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Hyperbolic Techniques in Modelling, Analysis and Numerics

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Abstract. Invariants of topological spaces of dimension three play a major role in many areas, in particular ...

Introduction by the Organisers

The workshop *Invariants of topological spaces of dimension three*, organised by Max Muster (München) and Bill E. Xample (New York) was well attended with over 30 participants with broad geographic representation from all continents. This workshop was a nice blend of researchers with various backgrounds ...

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Workshop: Hyperbolic Techniques in Modelling, Analysis and Numerics

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Abstracts

A Traffic Model with Phase Transitions at a Junction

Francesca Marcellini
(joint work with Mauro Garavello)

We consider the Phase Transition traffic model in [6], based on a non-smooth 2×2 system of conservation laws,

$$(1)\qquad \begin{cases} \partial_t\rho + \partial_x\left(\rho\, v(\rho,w)\right) = 0 \\ \partial_t(\rho\, w) + \partial_x\left(\rho\, w\, v(\rho,w)\right) = 0 \end{cases} \qquad \text{with} \qquad v = \min\left\{V_{\max}, w\,\psi(\rho)\right\}\,,$$

where ρ is the traffic density, $w = w(t,x)$ is the maximal speed of each driver, ψ is a $\mathbf{C^2}$ function and $V_{\max}$ is a uniform bound on the speed. This is a macroscopic description displaying 2 phases, the *Free* phase F and *Congested* phase C, described by the sets

$$\begin{aligned} F &= \left\{(\rho,w)\in[0,R]\times[\check{w},\hat{w}]\colon v(\rho,\rho w) = V_{\max}\right\}, \\ C &= \left\{(\rho,w)\in[0,R]\times[\check{w},\hat{w}]\colon v(\rho,\rho w) = w\,\psi(\rho)\right\}\,, \end{aligned}$$

where R is the maximal traffic density. This model is an extension of the classical Lighthill-Whitham [12] and Richards [14] model and it falls into the class of second order traffic models introduced by Aw and Rascle in [1] and independently by Zhang in [15]. In 2002, Colombo proposed the first second order model with two different phases in [4, 5]. See also the phase transition models in [2, 10, 13].

1. The Riemann Problem at a Junction

We propose a Riemann solver at a junction for the model in (1) which conserves the number of cars and also the maximal speed w of each vehicle, see [8]. Note that w is a peculiar characteristic of (1), being a specific feature of every single driver.

We consider a junction with n incoming arcs $I_1,\ldots,I_n$ and m outgoing arcs $I_{n+1},\ldots,I_{n+m}$, where each incoming arc is given by $I_i = \left]-\infty,0\right]$ and each outgoing arc is $I_j = \left[0,+\infty\right[$, see [3, 7, 9, 11]. On each arc we consider the phase transition model in (1) with the change of variable $\eta = \rho w$; we get a system where the conserved variables are ρ and η and the speed is $v(\rho,\eta) = \min\left\{V_{\max}, \frac{\eta}{\rho}\,\psi(\rho)\right\}$.

We consider the following Riemann problem

$$(2)\qquad \begin{cases} \begin{cases} \partial_t\rho + \partial_x\left(\rho\, v(\rho,\eta)\right) = 0 \\ \partial_t\eta + \partial_x\left(\eta\, v(\rho,\eta)\right) = 0 \end{cases} & (\rho,\eta)\in I_i \\ \begin{cases} \partial_t\rho + \partial_x\left(\rho\, v(\rho,\eta)\right) = 0 \\ \partial_t\eta + \partial_x\left(\eta\, v(\rho,\eta)\right) = 0 \end{cases} & (\rho,\eta)\in I_j \\ (\rho_i,\eta_i)(0,x) = (\bar\rho_i,\bar\eta_i) \\ (\rho_j,\eta_j)(0,x) = (\bar\rho_j,\bar\eta_j)\,, \end{cases}$$

where $(\bar\rho_i,\bar\eta_i)\in F\cup C$ are the initial data in each incoming arc I_i, $i = 1,\ldots,n$, and $(\bar\rho_j,\bar\eta_j)\in F\cup C$ are the initial data in each outgoing arc I_j, $j = 1,\ldots,m$.

We define the concept of Riemann solver at a generic junction.

Definition 1.1. *A Riemann solver at a junction is a function*

$$\begin{array}{rcl} \mathcal{RS}_J : \prod_{i=1}^{n+m} (F \cup C) & \longrightarrow & \prod_{i=1}^{n+m} (F \cup C) \\ ((\rho_1, \eta_1), \cdots, (\rho_{n+m}, \eta_{n+m})) & \longmapsto & \left((\rho_1^*, \eta_1^*), \cdots, (\rho_{n+m}^*, \eta_{n+m}^*)\right) \end{array}$$

satisfying the following properties.

(1) *The consistency condition holds, i.e.:*

$$\mathcal{RS}_J \left((\rho_1^*, \eta_1^*), \cdots, (\rho_{n+m}^*, \eta_{n+m}^*)\right) = \left((\rho_1^*, \eta_1^*), \cdots, (\rho_{n+m}^*, \eta_{n+m}^*)\right) .$$

(2) *For every $i \in \{1, \ldots, n\}$, the Riemann problem in* (2) *with initial data $(\rho, \eta)(0, x) = (\rho_i, \eta_i)$, with $x < 0$, is solved with waves with negative speed.*

(3) *For every $i \in \{n + 1, \ldots, n + m\}$, the Riemann problem in* (2) *with initial data $(\rho, \eta)(0, x) = (\rho_i, \eta_i)$, with $x > 0$, is solved with waves with positive speed.*

(4) *The traffic distribution*

$$A \begin{bmatrix} \rho_1^* \, v(\rho_1^*, \eta_1^*) \\ \vdots \\ \rho_n^* \, v(\rho_n^*, \eta_n^*) \end{bmatrix} = \begin{bmatrix} \rho_{n+1}^* \, v(\rho_{n+1}^*, \eta_{n+1}^*) \\ \vdots \\ \rho_{n+m}^* \, v(\rho_{n+m}^*, \eta_{n+m}^*) \end{bmatrix}$$

holds, where $A = (\alpha_{i,j})_{i=1,\ldots,n;\, j=n+1,\ldots,n+m}$, whose coefficients indicate the percentage of traffic that passes from I_i to I_j, with $\sum_{j=n+1}^{n+m} \alpha_{ij} = 1$.

(5) *The mass conservation holds, i.e. $\sum_{i=1}^{n} \rho_i^* v\left(\rho_i^*, \eta_i^*\right) = \sum_{i=n+1}^{n+m} \rho_i^* v\left(\rho_i^*, \eta_i^*\right)$.*

(6) *The distribution of the maximal speed holds, i.e.:*

$$w_{n+1}^* = \frac{1}{\sum_{i=1}^{n} \alpha_{i,n+1} \gamma_i^*} \left[\alpha_{1,n+1} \gamma_1^* w_1^* + \ldots + \alpha_{n,n+1} \gamma_n^* w_n^*\right] ,$$

$$\vdots$$

$$w_{n+m}^* = \frac{1}{\sum_{i=1}^{n} \alpha_{i,n+m} \gamma_i^*} \left[\alpha_{1,n+m} \gamma_1^* w_1^* + \ldots + \alpha_{n,n+m} \gamma_n^* w_n^*\right] ,$$

where $w_i^* = \frac{\eta_i^*}{\rho_i^*}$ and $\gamma_i^* = \rho_i^* v\left(\rho_i^*, \eta_i^*\right)$ for every $i \in \{1, \ldots, n + m\}$.

For special junctions, the cases of $1 \times m$ and 2×1 junctions, we prove that the Riemann solver is well defined. The following result holds (see [8] for the proof).

Theorem 1.2. *Under the assumptions*

(H-1): *$R, \check{w}, \hat{w}, V_{\max}$ are positive constants, with $\check{w} < \hat{w}$; $\check{w}$ and $\hat{w}$ are the minimum, respectively, maximum, of the maximal speeds of each vehicle;*

(H-2): *$\psi \in \mathbf{C^2}\left([0, R]; [0, 1]\right)$ with $\psi(0) = 1$, $\psi(R) = 0$, $\psi'(\rho) \leq 0$ and $\frac{d^2}{d\rho^2} \left(\rho \, \psi(\rho)\right) \leq 0$, for all $\rho \in [0, R]$;*

(H-3): *$\check{w} > V_{\max}$;*

(H-4): *the waves of the first family in C have negative speed,* *the Riemann solver* $\mathcal{RS}_J$ *for the cases of* $1 \times m$ *and* 2×1 *junctions, constructed as in* [8, Section 4, Section 5]*, satisfies all the conditions of Definition 1.1 and produces a solution to the Riemann problem (2).*

Remark 1.3. *We note that the distribution of the maximal speed in (6) of Definition 1.1 is given by*

$$w_2^* = \ldots = w_{1+m}^* = \bar{w}_1 ,$$

in the case of $1 \times m$ *junction and is given by*

$$w_3 = \frac{\gamma_1}{\gamma_1 + \gamma_2} \bar{w}_1 + \frac{\gamma_2}{\gamma_1 + \gamma_2} \bar{w}_2 ,$$

where $\gamma_1 = \rho_1 v(\rho_1, \eta_1)$ *and* $\gamma_2 = \rho_2 v(\rho_2, \eta_2)$*, in the case of* 2×1 *junction, see* [8].

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