Mixed Systems in the Description of Traffic Flow

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We present mixed systems in the description of dynamics of traffic flow. In particular, we consider two different frameworks, both consisting in the coupling of systems of different types and both displaying 2 phases.

The first one is the Free–Congested model, see [9], where a scalar conservation law is coupled with a 2×2 system. The result is a macroscopic model based on a non-smooth 2×2 system of conservation laws and displaying 2 distinct phases: *Free* and *Congested*. Then, we present the coupling of a micro- and a macroscopic models, the former consisting in a system of ordinary differential equations and the latter in a scalar conservation law, see [8].

We recall at first the classical Lighthill-Whitham [14] and Richards [16] (LWR) traffic model

$$\partial_t \rho + \partial_x (\rho V) = 0 \tag{1}$$

which is a scalar conservation law, where $\rho = \rho(t,x)$ is the (mean) traffic density and $V = V(\rho)$ is the (mean) traffic speed.

Concerning the Free–Congested model, we consider the LWR model and than we extend this model with two different assumptions on the speed V. At first, we assume that, at a given density, different drivers may differ in their *maximal* speed w, so that $V = w\,\psi(\rho)$, with $w \in [\check{w}, \hat{w}], \check{w} > 0$. The function ψ describes the attitude of drivers to choose their speed depending on the traffic density at their location and the maximal speed w is a specific feature of every single driver. Thus we are lead to study the system:

$$\left\{ \begin{array}{l} \partial_t \rho + \partial_x(\rho v) = 0 \\ \partial_t w + v\, \partial_x w = 0 \end{array} \right. \qquad \text{with} \qquad v = w\,\psi(\rho)\,. \tag{2}$$

The role of the second equation above is to let the maximal velocity w be propagated with the traffic speed. We identify the different behaviors of the different drivers by means of their maximal speed, see also [4, 5].
The second assumption on the speed is the introduction of a uniform bound, a constant $V_{\max}$ that the drivers do not exceed. We obtain the following model:

$$\left\{ \begin{array}{l} \partial_t \rho + \partial_x(\rho v) = 0 \\ \partial_t w + v\, \partial_x w = 0 \end{array} \right. \qquad \text{with} \qquad v = \min\left\{V_{\max},\, w\,\psi(\rho)\right\}\,. \tag{3}$$

As a consequence of the introduction of this speed bound we have the formation of the two distinct phases, *Free* and *Congested*. The phases are presented in Figure 1; see [9, Section 2] for the notations.

The system in (3) is not in conservation form for the second equation, so similarly to [2, formula (3.1)], [3, formula (2.2)], [13, formula (1)], [15], we choose to reformulate (3) in conservation form, as follows:

$$\left\{ \begin{array}{l} \partial_t \rho + \partial_x \left(\rho\, v(\rho,\eta)\right) = 0 \\ \partial_t \eta + \partial_x \left(\eta\, v(\rho,\eta)\right) = 0 \end{array} \right. \qquad \text{with} \qquad v(\rho,\eta) = \min\left\{V_{\max}, \frac{\eta}{\rho}\,\psi(\rho)\right\}. \tag{4}$$

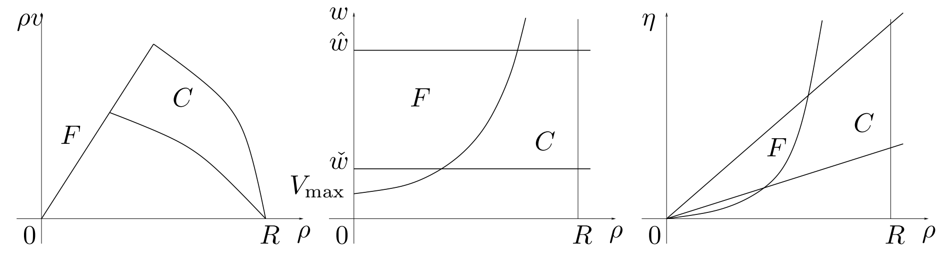

FIGURE 1. The phases F and C in the coordinates $(\rho, \rho v)$, (ρ, w) and (ρ, η).

This model consists of a 2×2 system of conservation laws with a $\mathbf{C^{0,1}}$ but not $\mathbf{C^1}$ flow. Note in fact that $\frac{\eta}{\rho} = w \in [\check{w}, \hat{w}]$.

We study the Riemann Problem for (4), see [9, Section 2]. This model is also compared with other models of the same type in the current literature [2, 7, 17], as well as with a kinetic one [5].

The second traffic flow model that we present is the Micro–Macro model, see [8], consisting of a macroscopic and a microscopic descriptions glued together. The macroscopic part is described again through the LWR traffic model, as in (1), and the microscopic part through a Follow–the–Leader (FtL) model, see [1].

Microscopic models for vehicular traffic consist of a finite set of ordinary differential equations, describing the motion of each vehicle in the traffic flow. Below we consider a first order FtL model, where each driver adjusts his/her velocity to the vehicle in front, that is

$$\dot{p}_i = v\left(\frac{\ell}{p_{i+1} - p_i}\right). \tag{5}$$

Here, $p_i = p_i(t)$ is the position of the i-th driver, for $i = 1, \ldots, n$, and $p_{i+1} - p_i \geq \ell$ for all $i = 1, \ldots, n-1$, the fixed parameter ℓ denoting the (mean) vehicles' length. Here, $\ell/(p_{i+1} - p_i)$ is the local traffic density seen by the driver p_i. Equation (5) needs to be closed with the trajectory of the first driver p_n. Throughout, we carefully select assumptions allowing us to prove that all speeds are bounded.

Our aim is to consider a general situation in which the two descriptions (1) and (5) are alternatively used in different segments of the real line. A similar approach to traffic modeling is in [12], where the interface between the micro- and macro description is kept fixed and the model in [2, 17] plays the role here played by the LWR one. See also [10] for the case $n = 1$.

Some numerical results complete the study of the model and prove the reasonableness of it's solutions: in particular they expain how the two micro- and macroscopic descriptions coexist in a single model, although being separated.

The Free–Congested and the Micro–Macro descriptions, both consist in the coupling of systems of different types and both display 2 phases. Moreover both the two descriptions display "free boundaries" models to be determined. Another analogy is the study of phase transitions, see also [6, 7, 9, 11]. In the Free–Congested model a vehicle can enter in or exit from one of the two distinct phases, depending on the traffic conditions. This does not occur in the Micro-Macro model: there is a backward propagating exchange of information between the different phases, although there is no exchange of mass.

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