I \times \mathbb{R}^N; \mathbf{L}^\infty(\mathcal{U}_U \times \mathcal{U}_U^m; \mathbb{R}^{N+m \times n}))}. \end{aligned} \tag{9}$$

Step 3: Dependence on Φ . Assume $\varphi^1 = \varphi^2$ and $\vartheta^1 = \vartheta^2$, so that $f = g$ in (8). Then, again with reference to the notation in [10, Theorem 2.5], we have $\kappa^* = \|\partial_u F\|_{\mathbf{L}^\infty(I \times \mathbb{R}^N \times \mathcal{U}_U; \mathbb{R})} \leq C_U$ so that

$$\|u^1(t) - u^2(t)\|_{\mathbf{L}^1(\mathbb{R}^N; \mathbb{R}^n)} \leq C_U T \|\Phi^1 - \Phi^2\|_{\mathbf{L}^1(I \times \mathbb{R}^N; \mathbf{L}^\infty(\mathcal{U}_U \times \mathcal{U}_U^m; \mathbb{R}))}. \tag{10}$$

Step 4: Dependence on ϑ . We are left with the case $\varphi^1 = \varphi^2 = \varphi$ and $\Phi^1 = \Phi^2 = \Phi$. We fix for simplicity the notations $\varphi^{\vartheta^1} = \varphi^1(t, x, u, \vartheta^1 * w(t, x))$, $\varphi^{\vartheta^2} = \varphi^1(t, x, u, \vartheta^2 * w(t, x))$ and $\Phi^{\vartheta^1} = \Phi^1(t, x, u, \vartheta^1 * w(t, x))$, $\Phi^{\vartheta^2} = \Phi^1(t, x, u, \vartheta^2 * w(t, x))$. Then, always with reference to [10],

$$\begin{aligned} \kappa^* &\leq \|\partial_u F\|_{\mathbf{L}^\infty(I \times \mathbb{R}^N \times \mathcal{U}_U; \mathbb{R})} + \|\partial_u \operatorname{div}_x (f - g)\|_{\mathbf{L}^\infty(I \times \mathbb{R}^N \times \mathcal{U}_U; \mathbb{R})} \\ &\leq \|\Phi^1\|_{\mathbf{W}^{1,\infty}(I \times \mathbb{R}^N \times \mathcal{U}_U \times \mathcal{U}_U^m; \mathbb{R})} + \left\| \partial_u \operatorname{div}_x \left(\varphi^{\vartheta^1} - \varphi^{\vartheta^2} \right) \right\|_{\mathbf{L}^\infty(I \times \mathbb{R}^N \times \mathcal{U}_U \times \mathcal{U}_U^m; \mathbb{R})} \\ &\quad + \left\| \partial_u \operatorname{grad}_A \varphi^{\vartheta^1} \right\|_{\mathbf{L}^\infty(I \times \mathbb{R}^N \times \mathcal{U}_U \times \mathcal{U}_U^m; \mathbb{R}^{m \times n})} \left\| \operatorname{div}_x (\vartheta^1 * w) \right\|_{\mathbf{L}^\infty(I \times \mathbb{R}^N; \mathbb{R}^m)} \\ &\quad + \left\| \partial_u \operatorname{grad}_A \varphi^{\vartheta^2} \right\|_{\mathbf{L}^\infty(I \times \mathbb{R}^N \times \mathcal{U}_U \times \mathcal{U}_U^m; \mathbb{R}^{m \times n})} \left\| \operatorname{div}_x (\vartheta^2 * w) \right\|_{\mathbf{L}^\infty(I \times \mathbb{R}^N; \mathbb{R}^m)} \\ &\leq C C_U (1 + RT), \end{aligned}$$

and $\kappa_0^* \leq C C_U (1 + RT)$. Similarly to the previous case,

$$\|\partial_u (f - g)\|_{\mathbf{L}^\infty(I \times \mathbb{R}^N \times \mathcal{U}_U; \mathbb{R})} \leq R \|\varphi\|_{\mathbf{W}^{1,\infty}(I \times \mathbb{R}^n \times \mathcal{U}_U \times \mathcal{U}_U^m; \mathbb{R})} \|\vartheta^1 - \vartheta^2\|_{\mathbf{L}^1(I \times \mathbb{R}^N; \mathbb{R}^{m \times n})}.$$

We also need to estimate

$$\begin{aligned} &\int_0^T \int_{\mathbb{R}^N} \|((F - G) - \operatorname{div}_x (f - g))\|_{\mathbf{L}^\infty(\mathcal{U}_U; \mathbb{R})} \, dx \, dt \\ &\leq C_U R \left(\|\Phi\|_{\mathbf{W}^{1,\infty}(I \times \mathbb{R}^N \times \mathcal{U}_U; \mathbb{R})} + \|\varphi\|_{\mathbf{W}^{2,\infty}(I \times \mathbb{R}^N \times \mathcal{U}_U; \mathbb{R})} \right) \|\vartheta^1 - \vartheta^2\|_{\mathbf{L}^1(I \times \mathbb{R}^N; \mathbb{R}^{m \times n})}, \end{aligned}$$

so that, following the same procedure used in Step 2,

$$\begin{aligned} \|u^1(t) - u^2(t)\|_{\mathbf{L}^1(\mathbb{R}^N; \mathbb{R})} &\leq (1 + C\Lambda(T, U)(1 + RT + R^2T^2)) C_U RT \operatorname{TV}(\bar{u}) e^{C C_U (1+RT)T} \\ &\quad \times \|\vartheta^1 - \vartheta^2\|_{\mathbf{L}^1(I \times \mathbb{R}^N; \mathbb{R}^{m \times n})} \\ &\quad + C_U e^{C C_U (1+RT)T} R \|\vartheta^1 - \vartheta^2\|_{\mathbf{L}^1(I \times \mathbb{R}^N; \mathbb{R}^{m \times n})}. \end{aligned} \tag{11}$$

Summing up the estimates (9), (10) and (11) we obtain the Lipschitz continuous dependence of \mathcal{T} defined in (7) on the parameter $p \equiv (\varphi, \Phi, \vartheta)$. A straightforward argument allows to conclude that also the fixed point of \mathcal{T} is Lipschitz continuous in p . \square

3. Application to the cutting of metal

The cutting of metal plates by means of a laser beam can be described through the following equations, introduced in [1, Section 3]:

$$\begin{cases} \partial_t h_m + \operatorname{div}_x (h_m V) = \mathcal{L} \\ \partial_t h_s = -\mathcal{L}. \end{cases} \quad (t, x) \in \mathbb{R}^+ \times \mathbb{R}^2. \tag{12}$$

In this 3D framework, the laser beam is parallel to the vertical z axis and its trajectory is prescribed by the map $x_L = x_L(t)$. Above, h_m is the height of the melted metal and h_s is the height of solid part, both measured along the z axis. The vector V is the projection of the melted material velocity on the horizontal (x, y) -plane. The source \mathcal{L} describes the laser position and intensity: it describes the net rate at which the solid part turns into melted. In particular,

$$\begin{aligned} V &= (w(t, x) - \tau_g(t, x)h_m) \frac{-\operatorname{grad}_x (\eta * h_s)}{\sqrt{1 + \|\operatorname{grad}_x (\eta * h_s)\|^2}}, & w(t, x) &= \mathcal{W}(\|x - x_L(t)\|), \\ \mathcal{L} &= \frac{i(t, x)}{1 + \|\operatorname{grad}_x (\eta * (h_s + h_m))\|^2}, & \tau_g(t, x) &= \mathcal{T}_g(\|x - x_L(t)\|), \\ & & i(t, x) &= \mathcal{I}(\|x - x_L(t)\|), \end{aligned} \tag{13}$$

where $\mathcal{T}_g = \mathcal{T}_g(x)$ is related to the shear stress, see [1, Formula (3.7)] and [11]; $\mathcal{W} = \mathcal{W}(x)$ describes the effect on the horizontal (x, y) -plane of the vertical wind that pushes the melted material and is produced around the laser beam at $x = x_L(t)$; the laser intensity is $\mathcal{I} = \mathcal{I}(x)$, again centered at the moving laser position $x = x_L(t)$; the convolution kernel η is chosen smooth, compactly supported and with total mass 1. The system in (12) fits into (1) as shown by the following proposition.

Proposition 3.1 ([1, Proposition 3.1]). *Model (12)–(13) fits into (1) setting $u \equiv (h_m, h_s)$ and*

$$\begin{aligned}
 N &= 2 & \vartheta(x) &= \begin{bmatrix} \partial_{x_1} \eta(x) & 0 \\ 0 & \partial_{x_1} \eta(x) \\ \partial_{x_2} \eta(x) & 0 \\ 0 & \partial_{x_2} \eta(x) \end{bmatrix} & \varphi_1(t, x, u_1, A) &= \frac{-(w(t, x) - \tau_g(t, x)u_1) u_1}{\sqrt{1 + (A_3)^2 + (A_4)^2}} \begin{bmatrix} A_3 \\ A_4 \end{bmatrix} \\
 n &= 2 & & & \varphi_2(t, x, u_2, A) &= 0 \\
 m &= 4 & & & \Phi_1(t, x, u, A) &= \frac{i(t, x)}{\sqrt{1 + (A_1 + A_3)^2 + (A_2 + A_4)^2}} \\
 u_1 &= h_m & & & \Phi_2(t, x, u, A) &= -\frac{i(t, x)}{\sqrt{1 + (A_1 + A_3)^2 + (A_2 + A_4)^2}}. \\
 u_2 &= h_s & & & &
 \end{aligned}$$

Moreover, if $x_L \in (\mathbf{C}^2 \cap \mathbf{W}^{2,\infty})([0, \hat{T}]; \mathbb{R}^2)$, $\mathcal{W}, \mathcal{I}, \mathcal{T}_g \in \mathbf{C}_c^2(\mathbb{R}; \mathbb{R})$ and $\eta \in \mathbf{C}_c^3(\mathbb{R}^2; \mathbb{R})$ for a positive \hat{T} , then, assumptions (φ) , (Φ) , (ϑ) and (3) hold.

(The verification of (3) is immediate and hence omitted.)

Theorem 2.3 ensures the stability of solutions to (12)–(13) with respect to variations in the functions defining the model, namely: $x_L, \mathcal{T}_g, \mathcal{W}, \mathcal{I}$ and η . A numerical investigation of the role of \dot{x}_L is presented in [3]. Here, on the basis of Theorem 2.3, we can specify how the solutions to (12)–(13) depend on \mathcal{W} and \mathcal{T}_g . Indeed, the bound (4) ensures that, calling (h_m^ℓ, h_s^ℓ) for $\ell = 1, 2$ solutions to (12)–(13) corresponding to functions $\mathcal{W}^\ell, \mathcal{T}_g^\ell$

$$\left\| \begin{pmatrix} h_m^1 - h_m^2 \\ h_s^1 - h_s^2 \end{pmatrix} (t) \right\|_{\mathbf{L}^1(\mathbb{R}^2; \mathbb{R})} \leq C^* \left[\|\mathcal{W}^1 - \mathcal{W}^2\|_{\mathbf{W}^{1,\infty}(\mathbb{R}; \mathbb{R})} + \|\mathcal{T}_g^1 - \mathcal{T}_g^2\|_{\mathbf{W}^{1,\infty}(\mathbb{R}; \mathbb{R})} \right].$$

The constant C^* , given in Theorem 2.3, grows exponentially in C, C_U, R and T as in the estimates (9), (10) and (11).

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References

[1] R.M. Colombo, F. Marcellini, Nonlocal systems of balance laws in several space dimensions with applications to laser technology, *J. Differential Equations* 259 (11) (2015) 6749–6773.
 [2] R.M. Colombo, G. Guerra, M. Herty, F. Marcellini, A hyperbolic model for the laser cutting process, *Appl. Math. Model.* 37 (14–15) (2013) 7810–7821.
 [3] R.M. Colombo, F. Marcellini, E. Rossi, Biological and industrial models motivating nonlocal conservation laws: a review of analytic and numerical results, *Netw. Heterog. Media* (11) (2016) 49–67.
 [4] S. Göttlich, S. Hoher, P. Schindler, V. Schleper, A. Verl, Modeling, simulation and validation of material flow on conveyor belts, *Appl. Math. Model.* 38 (13) (2014) 3295–3313.
 [5] R.M. Colombo, M. Garavello, M. Lécureux-Mercier, A class of nonlocal models for pedestrian traffic, *Math. Models Methods Appl. Sci.* 22 (4) (2012) 1150023, 34.

- [6] M. Di Francesco, P.A. Markowich, J.-F. Pietschmann, M.-T. Wolfram, On the Hughes' model for pedestrian flow: the one-dimensional case, *J. Differential Equations* 250 (3) (2011) 1334–1362.
- [7] R.L. Hughes, A continuum theory for the flow of pedestrians, *Transp. Res. B* 36 (2002) 507–535.
- [8] S.N. Kruzhkov, First order quasilinear equations with several independent variables, *Mat. Sb. (N.S.)* 81 (123) (1970) 228–255.
- [9] R.M. Colombo, M. Mercier, M.D. Rosini, Stability and total variation estimates on general scalar balance laws, *Commun. Math. Sci.* 7 (1) (2009) 37–65.
- [10] M. Lécureux-Mercier, Improved stability estimates on general scalar balance laws, *ArXiv e-prints*, <https://arxiv.org/pdf/1010.5116v2.pdf>, Oct. 2010, last revised Jul 2013.
- [11] G. Vossen, J. Schüttler, Mathematical modelling and stability analysis for laser cutting, *Math. Comput. Modelling Dyn. Syst.* 18 (4) (2012) 439–463.