Generation of finite groups

with applications to computing normalisers

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Bound is best possible:

- If n is even then $C_2^{n/2} \leq \mathsf{S}_n$, and $d(C_2^{n/2}) = n/2$.

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Key ingredient of proof is:

Lemma (Wielandt)

Let $P \leq \mathsf{S}_{p^m}$ be a transitive p-group.

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Can take $c = 0.92$, or $\sqrt{3}/2$ with finitely many exceptions.

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Corollary

If there exists a d s.t. a generic subgroup G of S_n lies in Γ_d, then the set stabiliser and intersection problem are generically polynomial-time.

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- Testing conjugacy of subgroups is hard in general, however there are various quick tests to show that two groups are NOT conjugate.

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- Once the image of $k = 3$ points in $\{1, \ldots, n/2\}$ are specified under $g \in S_n$, we know x^g for all $x \in H$ (up to E). So we can decide whether $g \in N$.
- N permutes sets of linearly dependent columns of M_H.
- For all $\alpha_i \in \{1, \ldots, n\}$, point stabilisers $H_{(\alpha_1, \alpha_2, \ldots, \alpha_i)}$ can be efficiently calculated from M_H.
- Let K_H be a row rank $(n/2 - k)$ matrix whose nullspace is H. Then N also acts naturally on columns of K_H, and similar observations apply.

Some timings

Our methods for the normaliser of subgroups of $C_p^{n/p}$ are still worst case exponential.

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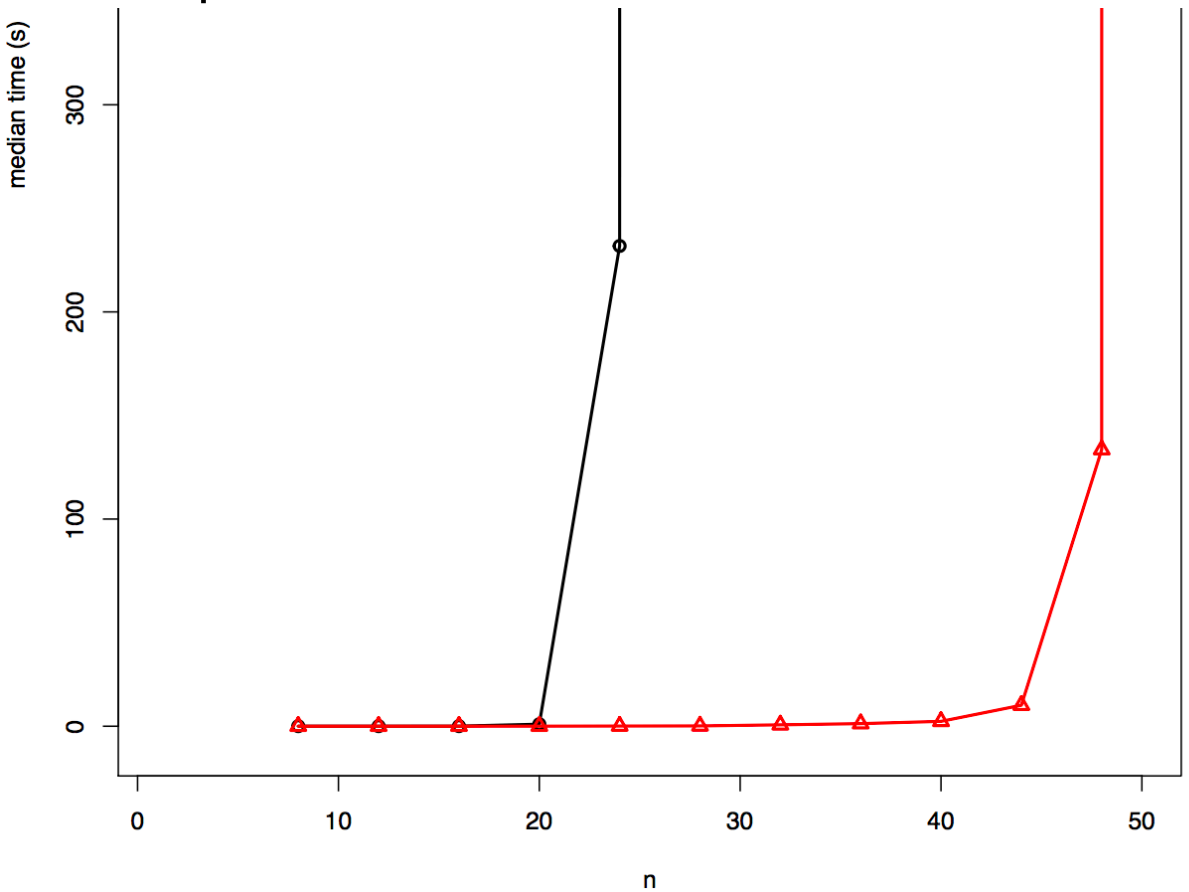

Red = our algorithm

Black = standard algorithm.

Time recorded is median over 20 random instances, with a 600 second timeout.