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A traffic model aware of real time data

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Nowadays, traffic monitoring systems have access to real time data, e.g. through GPS devices. We propose a new traffic model able to take into account these data and, hence, able to describe the effects of unpredictable accidents. The well-posedness of this model is proved and numerical integrations show qualitative features of the resulting solutions. As a further motivation for the use of real time data, we show that the inverse problem for the Lighthill–Whitham and Richards (LWR) model is ill-posed.

Keywords: Macroscopic traffic models; hyperbolic systems of conservation laws.

AMS Subject Classification: 35L65, 90B20

1. Introduction

Differently from fluid dynamics, traffic dynamics does not rely on well-established fundamental principles like the conservation of momentum or energy. Apart from the conservation of the total number of vehicles, the many traffic models available in the literature have to rely on assumptions on the drivers' behavior and these assumptions always contain some arbitrariness.

On the other hand, present-day measurement devices allow a detailed knowledge of the traffic situation essentially in real time. This leads to the possibility of improving models by means of real time data. Here, we propose a model aware of real time data or, in other words, that encodes these data. We stress that no deterministic model whatsoever can predict the insurgence of an accident. On the

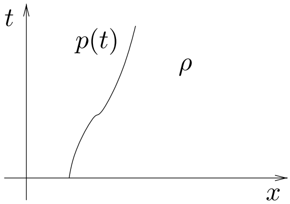

Fig. 1. Situation considered by problem (1.1). The trajectory $p = p(t)$ is measured, while the density $\rho = \rho(t, x)$ solves (1.2).

other hand, the present model is able to take into account such an event and to describe its effects.

In the current literature, three different approaches are mainly used to model traffic phenomena: microscopic, macroscopic and kinetic. For an overview of vehicular traffic models at all scales, we refer to Refs. 4, 5, 7, 12 and 21. Here, we are concerned with *macroscopic* models, where traffic is described through the fraction $\rho = \rho(t, x)$ of space occupied by vehicles at time t and at position x.

From an analytic point of view, a justification of our insertion of real time traffic data in the very formulation of the model is provided by the difficulties inherent to the solution of the inverse problem for a 1D scalar conservation law. Indeed, a rigorous approach to the issue of finding the "*right*" speed law $\mathcal{V} = v(\rho)$ leads to an inverse problem that can be stated as follows. Find the function $\mathcal{V} = v(\rho)$ so that the solution to (1.2) best approximates the observed traffic dynamics. More formally, we are led to consider the inverse problem, (see Fig. 1):

$$\begin{gathered} \text{find } v \text{ so that } \int_0^T |\dot{p}(t) - v(\rho(t, p(t)+)| \, \mathrm{d}t \text{ is minimal,} \\ \text{where } \begin{cases} \partial_t \rho + \partial_x (\rho v(\rho)) = 0, \\ \rho(0, x) = \rho_0(x). \end{cases} \end{gathered} \tag{1.1}$$

By means of an example, we show below that problem (1.1) is in general ill-posed. Moreover, a positive result in this direction is obtained, see Proposition 2.1, but it relies on assumptions that can be hardly acceptable in a real situation.

The present macroscopic model consists of a conservation law of the type,

$$\partial_t \rho + \partial_x (\rho \mathcal{V}(t, x, \rho)) = 0, \tag{1.2}$$

where a measured trajectory $p = p(t)$ is encoded in the time and space-dependent speed law $\mathcal{V} = \mathcal{V}(t, x, \rho)$. To define it, introduce first a smooth, non-negative function $\chi = \chi(\xi)$ attaining the value 1 for $|\xi| \leq \ell$ and vanishing when $|\xi| \geq L$, for two fixed constants ℓ, L, with $\ell < L$. *Far* from the measuring vehicle, i.e. for $|x - p(t)| > L$, the speed law $\mathcal{V}$ coincides with any speed law $v = v(\rho)$ that can be used in the framework of the LWR model. Typically, v is Lipschitz continuous, non-increasing and vanishes at the maximal density, which corresponds to a bumper-to-bumper standing queue. *Near* to the measuring vehicle, i.e. for $|x - p(t)| < \ell$, the speed law $\mathcal{V}$ is the harmonic mean of the measured speed $\dot{p}(t)$ and of the LWR speed $v(\rho(t,x))$. Therefore,

$$\mathcal{V}(t,x,\rho) = \chi(x-p(t))\frac{2\dot{p}(t)v(\rho)}{\dot{p}(t)+v(\rho)} + (1-\chi(x-p(t)))v(\rho). \tag{1.3}$$

The choice of the harmonic mean is due to our aim of providing good descriptions in particular in case of queues. Indeed, when the measuring vehicle stops, i.e. $\dot{p}(t)=0$, the speed $\mathcal{V}(t,x,\rho(t,x))$ vanishes for $|x-p(t)|<\ell$ and the presence of a standing queue revealed by the measuring vehicle is encompassed in (1.2)–(1.3). It is also immediate to see that $\mathcal{V}$ inherits reasonable properties of v and $\dot{p}$ such as, for instance, positivity and boundedness.

Below, we prove existence, uniqueness and Lipschitz continuous dependence from initial data of the solutions to the Cauchy problem for (1.2)–(1.3). By means of a few numerical integrations, we show below qualitative properties of the solutions to (1.2)–(1.3). Remark that the extension to the case of several measured trajectories is of a merely technical nature, both at the analytic and at the numeric levels. Related results on models based on mixed microscopic–macroscopic descriptions are found in Refs. 10, 14, 18 and 17.

As a further remark, we observe that the use of real time data makes the model "*less falsifiable*". On the other hand, we gain the possibility of describing the effects of unpredictable accidents. The current mathematical and engineering literatures show an increasing interest in this direction, see also Refs. 1, 3, 8 and 23.

The paper is organized as follows: Sec. 2 is devoted to the inverse problem for a standard LWR[16,22] model. In Sec. 3, we study the Cauchy problem for system (1.3) and in Sec. 4, we present some numerical integrations of this model. All proofs are gathered in the last section.

Throughout, we denote $\mathbb{R}^+ = [0,+\infty[$ and $\mathring{\mathbb{R}}^+ =]0,+\infty[$. A Lipschitz constant of the map f is denoted $\mathbf{Lip}(f)$. The maximal density (or occupancy), i.e. the density at which vehicles are bumper-to-bumper and cannot move, is normalized to 1.

2. On an Inverse Problem for the LWR Model

As a first step, we show that in general a solution to the inverse problem (1.1) may fail to exist. Indeed, for $\varepsilon \in [-1, 1]$, consider the speed law and the flow

$$v_\varepsilon(\rho) = (1+\varepsilon\rho)(1-\rho) \quad \text{and} \quad f_\varepsilon(\rho) = (1+\varepsilon\rho)(1-\rho)\rho \tag{2.1}$$

and choose the trajectory $p(t)=t/2$. Note that $f''_\varepsilon(\rho)<0$ for all $\rho\in[0,1]$ and $\varepsilon\in[-1/3,1/3]$. For simplicity, we consider the Riemann problem,

$$\begin{cases} \partial_t\rho + \partial_x(\rho\, v_\varepsilon(\rho)) = 0, \\ \rho(0,x) = \begin{cases} 1/8 & x<0, \\ 3/8 & x>0, \end{cases} \end{cases} \tag{2.2}$$

whose solution is depicted in Fig. 2 in the two cases $\varepsilon<0$ and $\varepsilon>0$.

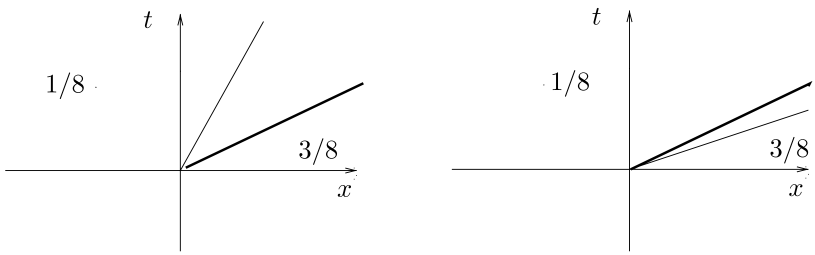

Fig. 2. Solution to the Riemann problem (2.2) with v_ε and f_ε as in (2.1). Left, the case $\varepsilon < 0$ and, right, $\varepsilon > 0$.

The discrepancy between measured data and the description provided by the LWR model can be estimated through straightforward computations as follows:

$$\int_0^T |\dot{p}(t) - v_\varepsilon(\rho_\varepsilon(t, p(t)))| \, \mathrm{d}t = \begin{cases} \left(\dfrac{9}{8} + \dfrac{15\varepsilon}{64}\right) T & \text{if } \varepsilon \le 0, \\ \left(\dfrac{3}{8} + \dfrac{7\varepsilon}{64}\right) T & \text{if } \varepsilon > 0. \end{cases}$$

This shows that the map:

$$\begin{aligned} \varphi \colon [-1/3,\, 1/3] &\to \mathbb{R} \\ \varepsilon &\to \int_0^T |\dot{p}(t) - v_\varepsilon(\rho_\varepsilon(t, p(t)+))| \, \mathrm{d}t \end{aligned} \tag{2.3}$$

does not attain a minimum, see Fig. 3. Clearly, this example can be easily extended to more general, non-constant, functions p and to more general Cauchy initial data.

Due to the example above, one is led to consider problem (1.1) in a specific class of speed laws. A positive result is available in the class of speed laws $v_{_V}(\rho) = V(1-\rho)$ with $V \in [\check{V}, \hat{V}]$ being the maximal speed of cars along the considered road. Clearly, we assume throughout that $\hat{V} > \check{V} > 0$.

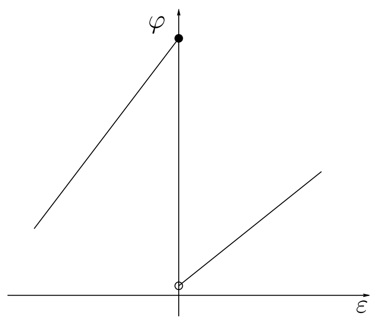

Fig. 3. The map φ defined in (2.3) does not attain a minimum for $\varepsilon \in [-1/3,\, 1/3]$. Above, $\varphi(0+) = 3/8$ and $\varphi(0-) = 9/8$.

The following lemma plays a key role to obtain the basic continuity estimate on the dependence of the error functional $\gamma \to \int_0^T |\dot{p}(t) - v(\rho(t,\gamma(t)))|\, dt$ from a generic (non-characteristic) curve $\gamma = \gamma(t)$.

Lemma 2.1. *Fix $T > 0$, $f \in \mathbf{C^1}(\mathbb{R};\mathbb{R})$ and $\rho_o \in (\mathbf{L^1} \cap \mathbf{BV})(\mathbb{R};[0,1])$. Call ρ the solution to*

$$\begin{cases} \partial_t \rho + \partial_x f(\rho) = 0, \\ \rho(0,x) = \rho_o(x). \end{cases} \tag{2.4}$$

Choose two curves $\gamma_1, \gamma_2 \in \mathbf{W^{1,\infty}}([0,T];\mathbb{R})$, non-characteristic in the sense that there exists a $c > 0$ such that

$$\dot{\gamma}_i(t) > f'(\rho(t,\gamma_i(t))) + c \quad \textit{for all } t \in [0,T] \quad \textit{and} \quad i = 1,2.$$

Then,

$$\int_0^T |\rho(t,\gamma_1(t)) - \rho(t,\gamma_2(t))|\, dt \leq \frac{1}{c} \mathrm{TV}(\rho_o) \|\gamma_1 - \gamma_2\|_{\mathbf{C^0}([0,T];\mathbb{R})}.$$

The proof is deferred to Sec. 5.

We are now ready to prove the continuity result that implies the existence of a solution to the inverse problem (1.1).

Proposition 2.1. *Let $T > 0$, $\hat{V}, \check{V}$ be such that $\hat{V} > \check{V} > 0$. Fix $\check{\rho} \in \,]0,1[$. If the initial datum $\rho_o \in (\mathbf{L^1} \cap \mathbf{BV})(\mathbb{R};[0,1])$ and the path $p \in \mathbf{W^{1,\infty}}$ are such that*

$$\operatorname*{ess\,inf}_{x\in\mathbb{R}} \rho_o > \check{\rho} \quad \textit{and} \quad \operatorname*{ess\,inf}_{t\in[0,T]} \dot{p} \geq \hat{V}(1 - 2\check{\rho}), \tag{2.5}$$

then the map

$$\begin{aligned} \mathcal{E} \colon [\check{V}, \hat{V}] &\to \mathbb{R}, \\ V &\to \int_0^T \left|\dot{p}(t) - v_V\left(\rho_V(t,p(t))\right)\right| dt \quad \textit{where} \\ &\qquad \begin{cases} \partial_t \rho_V + \partial_x \left[\rho_V V \left(1 - \rho_V\right)\right] = 0 \\ \rho_V(0,x) = \rho_o(x) \end{cases} \end{aligned}$$

is continuous.

The proof is deferred to Sec. 5.

The existence of solution to the inverse problem (1.1) now follows through a standard Weierstraß argument. Note however that the two conditions in (2.5) do not agree with the needs of a real application of this result. The former inequality requires the vehicular density to be *high*. At the same time, the latter inequality in (2.5) imposes that the measured speed be greater than $\hat{V}(1 - \check{\rho})$ uniformly in ρ, for $\rho \in [\check{\rho}, 1]$. These two conditions are somewhat contradictory to expecting that $\dot{p}$ is well approximated by $V(1-\rho)$ with $V \in [\check{V}, \hat{V}]$.

3. A Traffic Model Encoding Real Time Data

This section is devoted to the Cauchy problem for (1.3), more precisely,

$$\begin{cases} \partial_t \rho + \partial_x f(t,x,\rho) = 0, \\ \rho(0,x) = \rho_o(x), \end{cases} \tag{3.1}$$

where $\rho_o \in \mathbf{L}^\infty(\mathbb{R};[0,1])$ and

$$f(t,x,\rho) = \rho\,\mathcal{V}(t,x,\rho),$$
$$\mathcal{V}(t,x,\rho) = \begin{cases} \chi(x-p(t))\dfrac{2\dot p(t)v(\rho)}{\dot p(t)+v(\rho)} + [1-\chi(x-p(t))]v(\rho), & (\dot p(t),v(\rho)) \neq (0,0), \\ 0 & (\dot p(t),v(\rho)) = (0,0). \end{cases} \tag{3.2}$$

We posit below the following assumptions on the functions appearing in (3.1)–(3.2). Throughout, the maximal speed V is a fixed positive constant.

(v) $v \in \mathbf{C^{0,1}}([0,1];[0,V])$ is such that:

$$\begin{cases} v(1) = 0 & \\ \rho \to v(\rho) & \text{is non-increasing,} \\ \rho \to \rho v(\rho) & \text{is strictly concave,} \\ \rho \to \dfrac{\rho w v(\rho)}{w+v(\rho)} & \text{is strictly concave, } \forall\, w > 0. \end{cases}$$

(p) $p \in \mathbf{C^{1,1}}(\mathbb{R}^+;\mathbb{R})$ is such that $\dot p \geq 0$ for a.e. $t \in \mathbb{R}^+$.
($\boldsymbol{\chi}$) $\chi \in \mathbf{C^1_c}(\mathbb{R}^+;[0,1])$.

Clearly, the usual choice $v(\rho) = V(1-\rho)$, for $V > 0$, satisfies **(v)**. Moreover, as soon as $v \in \mathbf{C^2}$, the requirement that the map $\rho \to \frac{\rho w v(\rho)}{w+v(\rho)}$ be strictly concave follows from the other two requirements on the function v, see Lemma 5.1 in Sec. 5. Condition **(p)** simply states that the acceleration of the measured trajectory is bounded and the speed has a definite sign. The interpolating function χ needs only to be sufficiently regular and to attain values in $[0,1]$, so that $\mathcal{V}$ varies smoothly between $v(\rho)$, far from $p(t)$, and the harmonic mean between $v(\rho)$ and $\dot p$, near to $p(t)$.

The current literature, e.g. Refs. 2, 11, 13, 19 and 20, offers different definitions of solution to (3.1)–(3.2). Following Refs. 11 and 13, we first recall the classical Kružkov definition.

Definition 3.1. (Definition 1 in Ref. 13) Let $\rho_o \in \mathbf{L}^\infty(\mathbb{R};[0,1])$. A map $\rho \in \mathbf{L}^\infty(\mathbb{R}^+ \times \mathbb{R};[0,1])$ is a *Kružkov solution* to (3.1) if for any $k \in \mathbb{R}$ and for any

$\varphi \in \mathbf{C_c^\infty}(\mathring{\mathbb{R}}^+ \times \mathbb{R}; \mathbb{R}^+)$,

$$\begin{aligned}\int_{\mathbb{R}^+}\int_{\mathbb{R}} [|\rho(t,x) - k|\, \partial_t \varphi(t,x) \\ + \operatorname{sgn}(\rho(t,x) - k)(f(t,x,\rho(t,x)) - f(t,x,k))\partial_x \varphi(t,x) \\ - \operatorname{sgn}(\rho(t,x) - k)\partial_x f(t,x,k)\varphi(t,x)]\, \mathrm{d}x\, \mathrm{d}t \geq 0 \end{aligned} \tag{3.3}$$

and there exists a set $\mathcal{E} \subset \mathbb{R}$ of zero Lebesgue measure such that

$$\lim_{t \to 0+, t \in [0,T] \setminus \mathcal{E}} \int_{\mathbb{R}} |\rho(t,x) - \rho_o(x)|\, \mathrm{d}x = 0. \tag{3.4}$$

The weaker concept of solution proposed by Panov is of use in the proofs below.

Definition 3.2. (Definition 3 in Ref. 20) For any $k \in \mathbb{R}$, call μ_c^k, respectively μ_s^k, the continuous, respectively singular, part of the distributional derivative $(t,x) \to \partial_x f(t,x,k)$ of the map $(t,x) \to f(t,x,k)$. A map $\rho \in \mathbf{L}^\infty(\mathbb{R}^+ \times \mathbb{R}; [0,1])$ is a *Panov solution* to (3.1)–(3.2) if for any $k \in \mathbb{R}$ and for any $\varphi \in \mathbf{C_c^\infty}(\mathbb{R}^+ \times \mathbb{R}; \mathbb{R}^+)$,

$$\begin{aligned}\int_{\mathbb{R}^+}\int_{\mathbb{R}} [|\rho(t,x) - k|\partial_t \varphi(t,x) \\ + \operatorname{sgn}(\rho(t,x) - k)(f(t,x,\rho(t,x)) - f(t,x,k))\partial_x \varphi(t,x)]\, \mathrm{d}x\, \mathrm{d}t \\ - \int_{\mathbb{R}^+}\int_{\mathbb{R}} \operatorname{sgn}(\rho(t,x) - k)\varphi(t,x)\mathrm{d}\mu_c^k(t,x) + \int_{\mathbb{R}^+}\int_{\mathbb{R}} \varphi(t,x)\mathrm{d}\mu_s^k(t,x) \\ + \int_{\mathbb{R}^+} |\rho_o(x) - k|\varphi(0,x)\mathrm{d}x \geq 0. \end{aligned} \tag{3.5}$$

The well-posedness of (3.1) is obtained through the following propositions, whose proofs are detailed in Sec. 5. First, the existence of Panov solutions is obtained.

Proposition 3.1. *Let* **(v)**, **(p)** *and* **(χ)** *hold. Let* $\rho_o \in \mathbf{L}^\infty(\mathbb{R}; [0,1])$. *Then, Theorem* 2 *in Ref.* 20 *applies, so that problem* (3.1)–(3.2) *admits a Panov solution in the sense of Definition* 3.2.

Now, we verify that in (3.1), Panov solutions are also Kružkov solutions.

Proposition 3.2. *Let* **(v)**, **(p)** *and* **(χ)** *hold. Let* $\rho_o \in \mathbf{L}^\infty(\mathbb{R}; [0,1])$. *If* ρ *is a Panov solution to* (3.2)–(3.2) *in the sense of Definition* 3.2, *then* ρ *is also a Kružkov solution to* (3.1)–(3.2), *in the sense of Definition* 3.1.

We are thus left with the task of proving the uniqueness of Kružkov solution and its stability with respect to the initial datum. Note that Theorem 1.1 in Ref. 11 almost applies to the present case, see Lemma 5.4 for details.

Proposition 3.3. *Let* **(v)**, **(p)** *and* **(χ)** *hold. Let* $\rho_o \in \mathbf{L}^\infty(\mathbb{R}; [0,1])$. *Then, problem* (3.1)–(3.2) *admits at most a unique Kružkov solution in the sense of Definition* 3.1. *Moreover, if* $\rho'_o \in \mathbf{L}^\infty(\mathbb{R}; [0,1])$ *is another initial datum and* $\rho' = \rho'(t,x)$ *is* *the corresponding solution, the following Lipschitz estimate holds*:

$$\|\rho'(t) - \rho(t)\|_{\mathbf{L}^1(\mathbb{R};\mathbb{R})} \leq e^{Ct}\|\rho'_o - \rho_o\|_{\mathbf{L}^1(\mathbb{R};\mathbb{R})}, \tag{3.6}$$

where C *is defined in* (5.8), *independent from* ρ_o *and* ρ'_o.

Together, the above propositions give our main result.

Theorem 3.1. *Let* **(v)**, **(p)** *and* **(χ)** *hold. Then, for any* $\rho_o \in \mathbf{L}^\infty(\mathbb{R};[0,1])$, *problem* (3.1)–(3.2) *admits a unique Kružkov solution in the sense of Definition* 3.1 *and the Lipschitz continuity estimate* (3.6) *holds.*

The proof directly follows from Propositions 3.1–3.3.

4. Numerical Integrations

To numerically integrate (3.1)–(3.2) we use the standard Lax–Friedrichs algorithm, see Sec. 12.1 in Ref. 15. Throughout, we use a fixed space mesh size $\Delta x = 2.5 \times 10^{-3}$.

As a first example, in Fig. 4, we choose the speed law

$$v(\rho) = 1 - \rho \tag{4.1}$$

and the constant initial datum

$$\rho_o(x) = 0.5. \tag{4.2}$$

Both at the analytic and at the numeric levels, the extension of (3.1)–(3.2) to more than one measuring vehicle is immediate. Here, we consider that two cars p and q, which we imagine equipped with a GPS measuring device, follow trajectories with the same speed but exiting different initial positions, say

$$\dot{p}(t) = \dot{q}(t) = 0.5\boldsymbol{\chi}_{[0,5[}(t) + 0.6\boldsymbol{\chi}_{[5,6]}(t) + 0.2\boldsymbol{\chi}_{[8,11]}(t) + 0.4\boldsymbol{\chi}_{[13,18]}(t), \quad \begin{matrix} p(0) = 0, \\ q(0) = 2. \end{matrix} \tag{4.3}$$

In the time interval $[0,5]$, the speeds of p and q equal that resulting from the LWR model at the initial density (4.2). Therefore, the two measuring cars have no effect whatsoever on the evolution prescribed by the partial differential equation, see Fig. 4.

The model allows to observe queues unpredictable for the LWR model, thanks to the (supposedly) real time data provided by p and q. Behind the measuring cars, the maximal density is reached, while in front of them the road empties. Later, the jams disappear. Between the two cars, the queue behind q interacts with the rarefaction formed in front of p.

As a second example, we choose the non-constant initial datum

$$\rho_o(x) = \boldsymbol{\chi}_{[0.001,0.1]}(x) + \boldsymbol{\chi}_{[0.2,0.4]}(x) + \boldsymbol{\chi}_{[0.5,0.8]}(x), \tag{4.4}$$

for (3.1) and assign to the unique measuring vehicle p the speed resulting from the speed law (4.1), see Fig. 5. The evolution prescribed by (1.3) is the same as the one provided by the LWR model.

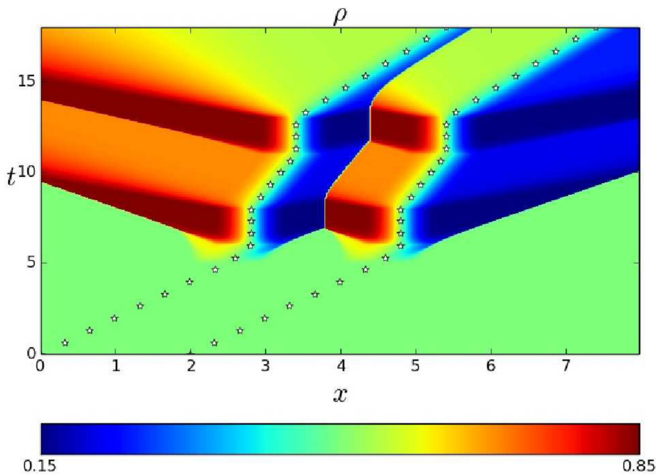

Fig. 4. Numerical integration of the model (3.1)–(3.2)–(4.2)–(4.3). For $t \in [0, 5]$, the knowledge of the trajectories of p and q has no effects on the LWR description. For $t > 5$, the effects of the jams revealed by the two cars are evident.

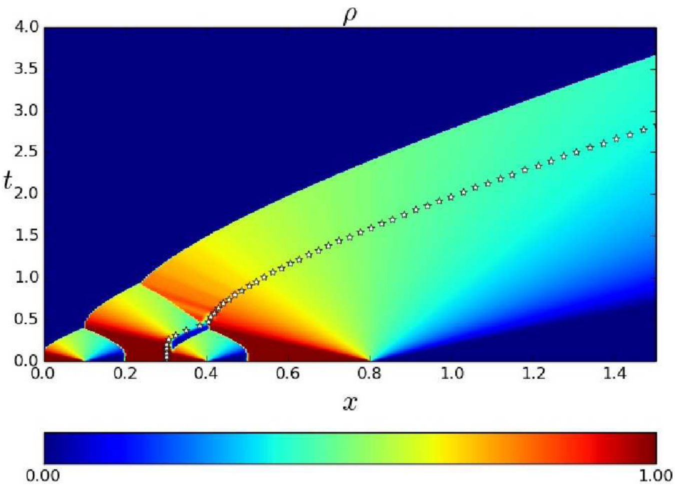

Fig. 5. Numerical integration of (3.1)–(3.2)–(4.4) with $\dot{p} = v(\rho(t, p(t)+))$, resulting in the usual LWR evolution.

Assume now that at time $t = 2.0$ the measuring car stops, due for instance to some sort of accident. Then, Eq. (1.3) displays the formation of a standing queue, see Fig. 6.

A slightly different situation is in Fig. 7. Here, the measuring vehicle travels according to $\dot{p} = v(\rho(t, p(t)+))$ for $t \in [0, 0.75]$. Then, the GPS data show the presence of a standing queue at the location of p during the time interval $[0.75, 1.50]$. For $t > 1.50$, the measuring vehicle travels again coherently with the prescription of

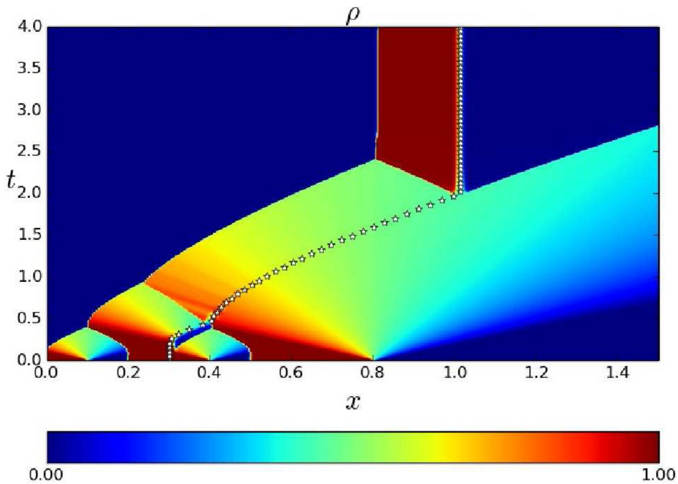

Fig. 6. Here, the experimental data are very different from the predictions of the LWR model. The measuring vehicle p moves according to $\dot{p} = v(\rho(t, p(t)+))$ for $t \in [0, 2.0]$ and stops at $t = 2.0$ due to, say, an accident. The model (3.1)–(3.2)–(4.4) is able to describe the resulting queue.

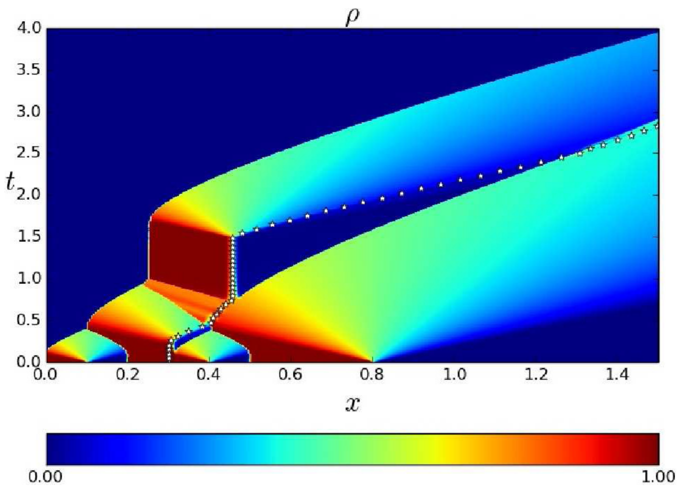

Fig. 7. The measuring vehicle p moves according to $\dot{p} = v(\rho(t, p(t)+))$ for $t \in [0, 0.75]$ and stops for $t \in [0.75, 1.50]$.

the LWR model: first at the maximal speed and then, after about $t \approx 2.50$, slowing down due to its reaching the vehicles in front.

The last two examples displayed in Figs. 6 and 7, when compared with Fig. 5, show the dramatic changes in the description of traffic due to the exploitation of real data. Anticipating the place and moment of an accident is not possible, but within the framework of (3.1)–(3.2) taking into account its consequences is practically feasible.

5. Technical Details

Proof of Lemma 2.1. We follow some ideas of the proof of Proposition 2.3 in Ref. 9.

First, following Chap. 6 in Ref. 6, we use the wave front algorithm to obtain a sequence of piecewise constant approximate solutions to (2.4).

Fix $n \in \mathbb{N}\backslash\{0\}$ and call $f_n \in \mathbf{C^{0,1}}(\mathbb{R};\mathbb{R})$, with $\mathbf{Lip}(f_n) \leq \mathbf{Lip}(f)$, the piecewise linear and continuous function such that $f_n(\rho) = f(\rho)$ for all $\rho \in 2^{-n}\mathbb{Z}$. Approximate the initial datum ρ_o with a piecewise constant $\rho_o^n \in (\mathbf{L^1} \cap \mathbf{BV})(\mathbb{R};\mathbb{R})$ such that $\rho_o^n(\mathbb{R}) \subseteq 2^{-n}\mathbb{Z}$ and $\mathrm{TV}(\rho_o^n) \leq \mathrm{TV}(\rho_o)$. Call ρ^n the exact solution to

$$\begin{cases} \partial_t \rho^n + \partial_x(f^n(\rho^n)) = 0, \\ \rho^n(0,x) = \rho_o^n(x), \end{cases}$$

obtained gluing the solutions to the Riemann problems at the jumps in ρ_o^n, see Chap. 6 in Ref. 6. Following Lemma 4.4 in Ref. 9, locally the map $t \to \rho^n(t, \gamma_1(t))$ can be written

$$\rho^n(t,\gamma_1(t)) = \sum_\alpha \rho_\alpha \chi_{[t_\alpha, t_{\alpha+1}[}(t) \quad \text{with} \quad \gamma_1(t_\alpha) = \lambda_\alpha t_\alpha + x_\alpha, \tag{5.1}$$

where $t \to \lambda_\alpha t + x_\alpha$ supports a discontinuity in ρ^n crossed by γ_1. Here, we intend that all states attained by ρ^n in a neighborhood of (t_α, x_α) appear in the sum (5.1), possibly with $t_{\alpha+1} = t_\alpha$.

If $\eta \in \mathbf{C^1}(\mathbb{R};\mathbb{R})$ and $\|\eta\|_{\mathbf{C^1}}$ is sufficiently small, there exists times t'_α such that

$$\rho^n(t,\gamma_1(t)+\eta(t)) = \sum_\alpha \rho_\alpha \chi_{[t'_\alpha, t'_{\alpha+1}[}(t) \quad \text{with} \quad \gamma_1(t'_\alpha) = \lambda_\alpha t'_\alpha + x_\alpha. \tag{5.2}$$

Hence

$$\begin{aligned} [\gamma_1(t'_\alpha) + \eta(t'_\alpha)] - \gamma_1(t_\alpha) &= \lambda_\alpha(t'_\alpha - t_\alpha) \quad \text{on the other hand} \\ [\gamma_1(t'_\alpha) + \eta(t'_\alpha)] - \gamma_1(t_\alpha) &= [\gamma_1(t'_\alpha) - \gamma_1(t_\alpha)] + \eta(t'_\alpha) \\ &= \int_0^1 \dot\gamma_1(\vartheta t'_\alpha + (1-\vartheta)t_\alpha)\mathrm{d}\tau (t'_\alpha - t_\alpha) + \eta(t'_\alpha) \end{aligned}$$

so that

$$\begin{aligned} \lambda_\alpha(t'_\alpha - t_\alpha) &= \int_0^1 \dot\gamma_1(\vartheta t'_\alpha + (1-\vartheta)t_\alpha)\mathrm{d}\tau (t'_\alpha - t_\alpha) + \eta(t'_\alpha) \\ t'_\alpha - t_\alpha &= \frac{\eta(t'_\alpha)}{\lambda_\alpha - \int_0^1 \dot\gamma_1(\vartheta t'_\alpha + (1-\vartheta)t_\alpha)\mathrm{d}\tau} \\ |t'_\alpha - t_\alpha| &= \frac{|\eta(t'_\alpha)|}{\left|\lambda_\alpha - \int_0^1 \dot\gamma_1(\vartheta t'_\alpha + (1-\vartheta)t_\alpha)\mathrm{d}\tau\right|} \\ |t'_\alpha - t_\alpha| &\leq \frac{1}{c}\|\eta\|_{\mathbf{C^0}}. \end{aligned}$$

Integrating the modulus of the difference between the terms (5.1) and (5.2), we obtain a first Lipschitz type estimate:

$$\begin{aligned}\int_0^T & |\rho^n(t,\gamma_1(t)) - \rho^n(t,\gamma_1(t)+\eta(t))|\,\mathrm{d}t \\ &= \sum_\alpha |\rho_\alpha - \rho_{\alpha-1}|(t_\alpha - t'_\alpha) \le \frac{1}{c}\|\eta\|_{\mathbf{C^0}} \sum_\alpha |\rho_\alpha - \rho_{\alpha-1}| \\ &\le \frac{1}{c}\|\eta\|_{\mathbf{C^0}} \mathrm{TV}\{\rho^n(\cdot,\gamma_1(\cdot)),[0,T]\} \le \frac{1}{c}\|\eta\|_{\mathbf{C^0}}\mathrm{TV}(\rho_o^n) \le \frac{1}{c}\|\eta\|_{\mathbf{C^0}}\mathrm{TV}(\rho_o). \qquad (5.3)\end{aligned}$$

The proof is now completed as that of Lemma 4.4 in Ref. 9. Introduce $\psi\colon [0,1]\to\mathbb{R}$ by

$$\psi(\vartheta) = \int_0^T |\rho^n(t,\vartheta\gamma_2(t) + (1-\vartheta)\gamma_1(t))|\,\mathrm{d}t$$

and observe that the above estimate (5.3) ensures that ψ is locally Lipschitz continuous and moreover $|\dot\psi| \le \frac{1}{c}\mathrm{TV}(\rho_o)\|\gamma_2-\gamma_1\|_{\mathbf{C^0}([0,T];\mathbb{R})}$. Finally,

$$\begin{aligned}\int_0^T |\rho^n(t,\gamma_2(t)) - \rho^n(t,\gamma_1(t))|\,\mathrm{d}t &= \psi(1)-\psi(0) \\ &\le \|\dot\psi\|_{\mathbf{L}^\infty([0,1];\mathbb{R})} \\ &\le \frac{1}{c}\mathrm{TV}(\rho_o)\|\gamma_2-\gamma_1\|_{\mathbf{C^0}([0,T];\mathbb{R})}.\end{aligned}$$

Thanks to the convergence of ρ^n to ρ, an application of Lebesgue dominated convergence theorem completes the proof. □

Proof of Proposition 2.1. Let $\rho_{_V}$ solve $\partial_t\rho + \partial_x\left(\rho v_{_V}(\rho)\right) = 0$. Then, $\rho_{_{V_2}}(t,x) = \rho_{_{V_1}}(t,\frac{V_1}{V_2}\,x)$. Indeed,

$$\partial_t\rho_{_{V_2}}(t,x) = \partial_t\rho_{_{V_1}}\left(t,\frac{V_1}{V_2}x\right) \quad \text{and} \quad \partial_x\rho_{_{V_2}}(t,x) = \frac{V_1}{V_2}\,\partial_x\rho_{_{V_1}}\left(t,\frac{V_1}{V_2}\,x\right),$$

so that, setting $f_V(\rho) = \rho V(1-\rho)$, we have that $f_{V_2} = \frac{V_2}{V_1}f_{V_1}$ and

$$\begin{aligned}&\partial_t\rho_{_{V_2}}(t,x) + \partial_x\left(f_{V_2}\left(\rho_{_{V_2}}(t,x)\right)\right) \\ &\quad= \partial_t\rho_{_{V_2}}(t,x) + f'_{V_2}\left(\rho_{_{V_2}}(t,x)\right)\partial_x\rho_{_{V_2}}(t,x) \\ &\quad= \partial_t\rho_{_{V_1}}\left(t,\frac{V_1}{V_2}x\right) + \frac{V_2}{V_1}f'_{V_1}\left(\rho_{_{V_1}}\left(t,\frac{V_1}{V_2}x\right)\right)\frac{V_1}{V_2}\partial_x\rho_{_{V_1}}\left(t,\frac{V_1}{V_2}x\right)\end{aligned}$$

$$
\begin{aligned}
&= \partial_t \rho_{V_1}\left(t, \frac{V_1}{V_2}x\right) + f'_{V_1}\left(\rho_{V_1}\left(t, \frac{V_1}{V_2}x\right)\right)\partial_x \rho_{V_1}\left(t, \frac{V_1}{V_2}x\right) \\
&= \partial_t \rho_{V_1}\left(t, \frac{V_1}{V_2}x\right) + \partial_x\left(f_{V_1}\left(\rho_{V_1}\left(t, \frac{V_1}{V_2}x\right)\right)\right) \\
&= 0.
\end{aligned}
$$

We are led to consider

$$
\begin{aligned}
|\mathcal{E}(V_2) - \mathcal{E}(V_1)| &= \left| \int_0^T \left| \dot{p}(t) - v_{V_2}\Big(\rho_{V_2}(t, p(t))\Big) \right| \mathrm{d}t - \int_0^T \left| \dot{p}(t) - v_{V_1}\Big(\rho_{V_1}(t, p(t))\Big) \right| \mathrm{d}t \right| \\
&\le \left| \int_0^T \left| v_{V_2}\left(\rho_{V_2}(t, p(t))\right) - v_{V_1}\left(\rho_{V_1}(t, p(t))\right) \right| \mathrm{d}t \right| \\
&\le \left| \int_0^T \left| v_{V_2}\left(\rho_{V_2}(t, p(t))\right) - v_{V_2}\left(\rho_{V_1}(t, p(t))\right) \right| \mathrm{d}t \right| \\
&\quad + \left| \int_0^T \left| v_{V_2}\left(\rho_{V_1}(t, p(t))\right) - v_{V_1}\left(\rho_{V_1}(t, p(t))\right) \right| \mathrm{d}t \right| \\
&\le \left| V_2 \int_0^T \left| \rho_{V_2}(t, p(t)) - \rho_{V_1}(t, p(t)) \right| \mathrm{d}t \right| + |V_2 - V_1| t \\
&\le V_2 \left| \int_0^T \left| \rho_1\left(t, \frac{p(t)}{V_2}\right) - \rho_1\left(t, \frac{p(t)}{V_1}\right) \right| \mathrm{d}t \right| + |V_2 - V_1| t
\end{aligned}
$$

and to prove continuity we show that Lemma 2.1 can be applied. Indeed, by the maximum principle for conservation laws, $\rho(t,x) \ge \check{\rho}$ for a.e. $(t,x) \in [0,T] \times \mathbb{R}$. Moreover,

$$
\begin{aligned}
f'_1(\rho(t,p(t))) &= 1 - 2\rho(t,p(t)) \le 1 - 2 \operatorname*{ess\,inf}_{x\in\mathbb{R}} \rho_o \\
\frac{\dot{p}(t)}{V_i} &\ge \frac{\hat{V}}{V_i}(1 - 2\check{\rho}) \ge 1 - 2\check{\rho}.
\end{aligned}
$$

Choosing now $c > 0$ such that $c < 2(\operatorname{ess\,inf}_{x\in\mathbb{R}} \rho_o - \check{\rho})$, Lemma 2.1 can be applied:

$$
\begin{aligned}
\left| \int_0^T \left| \rho_1\left(t, \frac{p(t)}{V_2}\right) - \rho_1\left(t, \frac{p(t)}{V_1}\right) \right| \mathrm{d}t \right| &\le \frac{1}{c} \mathrm{TV}(\rho_o) \left\| \frac{p(t)}{V_2} - \frac{p(t)}{V_1} \right\|_{\mathbf{C^0}([0,T];\mathbb{R})} \\
&\le \frac{1}{c\check{V}} \mathrm{TV}(\rho_o) \|p\|_{\mathbf{C^0}([0,T];\mathbb{R})} |V_2 - V_1|,
\end{aligned}
$$

completing the proof. □

Lemma 5.1. *Let $v \in \mathbf{C^2}([0,1];[0,V])$ be such that $v' \le 0$ and $\rho \to \rho v(\rho)$ is strictly concave. Then, the map $\rho \to \frac{\rho w v(\rho)}{w + v(\rho)}$ is strictly concave for all $w > 0$.*

Proof. Call $q(\rho) = \rho v(\rho)$ and $q_w(\rho) = \frac{\rho w v(\rho)}{w+v(\rho)}$. By direct computations,

$$q_w'(\rho) = w\frac{v^2(\rho) + wv(\rho) + w\rho v'(\rho)}{(w + v(\rho))^2},$$

$$q_w''(\rho) = w^2\frac{(v(\rho) + w)q''(\rho) - 2\rho(v'(\rho))^2}{(v(\rho) + w)^3},$$

which clearly shows that $q_w''(\rho) \leq 0$, since $q'' \leq 0$. □

The next regularity result is of use below.

Lemma 5.2. *Let* **(v)**, **(p)** *and* **(χ)** *hold. Then*:

(i) $\mathcal{V}$ *as defined in* (3.2) *is continuous*;
(ii) $x \to \mathcal{V}(t, x, \rho)$ *is uniformly Lipschitz continuous for* $t \in \mathbb{R}^+$ *and* $\rho \in [0, 1]$;
(iii) $\rho \to \mathcal{V}(t, x, \rho)$ *is uniformly Lipschitz continuous for* $t \in \mathbb{R}^+$ *and* $x \in \mathbb{R}$;
(iv) $\rho \to \partial_x\mathcal{V}(t, x, \rho)$ *is uniformly Lipschitz continuous for* $t \in \mathbb{R}^+$ *and* $x \in \mathbb{R}$.

Proof. Call $P = \mathbf{Lip}(p)$ and observe that the map:

$$[0, P] \times [0, V] \to \mathbb{R},$$
$$(\dot{p}, v) \to \begin{cases} \dfrac{\dot{p}v}{\dot{p} + v} & (\dot{p}, v) \neq (0, 0), \\ 0 & (\dot{p}, v) = (0, 0) \end{cases}$$

is continuous and non-negative. The continuity of $\mathcal{V}$ immediately follows, proving (i). Call $M = \max_{[0,P]\times[0,V]} \frac{\dot{p}v}{\dot{p}+v}$. Then,

$$\begin{aligned}|\mathcal{V}(t, x_2, \rho) - \mathcal{V}(t, x_1, \rho)| &\leq (M + V)|\chi(x_2 - \dot{p}(t)) - \chi(x_1 - \dot{p}(t))| \\ &\leq (M + V)\mathbf{Lip}(\chi)|x_2 - x_1|\,,\end{aligned}$$

completing the proof of (ii). Direct computations show that a Lipschitz constant for the map $\rho \to \mathcal{V}(t, x, \rho)$ is $2\,\mathbf{Lip}(v)$, proving (iii). Finally, entirely analogous computations ensure that $\rho \to \partial_x\mathcal{V}(t, x, \rho)$ is Lipschitz continuous with Lipschitz constant $(1 + \mathbf{Lip}(\chi))\mathbf{Lip}(v)$. □

Lemma 5.3. *Let* **(v)** *hold and fix a positive* P. *Consider the map*:

$$g\colon [0, 1] \times [0, P] \to \mathbb{R},$$
$$(\rho, q) \to \frac{q\rho v(\rho)}{q + v(\rho)}. \tag{5.4}$$

Then, there exists a $L > 0$ *such that for all* $(\rho_1, q_1), (\rho_2, q_2) \in [0, 1] \times [0, P]$,

$$|g(\rho_1, q_1) - g(\rho_1, q_2) - g(\rho_2, q_1) + g(\rho_2, q_2)| \leq L|\rho_1 - \rho_2||q_1 - q_2|.$$

Proof. Compute first the partial derivative

$$\partial_q g(\rho, q) = \frac{\rho v^2(\rho)}{(q + v(\rho))^2}.$$

By **(v)**, the map $\rho \to \partial_q g(\rho, q)$ is Lipschitz continuous on $[0,1] \times [0,P]$, hence it is a.e. differentiable with respect to ρ. Moreover

$$\partial^2_{\rho q} g(\rho, q) = v(\rho)(v(\rho) + 2\rho v'(\rho)) - 2\frac{\rho v^2(\rho) v'(\rho)}{q + v(\rho)}$$

and, clearly, $\sup_{[0,1]\times[0,P]} \left|\partial^2_{\rho q} g(\rho, q)\right| < +\infty$. We can then write:

$$\begin{aligned}
&|g(\rho_1, q_1) - g(\rho_1, q_2) - g(\rho_2, q_1) + g(\rho_2, q_2)| \\
&\quad = \left| \int_0^1 \partial_q g(\rho_1, (1-\vartheta)q_1 + \vartheta q_2)\mathrm{d}\vartheta - \int_0^1 \partial_q g(\rho_2, (1-\vartheta)q_1 + \vartheta q_2)\mathrm{d}\vartheta \right| |q_1 - q_2| \\
&\quad = \left| \int_0^1 \int_0^1 \partial^2_{\rho q} g((1-\eta)\rho_1 + \eta\rho_2, (1-\vartheta)q_1 + \vartheta q_2)\, \mathrm{d}\eta\, \mathrm{d}\vartheta \right| |\rho_1 - \rho_2||q_1 - q_2| \\
&\quad \le \left(\sup_{[0,1]\times[0,P]} \left|\partial^2_{\rho q} g(\rho, q)\right| \right) |\rho_1 - \rho_2||q_1 - q_2|,
\end{aligned}$$

completing the proof. □

Proof of Proposition 3.1. In the present setting, the assumptions of Theorem 2 in Ref. 20 read:

(1) f is a Caratheodory vector field on $\mathbb{R}^+ \times \mathbb{R}$, i.e. for a.e. $(t,x) \in \mathbb{R}^+ \times \mathbb{R}$, the map $\rho \to f(t,x,\rho)$ is continuous and for all $\rho \in [0,1]$ the map $(t,x) \to f(t,x,u)$ is measurable.
(2) The map $\rho \to f(t,x,\rho)$ is non-degenerate, i.e. it is not affine on non-trivial intervals.
(3) For some $a, b \in [0,1]$, for all $(t,x) \in \mathbb{R}^+ \times \mathbb{R}$, $f(t,x,a) = f(t,x,b) = 0$ and the map $(t,x) \to \max_{\rho\in[a,b]} |f(t,x,\rho)|$ is in $\mathbf{L^q_{loc}}(\mathbb{R}^+ \times \mathbb{R}; \mathbb{R})$ for a $q > 2$.

Note that (1) follows from **(v)**, **(p)** and **(χ)**. The requirement (2) follows from **(v)**, **(χ)** and **(p)**, indeed they ensure that $\mathcal{V}$ is a convex combination of strictly concave functions. Condition (3) can be easily verified, with $a = 0$ and $b = 1$, thanks to **(v)** and since $f(t,x,\rho) \in [0, \max\{\mathbf{Lip}(\dot{p}), V\}]$. Hence, Theorem 2 in Ref. 20 applies. □

Proof of Proposition 3.2. Observe that by (ii) of Lemma 5.2, the distributional derivative of the map $(t,x) \to \partial_x f(t,x,k)$ has no singular part. Hence, $\mu^k s$ vanishes in (3.5). Moreover, choosing a test function with support in $\mathring{\mathbb{R}}^+ \times \mathbb{R}$ makes the latter addend in the left-hand side of (3.5) vanish. Hence, (3.5) implies (3.3). Finally, the condition (3.4) on the initial datum is known to be implied by the stronger (3.5), see for instance Formula (10) in Ref. 20. □

Lemma 5.4. *Theorem* 1.1 *in Ref.* 11 *does not apply to* (3.2), *since the one-sided Lipschitz condition stated in Formula* (1.7) *of Ref.* 11 *may fail to hold.*

Proof. In the present setting, due to the absence of the parabolic term and of the source on the right-hand side of (1.3), the assumptions in Ref. 11 necessary to apply Theorem 1.1 in Ref. 11 on the time interval $[0, T]$, for any $T > 0$, are the following:

(1) $f(t, x, 0) = \partial_x f(t, x, 0) = 0$ for a.e. $t \in \mathbb{R}^+$ and for all $x \in \mathbb{R}$.
(2) The map $(t, x) \to f(t, x, \rho)$ is in $\mathbf{L^1}(\mathbb{R}^+; \mathbf{W^{1,1}_{loc}}(\mathbb{R}; \mathbb{R}))$ for all $\rho \in [0, 1]$.
(3) For any $T > 0$, the map $(t, x) \to \partial_x f(t, x, \rho)$ is in $\mathbf{L^1}([0, T]; \mathbf{L}^\infty(\mathbb{R}; \mathbb{R}))$ for all $\rho \in [0, 1]$.
(4) There exists a positive L such that for a.e. $t \in \mathbb{R}^+$, for all $x \in \mathbb{R}$ and all $\rho_1, \rho_2 \in [0, 1]$,

$$|f(t, x, \rho_2) - f(t, x, \rho_1)| + |\partial_x f(t, x, \rho_2) - \partial_x f(t, x, \rho_1)| \leq L|\rho_2 - \rho_1|.$$

(5) Define $F(t, x, \rho_1, \rho_2) = \operatorname{sgn}(\rho_1 - \rho_2)(f(t, x, \rho_1) - f(t, x, \rho_2))$. There exists a positive C such that for a.e. $t_1, t_2 \in \mathbb{R}^+$, for all $x_1, x_2 \in \mathbb{R}$ and all $\rho_1, \rho_2 \in [0, 1]$,

$$(F(t_1, x_1, \rho_1, \rho_2) - F(t_2, x_2, \rho_1, \rho_2))(x_1 - x_2) \geq -C|\rho_1 - \rho_2|(x_1 - x_2)^2.$$

We first prove that (1)–(4) hold.

Note that (1) is immediate, by (3.2) and **(v)**. By (2) we mean that for any compact set $K \subset \mathbb{R}$, for any positive T and for any $\rho \in [0, 1]$, the map $t \to \int_K (|f(t, x, \rho)| + |\partial_x f(t, x, \rho)|)\mathrm{d}x$ is in $\mathbf{L^1}((0, T]; \mathbb{R})$, which is immediate since both f and $\partial_x f$ are uniformly bounded, thanks to (3.2) and Lemma 5.2. This uniform bound on $\partial_x f$ also proves (3). At (4), the Lipschitz continuity of $\rho \to f(t, x, \rho)$, respectively $\rho \to \partial_x f(t, x, \rho)$, is proved in (iii), respectively (iv), of Lemma 5.2.

Finally, we note that (5) may fail to hold, due to the dependence of the left-hand side on time. Assume, for instance, that:

$$\begin{aligned} v(\rho) &= 1 - \rho, & t_1 &= 0, & x_1 &= 0, & \rho_1 &= 1/2, \\ p(t) &= t, & t_2 &= 2, & x_2 &= \varepsilon, & \rho_2 &= 0, \end{aligned}$$

the condition (5) amounts to require the existence of a constant C such that $\varepsilon/6 \leq C\varepsilon^2$, which is not possible. □

Proof of Proposition 3.3. We exploit the doubling of variables method, see Ref. 13. To this aim, assume that ρ_1 and ρ_2 are two solutions to (3.1) in the sense of Definition 3.1. Let $\psi = \psi(t, x, s, y)$ be in $\mathbf{C^\infty_c}((\mathring{\mathbb{R}}^+ \times \mathbb{R})^2; \mathbb{R}^+)$, write (3.3) for $\rho = \rho_1(t, x)$ and for $k = \rho_2(s, y)$, integrate the resulting inequality on $\mathbb{R}^+ \times \mathbb{R}$, to obtain:

$$\int_{\mathbb{R}^+}\int_{\mathbb{R}}\int_{\mathbb{R}^+}\int_{\mathbb{R}} [|\rho_1(t,x) - \rho_2(s,y)|\partial_t\psi(t,x,s,y)$$
$$+ \operatorname{sgn}(\rho_1(t,x) - \rho_2(s,y))[f(t,x,\rho_1(t,x)) - f(t,x,\rho_2(s,y))]\partial_x\psi(t,x,s,y)$$
$$- \operatorname{sgn}(\rho_1(t,x) - \rho_2(s,y))\partial_x f(t,x,\rho_2(s,y))\,\psi(t,x,s,y)]\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}y\,\mathrm{d}s \geq 0.$$

Repeat now the same procedure exchanging the roles of ρ_1 and ρ_2, obtaining

$$\int_{\mathbb{R}^+}\int_{\mathbb{R}}\int_{\mathbb{R}^+}\int_{\mathbb{R}} [|\rho_1(t,x) - \rho_2(s,y)|\partial_s\psi(t,x,s,y)$$
$$+ \operatorname{sgn}(\rho_2(s,y) - \rho_1(t,x))[f(s,y,\rho_2(s,y)) - f(s,y,\rho_1(t,x))]\partial_y\psi(t,x,s,y)$$
$$- \operatorname{sgn}(\rho_2(s,y) - \rho_1(t,x))\partial_y f(s,y,\rho_1(t,x))\psi(t,x,s,y)]\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}y\,\mathrm{d}s \geq 0.$$

The sum of the latter two inequalities above yields

$$0 \leq \int_{\mathbb{R}^+}\int_{\mathbb{R}}\int_{\mathbb{R}^+}\int_{\mathbb{R}} [|\rho_1(t,x) - \rho_2(s,y)|(\partial_t\psi(t,x,s,y) + \partial_s\psi(t,x,s,y))$$
$$+ \operatorname{sgn}(\rho_1(t,x) - \rho_2(s,y))(f(t,x,\rho_1(t,x)) - f(t,x,\rho_2(s,y)))\partial_x\psi(t,x,s,y)$$
$$+ \operatorname{sgn}(\rho_1(t,x) - \rho_2(s,y))(f(s,y,\rho_1(t,x)) - f(s,y,\rho_2(s,y)))\partial_y\psi(t,x,s,y)$$
$$+ \operatorname{sgn}(\rho_1(t,x) - \rho_2(s,y))[\partial_y f(s,y,\rho_1(t,x))$$
$$- \partial_x f(t,x,\rho_2(s,y))]\psi(t,x,s,y)]\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}y\,\mathrm{d}s.$$

Following Ref. 11, since

$$\partial_x[(f(s,y,\rho_2(s,y)) - f(t,x,\rho_2(t,x)))\psi(t,x,s,y)]$$
$$= -\partial_x f(t,x,\rho_2(t,x)) + f(s,y,\rho_2(s,y))\partial_x\psi(t,x,s,y)$$
$$- f(t,x,\rho_2(t,x))\partial_x\psi(t,x,s,y),$$

we get

$$[f(t,x,\rho_1(t,x)) - f(t,x,\rho_2(s,y))]\partial_x\psi(t,x,s,y) - \partial_x f(t,x,\rho_2(s,y))\psi(t,x,s,y)$$
$$= [f(t,x,\rho_1(t,x)) - f(s,y,\rho_2(s,y))]\partial_x\psi(t,x,s,y)$$
$$+ \partial_x[(f(s,y,\rho_2(s,y)) - f(t,x,\rho_2(t,x)))\psi(t,x,s,y)]$$

and similarly

$$[f(s,y,\rho_1(t,x)) - f(s,y,\rho_2(s,y))]\partial_y\psi(t,x,s,y) - \partial_y f(s,y,\rho_1(t,x))\psi(t,x,s,y)$$
$$= [f(t,x,\rho_1(t,x)) - f(s,y,\rho_2(s,y))]\partial_y\psi(t,x,s,y)$$
$$- \partial_y[(f(t,x,\rho_1(t,x)) - f(s,y,\rho_1(t,x)))\psi(t,x,s,y)].$$

Thus, following where possible the notation of p. 1093 in Ref. 11, we have

$$\int_{\mathbb{R}^+}\int_{\mathbb{R}}\int_{\mathbb{R}^+}\int_{\mathbb{R}} (I_0 + I_1 + I_3)\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}y\,\mathrm{d}s \geq 0, \tag{5.5}$$

where

$$I_0 = |\rho_1(t,x) - \rho_2(s,y)|(\partial_t\psi(t,x,s,y) + \partial_s\psi(t,x,s,y)),$$

$$I_1 = \mathrm{sgn}(\rho_1(t,x) - \rho_2(s,y))[f(t,x,\rho_1(t,x)) - f(s,y,\rho_2(s,y))] \times (\partial_x\psi(t,x,s,y) + \partial_y\psi(t,x,s,y)),$$

$$I_3 = \mathrm{sgn}(\rho_1(t,x) - \rho_2(s,y))[\partial_x((f(s,y,\rho_2(s,y)) - f(t,x,\rho_2(s,y)))\psi(t,x,s,y)) - \partial_y((f(t,x,\rho_1(t,x)) - f(s,y,\rho_1(t,x)))\psi(t,x,s,y))].$$

We proceed toward the choice of the test functions introducing first a map

$$\delta \in \mathbf{C}^\infty_\mathbf{c}(\mathbb{R};\mathbb{R}^+) \quad \text{such that} \quad \mathrm{spt}\,\delta \subseteq [-1,1],$$
$$\delta(-\xi) = \delta(\xi), \quad \text{and} \quad \int_\mathbb{R} \delta(\xi)\mathrm{d}\xi = 1.$$

Moreover, for $r > 0$, let

$$\delta_r(t) = \delta(t/r)/r \quad \text{and} \quad \omega_r(x) = \delta(x^2/r^2)/(2r), \tag{5.6}$$

and for a $\varphi \in \mathbf{C}^\infty_\mathbf{c}(\mathbb{R}^+ \times \mathbb{R};\mathbb{R}^+)$, choose

$$\psi(t,x,s,y) = \varphi\left(\frac{t+s}{2}, \frac{x+y}{2}\right)\omega_r\left(\frac{x-y}{2}\right)\ \delta_r\left(\frac{t-s}{2}\right).$$

We then rewrite (5.5) as

$$\int_{\mathbb{R}^+}\int_\mathbb{R}\int_{\mathbb{R}^+}\int_\mathbb{R}\left[(\bar I_0 + \bar I_1 + \bar I_3)\omega_r\left(\frac{x-y}{2}\right)\delta_r\left(\frac{t-s}{2}\right) + \bar I_5\partial_x\omega_r\left(\frac{x-y}{2}\right)\right]\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}y\,\mathrm{d}s \geq 0, \tag{5.7}$$

where

$$\bar I_0 = |\rho_1(t,x) - \rho_2(s,y)|\left(\partial_t\varphi\left(\frac{t+s}{2}, \frac{x+y}{2}\right) + \partial_s\varphi\left(\frac{t+s}{2}, \frac{x+y}{2}\right)\right),$$

$$\bar I_1 = \mathrm{sgn}(\rho_1(t,x) - \rho_2(s,y))(f(t,x,\rho_1(t,x)) - f(s,y,\rho_2(s,y))) \times\left(\partial_x\varphi\left(\frac{t+s}{2}, \frac{x+y}{2}\right) + \partial_y\varphi\left(\frac{t+s}{2}, \frac{x+y}{2}\right)\right),$$

$$\bar I_3 = \mathrm{sgn}(\rho_1(t,x) - \rho_2(s,y)) \times\Bigg[(\partial_x[f(s,y,\rho_2(s,y)) - f(t,x,\rho_2(s,y))] - \partial_y[f(t,x,\rho_1(t,x)) - f(s,y,\rho_1(t,x))])\varphi\left(\frac{t+s}{2}, \frac{x+y}{2}\right)$$

$$+ (f(s,y,\rho_2(s,y)) - f(t,x,\rho_2(s,y)))\partial_x\varphi\left(\frac{t+s}{2},\frac{x+y}{2}\right)$$

$$+ (f(t,x,\rho_1(t,x)) - f(s,y,\rho_1(t,x)))\partial_y\varphi\left(\frac{t+s}{2},\frac{x+y}{2}\right)\Bigg],$$

$$\begin{aligned}\bar{I}_5 &= \operatorname{sgn}(\rho_1(t,x) - \rho_2(s,y)) \\ &\quad \times (f(t,x,\rho_1(t,x)) - f(t,x,\rho_2(s,y)) - f(s,y,\rho_1(t,x)) + f(s,y,\rho_2(s,y))) \\ &\quad \times \varphi\left(\frac{t+s}{2},\frac{x+y}{2}\right)\ \delta_r\left(\frac{t-s}{2}\right).\end{aligned}$$

We first estimate the term $\bar{I}_5$ as follows:

$$\begin{aligned}\bar{I}_5 &= \operatorname{sgn}(\rho_1(t,x) - \rho_2(s,y)) \\ &\quad \times (f(t,x,\rho_1(t,x)) - f(t,x,\rho_2(s,y)) - f(s,y,\rho_1(t,x)) + f(s,y,\rho_2(s,y))) \\ &\quad \times \varphi\left(\frac{t+s}{2},\frac{x+y}{2}\right)\ \delta_r\left(\frac{t-s}{2}\right).\end{aligned}$$

We now estimate the term in parentheses above:

$$\begin{aligned}&f(t,x,\rho_1(t,x)) - f(t,x,\rho_2(s,y)) - f(s,y,\rho_1(t,x)) + f(s,y,\rho_2(s,y)) \\ &\quad = \chi(x-p(t))\left(\frac{\dot{p}(t)\rho_1 v(\rho_1)}{\dot{p}(t)+v(\rho_1)} - \frac{\dot{p}(t)\rho_2 v(\rho_2)}{\dot{p}(t)+v(\rho_2)}\right) \\ &\qquad - \chi(y-p(s))\left(\frac{\dot{p}(s)\rho_1 v(\rho_1)}{\dot{p}(s)+v(\rho_1)} - \frac{\dot{p}(s)\rho_2 v(\rho_2)}{\dot{p}(s)+v(\rho_2)}\right) \\ &\qquad + (1-\chi(x-p(t)))(\rho_1 v(\rho_1) - \rho_2 v(\rho_2)) \\ &\qquad - (1-\chi(x-p(s)))(\rho_1 v(\rho_1) - \rho_2 v(\rho_2)) \\ &\quad = (\chi(x-p(t)) - \chi(y-p(s)))\left(\frac{\dot{p}(t)\rho_1 v(\rho_1)}{\dot{p}(t)+v(\rho_1)} - \frac{\dot{p}(t)\rho_2 v(\rho_2)}{\dot{p}(t)+v(\rho_2)}\right) \\ &\qquad + \chi(y-p(s))\left(\frac{\dot{p}(t)\rho_1 v(\rho_1)}{\dot{p}(t)+v(\rho_1)} - \frac{\dot{p}(t)\rho_2 v(\rho_2)}{\dot{p}(t)+v(\rho_2)} - \frac{\dot{p}(s)\rho_1 v(\rho_1)}{\dot{p}(s)+v(\rho_1)} + \frac{\dot{p}(s)\rho_2 v(\rho_2)}{\dot{p}(s)+v(\rho_2)}\right) \\ &\qquad - (\chi(x-p(t)) - \chi(x-p(s)))(\rho_1 v(\rho_1) - \rho_2 v(\rho_2)).\end{aligned}$$

To bound the absolute values of the terms above, use $(\boldsymbol{\chi})$, the Lipschitz continuity of the map $\rho \to (\rho v(r))/(\dot{p}+v(\rho))$ with Lipschitz constant L, the boundedness of $\dot{p}$ and the Lipschitz continuity of the map $\rho \to \rho v(\rho)$ with Lipschitz constant $\mathbf{Lip}(\rho v)$:

$$\begin{aligned}|\chi(x-p(t)) - \chi(y-p(s))| &\le \mathbf{Lip}(\chi)(|x-y| + |p(t)-p(s)|) \\ &\le \mathbf{Lip}(\chi)(1+\mathbf{Lip}(p))(|x-y| + |t-s|),\end{aligned}$$

$$\left|\frac{\dot p(t)\rho_1 v(\rho_1)}{\dot p(t)+v(\rho_1)} - \frac{\dot p(t)\rho_2 v(\rho_2)}{\dot p(t)+v(\rho_2)}\right| \le \mathbf{Lip}(p) L |\rho_1-\rho_2|,$$
$$|\rho_1 v(\rho_1) - \rho_2 v(\rho_2)| \le \mathbf{Lip}(\rho v)|\rho_1-\rho_2|,$$

while the remaining term is estimated by means of g as defined at (5.4) in Lemma 5.3:

$$\begin{aligned}&\left|\frac{\dot p(t)\rho_1 v(\rho_1)}{\dot p(t)+v(\rho_1)} - \frac{\dot p(t)\rho_2 v(\rho_2)}{\dot p(t)+v(\rho_2)} - \frac{\dot p(s)\rho_1 v(\rho_1)}{\dot p(s)+v(\rho_1)} + \frac{\dot p(s)\rho_2 v(\rho_2)}{\dot p(s)+v(\rho_2)}\right|\\ &\quad = |g(\rho_1,\dot p(t)) - g(\rho_2,\dot p(t)) - g(\rho_1,\dot p(s)) + g(\rho_2,\dot p(s))|\\ &\quad \le \mathbf{Lip}(g)|\dot p(t)-\dot p(s)||\rho_1-\rho_2|\\ &\quad \le \mathbf{Lip}(g)\mathbf{Lip}(\dot p)|t-s||\rho_1-\rho_2|,\end{aligned}$$

where we used the Lipschitz regularity of $t \to p(t)$. Going back to $\bar I_5$:

$$\bar I_5 \le C(|x-y|+|t-s|)|\rho_1(t,x)-\rho_2(s,y)|\varphi\left(\frac{t+s}{2},\frac{x+y}{2}\right)\delta_r\left(\frac{t-s}{2}\right),$$

where

$$C = \mathbf{Lip}(\chi)(1+\mathbf{Lip}(p))(\mathbf{Lip}(p)L + \mathbf{Lip}(\rho v)) + \mathbf{Lip}(g)\mathbf{Lip}(\dot p) \tag{5.8}$$

and

$$\begin{aligned}&\int_{\mathbb{R}^+}\int_{\mathbb{R}}\int_{\mathbb{R}^+}\int_{\mathbb{R}} \bar I_5 \partial_x\omega_r\left(\frac{x-y}{2}\right)\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}y\,\mathrm{d}s\\ &\quad \le C\int_{\mathbb{R}^+}\int_{\mathbb{R}}\int_{\mathbb{R}^+}\int_{\mathbb{R}}(|x-y|+|t-s|)|\rho_1(t,x)-\rho_2(s,y)|\\ &\qquad \varphi\left(\frac{t+s}{2},\frac{x+y}{2}\right)\delta_r\left(\frac{t-s}{2}\right)\partial_x\omega_r\left(\frac{x-y}{2}\right)\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}y\,\mathrm{d}s\\ &\quad \le C\int_{\mathbb{R}^+}\int_{\mathbb{R}}\int_{\mathbb{R}^+}\int_{\mathbb{R}}\frac{|x-y|+|t-s|}{r}|\rho_1(t,x)-\rho_2(s,y)|\\ &\qquad \varphi\left(\frac{t+s}{2},\frac{x+y}{2}\right)\delta_r\left(\frac{t-s}{2}\right)\frac{1}{r}\mathbf{1}_{[-r,r]}(x-y)\max|\delta'|\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}y\,\mathrm{d}s,\end{aligned}$$

so that

$$\begin{aligned}&\lim_{r\to 0}\int_{\mathbb{R}^+}\int_{\mathbb{R}}\int_{\mathbb{R}^+}\int_{\mathbb{R}} \bar I_5 \partial_x\omega_r\left(\frac{x-y}{2}\right)\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}y\,\mathrm{d}s\\ &\quad = C\int_{\mathbb{R}^+}\int_{\mathbb{R}}|\rho_1(t,x)-\rho_2(t,x)|\,\mathrm{d}x\,\mathrm{d}t.\end{aligned} \tag{5.9}$$

The other terms $\bar{I}_0, \bar{I}_1, \bar{I}_3$ in (5.7) are estimated exactly as in Formulae (3.40), (3.41) and (3.43) in Ref. 11. Therefore, we have

$$\begin{aligned}
&\lim_{r\to 0+} \int_{\mathbb{R}^+}\int_{\mathbb{R}}\int_{\mathbb{R}^+}\int_{\mathbb{R}} \bar{I}_0 \omega_r\left(\frac{x-y}{2}\right)\delta_r\left(\frac{t-s}{2}\right) \mathrm{d}x\,\mathrm{d}t\,\mathrm{d}y\,\mathrm{d}s \\
&\quad = \int_{\mathbb{R}^+}\int_{\mathbb{R}} |\rho_1(t,x)-\rho_2(t,x)|\partial_t\varphi(t,x)\,\mathrm{d}x\,\mathrm{d}t, \\
&\lim_{r\to 0+} \int_{\mathbb{R}^+}\int_{\mathbb{R}}\int_{\mathbb{R}^+}\int_{\mathbb{R}} \bar{I}_1 \omega_r\left(\frac{x-y}{2}\right)\delta_r\left(\frac{t-s}{2}\right) \mathrm{d}x\,\mathrm{d}t\,\mathrm{d}y\,\mathrm{d}s \\
&\quad = \int_{\mathbb{R}^+}\int_{\mathbb{R}} \mathrm{sgn}[\rho_1(t,x)-\rho_2(t,x)][f(t,x,\rho_1(t,x)) \\
&\qquad - f(t,x,\rho_2(t,x))]\partial_x\varphi(t,x)\,\mathrm{d}x\,\mathrm{d}t, \\
&\lim_{r\to 0+} \int_{\mathbb{R}^+}\int_{\mathbb{R}}\int_{\mathbb{R}^+}\int_{\mathbb{R}} \bar{I}_3 \omega_r\left(\frac{x-y}{2}\right)\delta_r\left(\frac{t-s}{2}\right) \mathrm{d}x\,\mathrm{d}t\,\mathrm{d}y\,\mathrm{d}s \\
&\quad = \int_{\mathbb{R}^+}\int_{\mathbb{R}} \mathrm{sgn}[\rho_1(t,x)-\rho_2(t,x)][\partial_x f(t,x,\rho_1(t,x)) \\
&\qquad - \partial_x f(t,x,\rho_2(t,x))]\varphi(t,x)\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}$$

We now closely follow the proof of Theorem 1.1 in Ref. 11. The latter relations, inserted in (5.7) together with (5.9), yield

$$\begin{aligned}
&-\int_{\mathbb{R}^+}\int_{\mathbb{R}} [|\rho_1(t,x)-\rho_2(t,x)|\partial_t\varphi(t,x) \\
&\quad + \mathrm{sgn}[\rho_1(t,x)-\rho_2(t,x)](f(t,x,\rho_1(t,x)) - f(t,x,\rho_2(t,x)))\partial_x\varphi(t,x) \\
&\quad + \mathrm{sgn}[\rho_1(t,x)-\rho_2(t,x)][\partial_x f(t,x,\rho_1(t,x)) - \partial_x f(t,x,\rho_2(t,x))]\varphi(t,x)]\,\mathrm{d}x\,\mathrm{d}t \\
&\quad \le C\int_{\mathbb{R}^+}\int_{\mathbb{R}} |\rho_1(t,x)-\rho_2(t,x)|\,\mathrm{d}x\,\mathrm{d}t,
\end{aligned}$$

for any test function $\varphi \in \mathbf{C}_{\mathbf{c}}^{\infty}$. By Lemma 3 in Ref. 13, the map

$$(\rho_1,\rho_2) \to \mathrm{sgn}(\rho_1-\rho_2)(\partial_x f(t,x,\rho_1) - \partial_x f(t,x,\rho_2))$$

is Lipschitz continuous, hence, possibly renaming the constant,

$$\begin{aligned}
&-\int_{\mathbb{R}^+}\int_{\mathbb{R}} (|\rho_1(t,x)-\rho_2(t,x)|\partial_t\varphi(t,x) \\
&\qquad + \mathrm{sgn}[\rho_1(t,x)-\rho_2(t,x)](f(t,x,\rho_1(t,x)) - f(t,x,\rho_2(t,x)))\partial_x\varphi(t,x))\,\mathrm{d}x\,\mathrm{d}t \\
&\quad \le C\int_{\mathbb{R}^+}\int_{\mathbb{R}} |\rho_1(t,x)-\rho_2(t,x)|\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}$$

Choose arbitrary t_1, t_2 in $]0,T[$ with $t_1 < t_2$ and the test function

$$\varphi(t,x) = \int_{-\infty}^{t} (\delta_r(\tau-t_1) - \delta_r(t_2-\tau))\mathrm{d}\tau \ \int_{-R}^{R} \delta_r(|x-y|)\mathrm{d}y$$

with δ_r as in (5.6) and in the limit $R \to +\infty$ and $r \to 0$, obtain

$$\|\rho_1(t_2) - \rho_2(t_2)\|_{\mathbf{L}^1(\mathbb{R};\mathbb{R})} \leq \|\rho_1(t_1) - \rho_2(t_1)\|_{\mathbf{L}^1(\mathbb{R};\mathbb{R})} + C\int_{t_1}^{t_2} \|\rho_1(\tau) - \rho_2(\tau)\|_{\mathbf{L}^1(\mathbb{R};\mathbb{R})}\, \mathrm{d}\tau.$$

An application of Gronwall lemma allows to conclude the proof, exactly as in the proof of Theorem 1.1 in Ref. 11. □

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