A PRODUCT FORMULA FOR SPHERICAL REPRESENTATIONS OF A GROUP OF AUTOMORPHISMS OF A HOMOGENEOUS TREE. I.

DONALD I. CARTWRIGHT¹, GABRIELLA KUHN² AND PAOLO M. SOARDI²

Abstract. Let $G = \text{Aut}(T)$ be the group of automorphisms of a homogeneous tree $T$, and let $\Gamma$ be a lattice subgroup of $G$. Let $\pi$ be the tensor product of two spherical irreducible unitary representations of $G$. We give an explicit decomposition of the restriction of $\pi$ to $\Gamma$. We also describe the spherical component of $\pi$ explicitly, and this decomposition is interpreted as a multiplication formula for associated orthogonal polynomials.

§1 Introduction and notation.

Let $G$ be the group of automorphisms of a homogeneous tree $T$. We fix a vertex $o$ of $T$, and let $K = \{g \in G : go = o\}$. As $G$ is a type I group [3, p. 112], each continuous unitary representation $\pi$ can be written in an essentially unique way as a direct integral $\int_\hat{G} \sigma \, dm(\sigma)$ [5, Thms. 2.15, 1.21]. Now (see [3]) $\hat{G}$ consists of the equivalence classes of (a) the spherical irreducible unitary representations (those having nonzero $K$-invariant vectors), (b) two special representations, and (c) an infinite sequence of cuspidal representations. The representations in (b) and (c) make up the discrete series of $G$. Thus the representation space $H_\pi$ of $\pi$ can be decomposed as an orthogonal direct sum

$$H_\pi = H_1 + H_2,$$

(1.1)

of $\pi$-invariant subspaces $H_1$ and $H_2$, where $H_1$ is the closed linear span in $H_\pi$ of the set of vectors $\pi(g)\xi$, where $g \in G$ and where $\xi$ is $K$-invariant. The $H_1$ component $\pi_1$ of $\pi$ must be a direct integral

$$\pi_1 = \int_\hat{G} \sigma \, dm(\sigma),$$

(1.2)

where $m$ is supported on the spherical part of $\hat{G}$, while the $H_2$ component $\pi_2$ of $\pi$ must be a direct sum

$$\pi_2 = \sum_k m_k\sigma_k,$$

(1.3)

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over the distinct discrete series representations \(\sigma_k\) of \(G\), where \(m_k \in \{0, 1, \ldots, \infty\}\) for each \(k\).

In this paper we consider the case when \(\pi\) is the tensor product of two spherical irreducible unitary representations of \(G\). We give an explicit description of \(\pi_1\) in Theorem 4.2 below. We defer the detailed description of \(\pi_2\) to [1], where somewhat different methods are needed. However, if \(\Gamma\) is a lattice subgroup of \(G\), we completely describe the restriction \(\pi|_\Gamma\) of \(\pi\) to \(\Gamma\) (see Theorem 5.6 below). This does not require a detailed knowledge of \(\pi_2\).

The groups \(G\) and \(\Gamma\) considered in this paper share with such groups as \(SL_2(\mathbb{R})\) and \(SL_2(\mathbb{Q}_p)\) the following feature: either exactly one of the complementary series representations appears in the decomposition of the tensor product, or the complementary series does not appear at all (see [6] and [8]).

Our method for finding a description for \(\pi_1\) is based on a formula for the product of the spherical functions associated with spherical representations (see below). This product formula is interpreted in Section 3 as a product formula for certain orthogonal polynomials, which turns out to be a special case of a product formula proved by other methods in [7].

We start by recalling the definition of a spherical representation of a group of automorphisms of a homogeneous tree. We shall mostly follow the notation in [3], but it will be convenient to use a different parametrization for these representations.

Let \(q \geq 2\) be an integer, let \(T\) be a homogeneous tree of degree \(q + 1\), and let \(d(x,y)\) denote the usual distance between vertices \(x, y\) of \(T\). For \(n \geq 0\) and \(x \in T\), the number \(N_n\) of vertices \(y\) satisfying \(d(x,y) = n\) is 1 if \(n = 0\), and \((q + 1)q^{n-1}\) if \(n \geq 1\). Let \(\Omega\) denote the space of ends of \(T\), i.e., equivalence classes \(\omega\) of infinite chains in \(T\). If \(x \in T\), let \([x,\omega]\) denote the unique infinite chain in the class \(\omega\) having initial vertex \(x\), and let \(\omega_i(x)\) denote the \(i\)th vertex of this chain. If \(x, y \in T\), let \(\Omega_x(y)\) denote the set of \(\omega \in \Omega\) such that \(y\) is a vertex in \([x,\omega]\). Fixing \(x \in T\), there is a totally disconnected compact topology on \(\Omega\) having the sets \(\Omega_x(y), y \in T\), as basis; this topology does not depend on \(x\). We denote by \(\nu_x\) the regular Borel probability measure on \(\Omega\) satisfying \(\nu_x(\Omega_x(y)) = 1/N_n\) whenever \(d(x,y) = n\). For \(x, y \in T\), \(\nu_x\) and \(\nu_y\) are mutually absolutely continuous, with

\[
P(x,y,\omega) := \frac{d\nu_y}{d\nu_x}(\omega) = q^{\delta(x,y,\omega)},
\]

where \(\delta(x,y,\omega)\) is the unique integer \(k\) such that \(\omega_i(y) = \omega_{i+k}(x)\) for all large \(i\) [3, p. 35].

Note that \(G\) acts on \(\Omega\) in a natural way. For \(g \in G\) write \(P(g,\omega)\) in place of \(P(o,go,\omega)\). Let \(K(\Omega)\) denote the space of locally constant functions on \(\Omega\) [3, p. 36]. For \(z \in \mathbb{C}\), we can define the spherical representation \(\pi_z\) of \(G\) on \(K(\Omega)\) by

\[
(\pi_z(g)f)(\omega) = P^z(g,\omega)f(g^{-1}\omega).
\]

Writing

\[
\langle f, g \rangle = \int_{\Omega} f(\omega)\overline{g(\omega)}\,d\nu_o(\omega),
\]

the “spherical function” defined on \(G\) by

\[
g \mapsto \langle \pi_z(g)\mathbf{1}, \mathbf{1} \rangle = \int_{\Omega} P^z(g,\omega)\,d\nu_o(\omega)
\]

for \(z \in \mathbb{C}\), is denoted by \(\phi_z\).
depends only on \( n = d(o, go) \). Here \( 1 \) is the function taking the constant value 1 on \( \Omega \).

The explicit formula for the spherical function is neatest if we now change the parametrization of the spherical representations. Write

\[ s = q^{\frac{k}{2} - z}, \quad (1.5) \]

and if \( s \in \mathbb{C} \setminus \{0\} \) and \( z \in \mathbb{C} \) are so related, we write \( \pi^s \) in place of \( \pi_z \). Thus

\[ (\pi^s(g)f)(\omega) = \left( \frac{s}{\sqrt{q}} \right)^{-\delta(o, go, \omega)} f(g^{-1} \omega) \quad \text{for} \ f \in \mathcal{K}(\Omega). \quad (1.6) \]

Write

\[ \varphi_n(s) = \langle \pi^s(g)1, 1 \rangle = \int_{\Omega} (s/\sqrt{q})^{-\delta(o, go, \omega)} d\nu_o(\omega) \quad \text{if} \ d(o, go) = n. \quad (1.7) \]

Then \( \varphi_n \) is analytic on \( \mathbb{C} \setminus \{0\} \), and (see [3, p. 43])

\[ \varphi_n(s) = q^{-n/2} (c(s)s^n + c(s^{-1})s^{-n}) \quad \text{if} \ s \neq 0, \pm 1, \quad (1.8) \]

where

\[ c(s) = \frac{qs - s^{-1}}{(q + 1)(s - s^{-1})} \quad \text{for} \ s \neq 0, \pm 1. \quad (1.9) \]

Also,

\[ \varphi_n(s) = \frac{s^n}{(q + 1)q^{n/2}} ((q + 1) + (q - 1)n) \quad \text{if} \ s = \pm 1. \]

For \( s \in \mathbb{T} = \{s \in \mathbb{C} : |s| = 1\} \) (equivalently, \( \Re z = 1/2 \)), \( \langle \cdot, \cdot \rangle \) is preserved by each \( \pi^s(g) \), and so \( \pi^s = \pi_z \) extends to a unitary representation on \( L^2(\Omega, \nu_o) \); these representations are called the principal series spherical representations. In addition, for \( s \) or \(-s\) in \( (q^{-1/2}, q^{1/2}) \setminus \{1\} \) (equivalently, \( 1/2 \neq \Re z \in (0, 1), \Im z = \pi k/\log q \) for some \( k \in \mathbb{Z} \)), there is an inner product \( \langle \cdot, \cdot \rangle \) on \( \mathcal{K}(\Omega) \) preserved by each \( \pi^s(g) \), and so \( \pi^s \) extends to a unitary representation on the completion \( H_s \) of \( \mathcal{K}(\Omega) \) with respect to this inner product. If \( s = \sqrt{q} \) or \(-\sqrt{q} \) (equivalently, \( z = 2k\pi i/\ln q \) or \( 1 + 2k\pi i/\ln q \) for some \( k \in \mathbb{Z} \)), then \( \varphi_n(s) = 1 \) for all \( n \), and we define \( \pi^s \) to be the trivial character of \( G \), instead of using the above definition. Also, if \( s = -\sqrt{q} \) or \(-1/\sqrt{q} \) (equivalently, \( z = (2k + 1)\pi i/\ln q \) or \( 1 + (2k + 1)\pi i/\ln q \)), then \( \varphi_n(s) = (-1)^n \) for all \( n \), and we define \( \pi^s \) to be the character \( g \mapsto (-1)^{d(o, go)} \) of \( G \), instead of using the above definition. The representations \( \pi^s \) for \( s \) or \(-s\) in \( [q^{-1/2}, q^{1/2}] \setminus \{1\} \) are called the complementary series spherical representations.

Parameter set for Complementary Series.

Parameter set for Principal Series.

Figure 1.
If $ds$ denotes the usual normalized measure on $\mathbb{T}$, let $\mu$ be the “Plancherel measure” on $\mathbb{T}$ given by

$$d\mu(s) = \frac{q}{2(q + 1)} \frac{1}{|c(s)|^2} \, ds$$

(1.10)

(notice that $1/|c(s)|^2$ extends to a continuous function on $\mathbb{T}$). Then it is well known and easy to verify that

$$\int_{\mathbb{T}} \varphi_m(s) \varphi_n(s) \, d\mu(s) = \frac{1}{N_n} \delta_{m,n}. \quad (1.11)$$

§2 Formulas for $\varphi_n(s_1)\varphi_n(s_2)$.

Proposition 2.1. If $s_1, s_2, s_3 \in \mathbb{C} \setminus \{0\}$ satisfy

$$|s_1^{\epsilon_1} s_2^{\epsilon_2} s_3^{\epsilon_3}| < \sqrt{q}$$

for each $\epsilon = (\epsilon_1, \epsilon_2, \epsilon_3) \in \{+1, -1\}^3$, (2.1)

then

$$\sum_{n=0}^{\infty} N_n \varphi_n(s_1) \varphi_n(s_2) \varphi_n(s_3) = \frac{q(q - 1)}{(q + 1)^2} \prod_{j=1}^{3} \left\{ \left( 1 - \frac{s_j^2}{q} \right) \left( 1 - \frac{s_{j+1}^2}{q} \right) \right\} \prod_{\epsilon} \left( 1 - \frac{1}{\sqrt{q}} s_1^{\epsilon_1} s_2^{\epsilon_2} s_3^{\epsilon_3} \right)$$

(2.2)

where the product is over all eight $\epsilon = (\epsilon_1, \epsilon_2, \epsilon_3) \in \{+1, -1\}^3$.

Proof. Assume first that $s_j \neq \pm 1$ for each $j$, so that $\varphi_n(s_j)$ is given by setting $s = s_j$ in (1.8). For $n \geq 1$, the $n$th summand on the left in (2.2) is then $(q + 1)/q$ times the sum of the eight terms

$$c(s_1^{\epsilon_1}) c(s_2^{\epsilon_2}) c(s_3^{\epsilon_3}) \left( s_1^{\epsilon_1} s_2^{\epsilon_2} s_3^{\epsilon_3} / \sqrt{q} \right)^n$$

Thus, assuming (2.1), the sum on the left in (2.2) is a sum of eight convergent geometric series. So the sum on the left in (2.2) is a sum of eight rational expressions in the $s_j$’s, which after some tedious algebra tidies up to the expression on the right in (2.2).

Suppose that $s_1, s_2, s_3 \in \mathbb{C}$ satisfy (2.1), but that one of the $s_j$’s, say $s_3$, equals $\pm 1$. Formula (1.7) implies that $|\varphi_n(s)| \leq \varphi_n(|s|)$, and so for arbitrary $s_3 \in \mathbb{T}$ we have $|\varphi_n(s_3)| \leq \varphi_n(1) \leq (n + 1)/\sqrt{q}$. Hence convergence of the left hand side of (2.2) is uniform with respect to $s_3 \in \mathbb{T}$ for any fixed $s_1, s_2$ satisfying $|s_1^{\epsilon_1} s_2^{\epsilon_2}| < \sqrt{q}$ for each $(\epsilon_1, \epsilon_2) \in \{+1, -1\}^2$. Hence the sum is continuous function of $s_3$. But the right hand side in (2.2) is also a continuous function of $s_3$ on $\mathbb{T}$ for any fixed such $s_1, s_2$, and so (2.2) is valid also at $s_3 = \pm 1$ (assuming (2.1), of course). Similarly, (2.2) remains valid if two or three of the $s_j$’s are $\pm 1$, and so the proposition is proved.

Let $K(s_1, s_2, s_3)$ denote the right hand side of (2.2) whenever the denominator is nonzero. Notice that $K(s_1, s_2, s_3)$ is symmetric in $s_1, s_2$ and $s_3$, and is unchanged if any $s_j$ is replaced by $s_j^{-1}$. Notice also that $K(s_1, s_2, s_3) > 0$ for all $s_3 \in \mathbb{T}$ when $s_1$ and $s_2$ are parameters corresponding to either the principal or complementary series, except that when $s_1$ or $s_2 \in \{ \pm \sqrt{q}, \pm 1/\sqrt{q} \}$, $K(s_1, s_2, s_3) = 0$ for all $s_3 \in \mathbb{T}$.2
If $|s_1^\epsilon s_2^\epsilon| \neq \sqrt{q}$ for each $\epsilon = (\epsilon_1, \epsilon_2) \in \{+1, -1\}^2$, then $K(s_1, s_2, s_3)$ is a continuous function of $s_3$ on $\mathbb{T}$, and is therefore integrable with respect to $\mu$.

**Proposition 2.2.** Consider $s_1, s_2 \in \mathbb{C} \setminus \{0\}$ satisfying

$$|s_1^\epsilon s_2^\epsilon| < \sqrt{q} \quad \text{for each } \epsilon = (\epsilon_1, \epsilon_2) \in \{+1, -1\}^2. \quad (2.3)$$

Then the following “multiplication formula” holds for each $m \in \mathbb{N}$:

$$\varphi_m(s_1)\varphi_m(s_2) = \int_\mathbb{T} K(s_1, s_2, s_3)\varphi_m(s_3) \, d\mu(s_3). \quad (2.4)$$

**Proof.** Apply (2.2) to $(s_1, s_2, s_3)$ for arbitrary $s_3 \in \mathbb{T}$. We multiply both sides of (2.2) by $\varphi_m(s_3)$, integrate with respect to $d\mu(s_3)$, and get (2.4). This is valid because the convergence of the sum on the left in (2.2) is uniform with respect to $s_3 \in \mathbb{T}$, as we saw in the proof of Proposition 2.1 above.

We next show that (2.4) holds in another case.

**Proposition 2.3.** Suppose that $s_1, s_2 \in \mathbb{C} \setminus \{0\}$, and that one of the $s_1^\epsilon s_2^\epsilon$’s, where $(\epsilon_1, \epsilon_2) \in \{+1, -1\}^2$, equals $\pm \sqrt{q}$, and the other three have modulus less than $\sqrt{q}$. Then $K(s_1, s_2, s_3)$ is integrable with respect to $d\mu(s_3)$, and (2.4) holds.

**Proof.** By the symmetry properties of $K(s_1, s_2, s_3)$, and because $\varphi_n(s) = \varphi_n(s^{-1})$, we may suppose that $s_1 s_2 = \sqrt{q}$ or $-\sqrt{q}$. For $s_3 \in \mathbb{T}$, we see from (1.9) and (2.2) that

$$\frac{q}{2(q + 1)} \frac{K(s_1, s_2, s_3)}{|c(s_3)|^2} = \frac{(q - 1) |s_3^2 - 1|^2 \prod_{j=1}^2 \{1 - \frac{s_j^2}{q}\}}{2(q + 1) \prod_i \left(1 - \frac{1}{\sqrt{q}} s_1^\epsilon s_2^\epsilon s_3^\epsilon\right)} \quad (2.5)$$

where the product in the denominator is over all eight $\epsilon = (\epsilon_1, \epsilon_2, \epsilon_3) \in \{+1, -1\}^3$.

The product of two of the factors in the denominator in (2.5) is

$$(1 - s_3)(1 - s_3^{-1}) \quad \text{or} \quad (1 + s_3)(1 + s_3^{-1}) \quad (2.6)$$

In either case, the factor $|s_3^2 - 1|^2$ in the numerator on the right in (2.5) allows us to cancel the two factors in (2.6). As the other six factors in the denominator are bounded away from zero, $K(s_1, s_2, s_3)$ is integrable with respect to $d\mu(s_3)$. Now let $0 < r < 1$. Then $r s_1, s_2$ satisfy (2.3) if $r$ is close to 1. Elementary calculus shows that $|1 - re^{i\theta}|/|1 - re^{i\theta}| \leq 2/(1 + r) < 2$ for $\theta \in \mathbb{R}$ and $1 \neq r > 0$. Using this, we see that for $r < 1$ close to 1, $|K(rs_1, s_2, s_3)| \leq M|K(s_1, s_2, s_3)|$ for a constant $M$. The Dominated Convergence Theorem shows that

$$\int_\mathbb{T} K(rs_1, s_2, s_3)\varphi_m(s_3) \, d\mu(s_3) \rightarrow \int_\mathbb{T} K(s_1, s_2, s_3)\varphi_m(s_3) \, d\mu(s_3) \quad \text{as } r \rightarrow 1. \quad (2.7)$$

Certainly the left hand side of (2.4) is continuous in $s_1$ and $s_2$, and so Proposition 2.2 and (2.7) show that (2.4) holds.

One can show that, apart from under the conditions of Propositions 2.2 and 2.3, there is only one other case when (2.4) holds: when one of the $s_1^\epsilon s_2^\epsilon$’s equals $\sqrt{q}$,
and another equals $-\sqrt{q}$. We omit the proof, as this cannot arise when $s_1, s_2$ are parameters corresponding to the principal or complementary series.

**Corollary 2.4.** Suppose that either

(i) $s_1, s_2$ are parameters corresponding to the principal series, or that

(ii) $s_1$ corresponds to the principal series and $s_2 \notin \{\pm \sqrt{q}, \pm 1/\sqrt{q}\}$ corresponds to the complementary series, or vice versa, or

(iii) $s_1, s_2 \notin \{\pm \sqrt{q}, \pm 1/\sqrt{q}\}$ correspond to the complementary series, and $|s_1^e s_2^f| \leq \sqrt{q}$ for each $(e_1, e_2) \in \{-1, +1\}^2$.

Then (2.4) holds, and $K(s_1, s_2, s_3) > 0$ for all $s_3 \in \mathbb{T}$.

We remark that when, say, $s_2 \in \{\sqrt{q}, 1/\sqrt{q}\}$, then $\varphi_m(s_2) = 1$ for all $m$, and there is a simple multiplication formula: $\varphi_m(s_1) \varphi_m(s_2) = \varphi_m(s_1)$. When $s_2 \in \{-\sqrt{q}, -1/\sqrt{q}\}$, then $\varphi_m(s_2) = (-1)^m$ for all $m$, and we have $\varphi_m(s_1) \varphi_m(s_2) = \varphi_m(-s_1)$.

Notice that part (iii) of Corollary 2.4, and the last remark, do not cover all cases when $s_1, s_2$ correspond to the complementary series. The region covered is indicated by $R_s$ in Figure 2. In terms of parameters $z_j$ satisfying $s_j = q^{1/2 - z_j}$, the region covered is the tilted square $R_z$ indicated in Figure 3.

![Figure 2](image1)

![Figure 3](image2)

We now consider modifications of (2.4) which hold when $|s_1^e s_2^f| > \sqrt{q}$ for some $(e_1, e_2)$. When $s_1, s_2$ correspond to the complementary series, this will cover the cases not covered above. We may assume that $|s_1 s_2|$ is the largest of the $|s_1^e s_2^f|$.

**Proposition 2.5.** Suppose that $s_1, s_2 \in \mathbb{C} \setminus \{0\}$ satisfy

$$|s_1 s_2| > \sqrt{q} > \max\{|s_1 s_2^{-1}|, |s_1^{-1} s_2|, |s_1^{-1} s_2^{-1}|\}. \quad (2.9)$$

Then for $A = c(s_1)c(s_2)/c(s_1 s_2/\sqrt{q})$, we have

$$\varphi_m(s_1) \varphi_m(s_2) = A \varphi_m(s_1 s_2/\sqrt{q}) + \int_\mathbb{T} K(s_1, s_2, s_3) \varphi_m(s_3) \, d\mu(s_3). \quad (2.10)$$

**Proof.** As is implicit in the proof of Proposition 2.1, we have

$$K(s_1, s_2, s_3) = 1 + \frac{q + 1}{q} \sum_{e} \frac{c(s_1^e) c(s_2^e) c(s_3^e) s_1^{e_1} s_2^{e_2} s_3^{e_3}}{\sqrt{q} - s_1^{e_1} s_2^{e_2} s_3^{e_3}}.$$
whenever \( s_1^{\epsilon_1} s_2^{\epsilon_2} s_3^{\epsilon_3} \neq \sqrt{q} \) for all \( \epsilon \) (the sum is over all eight \( \epsilon = (\epsilon_1, \epsilon_2, \epsilon_3) \in \{+1, -1\}^3 \)). Using (1.10) and (1.11), and \( c(s) = c(s^{-1}) \) for \( s \in \mathbb{T} \), and writing \( I_m = \int_{\mathbb{T}} K(s_1, s_2, s_3) \varphi_m(s_3) \, d\mu(s_3) \), we see that

\[
I_m = \delta_{m,0} + \frac{1}{2} \sum_{\epsilon} c(s_1^{\epsilon_1})c(s_2^{\epsilon_2})s_1^{\epsilon_1} s_2^{\epsilon_2} \int_{\mathbb{T}} \frac{s_3^{\epsilon_3} \varphi_m(s_3)}{\sqrt{q} - s_1^{\epsilon_1} s_2^{\epsilon_2} s_3^{\epsilon_3}} \, ds_3. \tag{2.11}
\]

Now using \( \int_{\mathbb{T}} f(s) \, ds = \int_{\mathbb{T}} f(s^{-1}) \, ds \) and \( \varphi_m(s) = \varphi_m(s^{-1}) \), we see that for given \( \epsilon_1, \epsilon_2 \in \{+1, -1\} \), the integral in (2.11) is the same for \( \epsilon_3 = +1 \) as for \( \epsilon_3 = -1 \). So

\[
I_m = \delta_{m,0} + \sum_{\epsilon_1, \epsilon_2 = \pm 1} c(s_1^{\epsilon_1})c(s_2^{\epsilon_2})s_1^{\epsilon_1} s_2^{\epsilon_2} \left\{ \frac{1}{2\pi i} \int_0^{2\pi} \frac{e^{i\theta} \varphi_m(e^{i\theta})}{(\sqrt{q} - s_1^{\epsilon_1} s_2^{\epsilon_2} e^{i\theta})c(e^{-i\theta})} \, d\theta \right\}.
\]

The expression is the braces equals

\[
\frac{1}{2\pi i} \oint_{|w|=1} \frac{\varphi_m(w)}{(\sqrt{q} - s_1^{\epsilon_1} s_2^{\epsilon_2} w)c(w^{-1})} \, dw, \tag{2.12}
\]

which we can evaluate using the residue theorem. Assuming \( |s_1^{\epsilon_1} s_2^{\epsilon_2}| < \sqrt{q} \), the singularity \( \sqrt{q}/(s_1^{\epsilon_1} s_2^{\epsilon_2}) \) of the integrand in (2.12) is outside the unit circle. Next write \( \varphi_m(w) = q^{-m/2}(c(w)w^m + c(w^{-1})w^{-m}) \). Now

\[
\oint_{|w|=1} \frac{c(w)w^m}{(\sqrt{q} - s_1^{\epsilon_1} s_2^{\epsilon_2} w)c(w^{-1})} \, dw = \oint_{|w|=1} \frac{w^m}{\sqrt{q} - s_1^{\epsilon_1} s_2^{\epsilon_2} w} \frac{qw^2 - 1}{w^2 - q} \, dw = 0,
\]

because the integrand is analytic on and inside the unit circle. On the other hand,

\[
\oint_{|w|=1} \frac{c(w^{-1})w^{-m}}{(\sqrt{q} - s_1^{\epsilon_1} s_2^{\epsilon_2} w)c(w^{-1})} \, dw = \oint_{|w|=1} \frac{1}{\sqrt{q}} \left\{ \sum_{k=0}^{\infty} \left( \frac{s_1^{\epsilon_1} s_2^{\epsilon_2} w}{\sqrt{q}} \right)^k \right\} \frac{1}{w^m} \, dw,
\]

which equals \( 2\pi i(s_1^{\epsilon_1} s_2^{\epsilon_2})^{m-1}/q^{m/2} \) if \( m \geq 1 \) and equals 0 if \( m = 0 \).

If, however, \( |s_1^{\epsilon_1} s_2^{\epsilon_2}| > \sqrt{q} \), then the simple pole \( \sqrt{q}/(s_1^{\epsilon_1} s_2^{\epsilon_2}) \) of the integrand in (2.12) is inside the unit circle, and the residue is

\[
- \frac{\varphi_m(\sqrt{q}/s_1^{\epsilon_1} s_2^{\epsilon_2})}{s_1^{\epsilon_1} s_2^{\epsilon_2} c(s_1^{\epsilon_1} s_2^{\epsilon_2}/\sqrt{q})}.
\]

Using \( \varphi_m(s) = \varphi_m(s^{-1}) \), the result is now clear.

When \( s_1, s_2 \) are parameters corresponding to the principal or complementary series, it cannot happen that \( |s_1^{\epsilon_1} s_2^{\epsilon_2}| > \sqrt{q} \) holds for two \( (\epsilon_1, \epsilon_2) \)'s. However, we mention for the sake of completeness that if \( s_2 \neq \pm 1 \) and \( |s_1 s_2| \geq |s_1^{-1} s_2| > \sqrt{q} \), then (2.10) holds, provided an extra term \( B \varphi_m(s_1 s_2^{-1}/\sqrt{q}) \), where \( B = c(s_1) c(s_2^{-1}) / c(s_1 s_2^{-1}/\sqrt{q}) \), is added on the right.
§3 Connection with orthogonal polynomials.

As is well known, and clear from the relation
\[ \varphi_1(s)\varphi_n(s) = \frac{q\varphi_{n+1}(s) + \varphi_{n-1}(s)}{q + 1}, \]
each function \( \varphi_n(s) \) is a polynomial \( p_n(t) \) in
\[ t = \varphi_1(s) = \frac{\sqrt{q}}{q + 1}(s + 1). \]
Note that \( \varphi_1 \) maps \( \mathbb{T} \) onto \( I = [-2\sqrt{q}/(q + 1), 2\sqrt{q}/(q + 1)] \). The image \( \tilde{\mu} \) of the Plancherel measure \( \mu \) under this map has density \( \sqrt{4q - (q + 1)^2t^2}/(2\pi(1 - t^2)) \) with respect to Lebesgue measure on \( I \). Formula (1.11) becomes
\[ \int_I p_m(t)p_n(t) \, d\tilde{\mu}(t) = \frac{1}{N_n}\delta_{m,n}. \]
Setting \( t_j = \varphi_1(s_j) \) for \( j = 1, 2, 3 \), we can express \( K(s_1, s_2, s_3) \) in terms of \( t_1, t_2, t_3 \):
\[ K(s_1, s_2, s_3) = \tilde{K}(t_1, t_2, t_3) \]
for
\[ \tilde{K}(t_1, t_2, t_3) = \frac{(q - 1)(1 - t_1^2)(1 - t_2^2)(1 - t_3^2)}{D}, \]
where
\[ D = (q + 1)^2t_1^2t_2^2t_3^2 - (q + 1)^2(t_1^2t_2^2t_3^2 + t_1t_2t_3t_2^3 + t_1t_2t_3^2t_3^2 + q(t_1^4 + t_2^4 + t_3^4)) + (q^2 + 1)(t_1^2t_2^2 + t_1^2t_3^2 + t_2^2t_3^2) - (q^2 - 6q + 1)t_1t_2t_3 - 2q(t_1^2 + t_2^2 + t_3^2) + q. \]
Thus, as a special case of Proposition 2.2, we know that \( \tilde{K}(t_1, t_2, t_3) > 0 \) for \( t_1, t_2, t_3 \in I \), and that, for any \( t_1, t_2 \in I \) and \( m \in \mathbb{N} \),
\[ p_m(t_1)p_m(t_2) = \int_I \tilde{K}(t_1, t_2, t_3)p_m(t_3) \, d\tilde{\mu}(t_3). \]
(3.1)
This formula is a special case of the product formula for \( q \)-ultraspherical polynomials found by Rahman and Verma [7, (1.20)]. The “\( q \)” here is not our \( q \). Let us temporarily write \( Q \) in place of our \( q \). Take the \( a \) and \( Q \) of [7, (1.20)] to be \( 1/\sqrt{Q} \) and 0, respectively. Then the polynomial \( p_n(x;a,a\sqrt{Q},-a,-a\sqrt{Q}) \) of [7] equals \( p_n(2\sqrt{Q}x/(Q + 1)) \), and [7, (1.20)] specializes to (3.1).

§4 Decomposing \( \pi^{s_1} \otimes \pi^{s_2} \).

Let \( \mathcal{K}(\Omega) \) and \( \mathcal{K}(\Omega \times \Omega) \) denote the space of locally constant functions on \( \Omega \) and \( \Omega \times \Omega \), respectively. We can identify the (algebraic) tensor product \( \mathcal{K}(\Omega) \otimes \mathcal{K}(\Omega) \) with \( \mathcal{K}(\Omega \times \Omega) \), the identification being induced by the assignment \( f_1 \otimes f_2 \mapsto F \), where \( F(\omega_1, \omega_2) = f_1(\omega_1)f_2(\omega_2) \). Thus if \( s_1, s_2 \in \mathbb{C} \setminus \{0\} \), we can consider the tensor product representation \( \pi = \pi^{s_1} \otimes \pi^{s_2} \) of \( G = \text{Aut}(T) \) as having representation space \( \mathcal{K}(\Omega \times \Omega) \), and given for \( F \in \mathcal{K}(\Omega \times \Omega) \) by
\[ (\pi(g)F)(\omega_1, \omega_2) = \left( \frac{s_1}{\sqrt{q}} \right)^{-\delta(o,go,\omega_1)} \left( \frac{s_2}{\sqrt{q}} \right)^{-\delta(o,go,\omega_2)} F(g^{-1}\omega_1, g^{-1}\omega_2). \]
We are mainly concerned with the case when \( s_1 \) and \( s_2 \) correspond to the principal or complementary series. Then there are inner products \( \langle \cdot , \cdot \rangle_{s_i} \) on \( K(\Omega) \) so that \( \pi^{s_i} \) extends to a unitary representation of \( G \) on the completion \( H_{s_i} \) of \( K(\Omega) \) with respect to \( \langle \cdot , \cdot \rangle_{s_i} \). Also, \( \pi \) extends to a unitary representation on the completion \( H_{s_1,s_2} \) of \( K(\Omega \times \Omega) \) with respect to the inner product \( \langle \cdot , \cdot \rangle_{s_1,s_2} \) determined by

\[
\langle f_1 \otimes f_2, g_1 \otimes g_2 \rangle_{s_1,s_2} = \langle f_1, g_1 \rangle_{s_1} \langle f_2, g_2 \rangle_{s_2} \quad \text{for} \quad f_1, f_2, g_1, g_2 \in K(\Omega).
\]

Our aim is to decompose \( \pi \) into irreducible representations. For this purpose, we may suppose that \( s_1, s_2 \neq \pm \sqrt{q}, \pm 1/\sqrt{q} \). For \( \pi \) is equivalent to \( \pi^{s_1} \) if \( s_2 = \sqrt{q} \) or \( s_2 = 1/\sqrt{q} \), while \( \pi \) is equivalent to \( \pi^{-s_1} \) if \( s_2 = -\sqrt{q} \) or \( s_2 = -1/\sqrt{q} \) (see the definition in the introduction of \( \pi^{s_2} \) in these special cases).

Let \( H_1 \) and \( H_2 \) be as in (1.1), and let \( \pi_1 \) and \( \pi_2 \) be the restrictions of \( \pi \) to these invariant subspaces.

**Theorem 4.1.** If \( s_1 \) and \( s_2 \) are parameters corresponding to the principal or complementary series, with \( s_1, s_2 \notin \{\pm \sqrt{q}, \pm 1/\sqrt{q}\} \), then \( H_1 \) equals the closure in \( H_{s_1,s_2} \) of the linear span of \( \{\pi(g)(1 \otimes 1) : g \in G\} \). Under the conditions of Corollary 2.4 above, we have

\[
\pi_1 \cong \int_{T}^{\oplus} \pi^s K(s_1, s_2, s) d\mu(s) \cong \int_{T}^{\oplus} \pi^s ds,
\]

while under the hypothesis of (2.9) we have

\[
\pi_1 \cong A \pi^{s_1 s_2/\sqrt{q}} + \int_{T}^{\oplus} \pi^s K(s_1, s_2, s) d\mu(s) \cong A \pi^{s_1 s_2/\sqrt{q}} + \int_{T}^{\oplus} \pi^s ds,
\]

where \( A = c(s_1)c(s_2)/c(s_1 s_2/\sqrt{q}) > 0 \). In both cases, \( K(s_1, s_2, s) > 0 \) for all \( s \in T \).

**Proof.** To show that \( H_1 \) equals the closure in \( H_{s_1,s_2} \) of the linear span of \( \{\pi(g)(1 \otimes 1) : g \in G\} \), we must show that if any \( K \)-invariant \( \xi \in H_{s_1,s_2} \) is in this closure. The orthogonal projection \( P \) of \( H_{s_1,s_2} \) onto the space of \( K \)-invariant vectors is \( \int_{K} \pi(k) \, dk \), where the integral is with respect to normalized Haar measure on \( K \), and it is clear that \( P \) leaves \( K(\Omega \times \Omega) \) invariant. So it is enough to check that each \( K \)-invariant \( F \in K(\Omega \times \Omega) \) belongs to the linear span of \( \{\pi(g)(1 \otimes 1) : g \in G\} \). This is proved in the next lemma.

To obtain the integral decompositions of \( \pi_1 \), it suffices to decompose the positive definite function

\[
\varphi(g) = \langle \pi(g)(1 \otimes 1), 1 \otimes 1 \rangle_{s_1,s_2} = \varphi_m(s_1) \varphi_m(s_2), \quad \text{where} \quad m = d(o, go),
\]

as a direct integral of spherical functions. This decomposition was carried out in Corollary 2.4 and Proposition 2.5. The second equivalences hold by [5, p. 87, Lemma] because \( s \mapsto K(s_1, s_2, s)/|c(s)|^2 \) is integrable with respect to \( ds \), and strictly positive (except at \( s = \pm i \)).

**Lemma 4.2.** Suppose that \( s_1, s_2 \in \mathbb{C} \setminus \{0, \pm \sqrt{q}\} \), and form \( \pi = \pi^{s_1} \otimes \pi^{s_2} \). Suppose that \( F \in K(\Omega \times \Omega) \) is \( K \)-invariant. Then \( F \) is in the linear span of \( \{\pi(g)(1 \otimes 1) : g \in G\} \).

**Proof.** For \( k \in K \) and \( F \in K(\Omega \times \Omega) \), \( (\pi(k)F)(\omega_1, \omega_2) = F(k^{-1} \omega_1, k^{-1} \omega_2) \). So \( F \) is \( K \)-invariant if and only if (i) \( F(\omega, \omega) \) is independent of \( \omega \in \Omega \), and (ii) for \( \omega_1 \neq \omega_2 \),
$F(\omega_1, \omega_2)$ depends only on $d(\omega_1 \land \omega_2, o)$. Here $\omega_1 \land \omega_2$ denotes the confluent of $\omega_1$ and $\omega_2$, i.e., the last vertex common to $[o, \omega_1)$ and $[o, \omega_2)$. For $n \in \mathbb{N}$, define $F_n \in \mathcal{K}(\Omega \times \Omega)$ by setting

$$F_n(\omega_1, \omega_2) = \begin{cases} 1 & \text{if } \omega_1 \neq \omega_2 \text{ and } d(\omega_1 \land \omega_2, o) = n, \\ 0 & \text{otherwise.} \end{cases}$$

So the linear span of $1 \otimes 1$ and the functions $F_n$ consists of the $K$-invariant functions $F \in \mathcal{K}(\Omega \times \Omega)$. To prove the lemma, it is sufficient to show that each function $F_n$ is in the linear span of $\{\pi(g)(1 \otimes 1) : g \in G\}$.

For $v \in T$, let $1_v \in \mathcal{K}(\Omega)$ denote the indicator function of $\Omega(v) = \Omega_o(v)$. Let us write $C_n$ for the set of vertices $v \in T$ such that $d(v, o) = n$. For $v \in T$, choose $g_v \in G$ such that $g_v o = v$. Suppose now that $v \in C_1$. For $s \in \mathbb{C} \setminus \{0\}$ and $\omega \in \Omega$, $(\pi^s(g_v)1)(\omega)$ equals $s/\sqrt{q}$ or $(s/\sqrt{q})^{-1}$ according as $\omega \not\in \Omega(v)$ or $\omega \in \Omega(v)$.

$$(\pi^s(g_v)1)(\omega) = \frac{s}{\sqrt{q}}1(\omega) + \left(\frac{\sqrt{q}}{s} - \frac{s}{\sqrt{q}}\right)1_v(\omega).$$

Suppose now $s_1, s_2 \in \mathbb{C} \setminus \{0\}$, $\omega_1, \omega_2 \in \Omega$ and $v \in C_1$. Then

$$(\pi^{s_1}(g_v)1) \otimes (\pi^{s_2}(g_v)1))(\omega_1, \omega_2)$$

$$= \left\{ \frac{s_1}{\sqrt{q}}1(\omega_1) + \left(\frac{\sqrt{q}}{s_1} - \frac{s_1}{\sqrt{q}}\right)1_v(\omega_1) \right\} \times \left\{ \frac{s_2}{\sqrt{q}}1(\omega_2) + \left(\frac{\sqrt{q}}{s_2} - \frac{s_2}{\sqrt{q}}\right)1_v(\omega_2) \right\}.$$ 

If we now sum this identity over the $q + 1$ $v$’s in $C_1$, and use $\sum_{v \in C_1} 1_v = 1$, we get

$$\sum_{v \in C_1} (\pi(g_v)(1 \otimes 1))(\omega_1, \omega_2) = \left(\frac{q - 1}{q} s_1 s_2 + \frac{s_1}{s_2} + \frac{s_2}{s_1}\right)(1 \otimes 1)(\omega_1, \omega_2) + \frac{(q - s_1^2)(q - s_2^2)}{qs_1 s_2} \sum_{v \in C_1} 1_v(\omega_1)1_v(\omega_2).$$

Now the sum $\sum_{v \in C_1} 1_v(\omega_1)1_v(\omega_2)$ equals 1 if and only if the geodesics $[o, \omega_1)$ and $[o, \omega_1)$ have a vertex in common other than $o$, and is zero otherwise. It therefore equals $(1 \otimes 1)(\omega_1, \omega_2) - F_0(\omega_1, \omega_2)$. Substituting this into the last expression, we get

$$\sum_{v \in C_1} (\pi(g_v)(1 \otimes 1))(\omega_1, \omega_2)$$

$$= \left(s_1 s_2 + \frac{q}{s_1 s_2}\right)(1 \otimes 1)(\omega_1, \omega_2) - \frac{(q - s_1^2)(q - s_2^2)}{qs_1 s_2} F_0(\omega_1, \omega_2).$$

As $(q - s_1^2)(q - s_2^2) \neq 0$ by hypothesis, we see that $F_0$ is in the linear span of $\{\pi(g)(1 \otimes 1) : g \in G\}$.

Now suppose that $n \geq 2$, and that we have shown that $F_0, \ldots, F_{n-2}$ are in the linear span of $\{\pi(g)(1 \otimes 1) : g \in G\}$. Suppose that $v \in C_n$. Let $v_0 = o, v_1, \ldots, v_n = v$ be the geodesic from $o$ to $v$. Then $(\pi^s(g_v)1)(\omega) = (s/\sqrt{q})^{n-2j}$ if $v_j$ is the last vertex common to $[o, \omega)$ and $[v, \omega)$. Thus

$$\pi^s(g_v)1 = \left(\frac{s}{\sqrt{q}}\right)^n 1_{\Omega \setminus \Omega(v_1)} + \sum_{j=1}^{n-1} \left(\frac{s}{\sqrt{q}}\right)^{n-2j} 1_{\Omega(v_j) \setminus \Omega(v_{j+1})} + \left(\frac{s}{\sqrt{q}}\right)^{-n} 1_{\Omega(v_n)}$$

$$= \sum_{j=0}^n c_j(s)1_{v_j}, \text{ say.}$$
Note that \( c_n(s) = \left( \frac{s}{\sqrt{q}} \right)^n (1 - \frac{s^2}{q}) \neq 0 \) if \( s \neq \pm \sqrt{q} \). Hence for \( \omega_1, \omega_2 \in \Omega, \)

\[
(\pi(g_v)(1 \otimes 1))(\omega_1, \omega_2) = (\pi^{s_1}(g_v)1)(\omega_1)(\pi^{s_2}(g_v)1)(\omega_2) = \sum_{j,k=0}^{n} c_j(s_1)c_k(s_2)1_{v_j}(\omega_1)1_{v_k}(\omega_2).
\]

We now wish to sum this last equation over all \( v \in C_n \), and so need to calculate

\[
\sum_{v \in C_n} 1_{v_j}(\omega_1)1_{v_k}(\omega_2).
\]

This clearly equals

\[
\sum_{u \in C_j} \sum_{u' \in C_k} N(u, u')1_u(\omega_1)1_{u'}(\omega_2), \quad (4.1)
\]

where for \( u \in C_j \) and \( u' \in C_k \), \( N(u, u') = \sharp \{ v \in C_n : u, u' \in [o, v] \} \).

To calculate \( N(u, u') \), first suppose that \( j \leq k \). Then \( N(u, u') = 0 \) if \( u \not\in [o, u'] \), while if \( u \in [o, u'] \), then \( N(u, u') \) is a number \( K_{n,k} \) independent of \( u' \in C_k \). Hence for each \( u' \in C_k \), there is only one \( u \in C_j \) such that \( N(u, u') \neq 0 \), namely the \( j \)th vertex \( u'_j \) on the geodesic \([o, u']\). Moreover, given \( \omega_2 \), there is only one \( u' \in C_k \) such that \( 1_{u'}(\omega_2) \neq 0 \), namely \( u' = (\omega_2)_k \), the \( k \)th vertex on the geodesic \([o, \omega_2]\). Thus the sum (4.1) equals

\[
K_{n,k}1_{u'_j}(\omega_1) = \begin{cases} K_{n,k} & \text{if } (\omega_1)_j = (\omega_2)_j, \\ 0 & \text{otherwise.} \end{cases}
\]

Thus

\[
\sum_{v \in C_n} (\pi(g_v)(1 \otimes 1))(\omega_1, \omega_2) = \sum_{j,k=0}^{n} c_j(s_1)c_k(s_2)K_{n,j \vee k} \delta_{(\omega_1),j \wedge k, (\omega_2),j \wedge k},
\]

where \( j \vee k = \max\{j, k\} \) and \( j \wedge k = \min\{j, k\} \). Now

\[
\delta_{(\omega_1),j \wedge k, (\omega_2),j \wedge k} = 1 \iff d(\omega_1 \wedge \omega_2, o) \geq j \wedge k \iff \left( 1 \otimes 1 - \sum_{\ell=0}^{j \wedge k-1} F_\ell \right)(\omega_1, \omega_2) = 1.
\]

It follows that \( \sum_{v \in C_n} \pi(g_v)(1 \otimes 1) \) is a linear combination of \( 1 \otimes 1 \) and \( F_0, \ldots, F_{n-1} \). The coefficient of \( F_{n-1} \) equals \( c_n(s_1)c_n(s_2)K_{n,n} = c_n(s_1)c_n(s_2) \neq 0 \). By the induction hypothesis we see that \( F_{n-1} \) is in the linear span of \( \{ \pi(g)(1 \otimes 1) : g \in G \} \).

\section{Restriction to a lattice subgroup.}

Throughout this section, let \( \Gamma \) denote a lattice subgroup of \( G \), i.e., a discrete subgroup of \( G \) such that \( G/\Gamma \) carries a finite \( G \)-invariant measure. The discreteness implies that \( \Gamma_o = \Gamma \cap K = \{ \gamma \in \Gamma : \gamma o = o \} \) is finite. Examples of lattice subgroups \( \Gamma \) include all discrete co-compact subgroups, which are the discrete subgroups having only a finite number of orbits on \( T \). Very special examples of these are the subgroups which act transitively and faithfully (so that \( \Gamma_o = \{1\} \)), which are described in \cite[Thm. I.6.3 and Appendix, Prop. 5.5]{3}. Here \( 1 \) denotes the identity.
automorphism of $T$. For examples of lattice subgroups which is not co-compact, see [9, p. 88] and [4]. Apart from discreteness, the only property of lattice subgroups we need is contained in the next lemma.

We shall use the notation $T_o(x) = \{y \in T : x \in [o, y]\}$, by analogy with the notation $\Omega_o(x)$, and let $\Gamma_x = \{\gamma \in \Gamma : \gamma x = x\}$ for $x \in T$.

**Lemma 5.1.** Assume that $\Gamma$ is a lattice subgroup of $G$. Then for any $\omega \in \Omega$ there is a sequence $(\gamma_n)$ in $\Gamma$ such that $\gamma_n o \to \omega$.

**Proof.** 1. We first show that, given $\omega \in \Omega$, there is a sequence $(x_n)$ in $T$ and there is a sequence $(\gamma_n)$ of distinct elements of $\Gamma$ such that $x_n \to \omega$ and $\gamma_n x_n \to \omega$.

Let $(g_\alpha)_{\alpha \in A}$ be a set of double coset representatives for $\Gamma \backslash G / K$. Let $v_\alpha = g_\alpha o$, and let $\Gamma_\alpha = \{v_\alpha = \Gamma \cap g_\alpha K g_\alpha^{-1}$. The fact that $\Gamma$ is a lattice in $G$ is equivalent to $\sum_{\alpha} 1/|\Gamma_\alpha| < \infty$ (see [9, p. 84]). In particular, either $A$ is finite, or the numbers $|\Gamma_\alpha|$ are unbounded.

Let $o, \omega = (y_0 = o, y_1, \ldots)$. We inductively choose $x_n \in T_o(y_n)$ and $\gamma_n \in \Gamma$ so that $\gamma_n x_n \in T_o(y_n)$ and $\gamma_0, \ldots, \gamma_n$ are distinct. We start by setting $x_0 = o$ and $\gamma_0 = 1$. Suppose that $n \geq 1$ and that $x_0, \ldots, x_{n-1}$ and $\gamma_0, \ldots, \gamma_{n-1}$ have been chosen. Now $\gamma_n v_\alpha \in T_o(y_n)$ for infinitely many pairs $(\gamma, \alpha) \in \Gamma \times A$. If there is an $\alpha$ so that $\gamma_n v_\alpha \in T_o(y_n)$ for infinitely many $\gamma \in \Gamma$, we can choose $\gamma, \gamma' \in \Gamma$ so that $\gamma_n v_\alpha, \gamma' v_\alpha \in T_o(y_n)$ and so that $\gamma' \gamma^{-1} \not\in \{\gamma_0, \ldots, \gamma_{n-1}\}$. Let $x_n = \gamma v_\alpha$ and $\gamma_n = \gamma' \gamma^{-1}$. If, for each $\alpha \in A$, $\gamma v_\alpha \in T_o(y_n)$ holds for only finitely many $\gamma \in \Gamma$, choose an $\alpha$ so that $|\Gamma_\alpha| > n$ and so that $\gamma v_\alpha \in T_o(y_n)$ for some $\gamma \in \Gamma$. Fix such a $\gamma$, and let $x_n = \gamma v_\alpha$. As $|\Gamma_\alpha| = |\gamma \Gamma_\alpha \gamma^{-1}| = |\Gamma_\alpha| > n$, we can choose $\gamma_n$ to be any element of $\Gamma_{x_n} \setminus \{\gamma_0, \ldots, \gamma_{n-1}\}$.

2. Let $\omega \in \Omega$ be given, and let $(x_n)$ and $(\gamma_n)$ be as in step 1. We show that a subsequence of $(\gamma_n o)$ or of $(\gamma_n^{-1} o)$ tends to $\omega$. Assuming that $\gamma_n^{-1} o \not\to \omega$, taking a subsequence if necessary we may assume that $\gamma_n^{-1} o \to \omega'$ for some $\omega' \neq \omega$. Let $N = d(o, z)$, where $z$ is the confluent of $[o, \omega)$ and $[o, \omega')$. Let $y \in [o, \omega)$ and $y' \in [o, \omega')$, with $d(o, y) = d(o, y') = m > N$. For some $n_m$ we have $x_n, \gamma_n x_n \in T_o(y)$ and $\gamma_n^{-1} o \in T_o(y')$ for all $n \geq n_m$. As the $\gamma_n$‘s are distinct, we know that $d(\gamma_n^{-1} o, o) \to \infty$, and so we may assume that also $d(\gamma_n y', o) \geq m$ for all $n \geq n_m$. Assume that $n \geq n_m$. Then $y'$ lies on the geodesic $[\gamma_n^{-1} o, x_n]$, and so $\gamma_n y'$ lies on the geodesic $[o, \gamma_n x_n]$. But $y$ lies on $[o, \gamma_n x_n]$, and $d(\gamma_n y', o) \geq m = d(y, o)$. Thus $y \in [o, \gamma_n y']$. Also, $y'$ lies on the geodesic $[o, \gamma_n^{-1} o]$, and so $\gamma_n y'$ lies on the geodesic $[o, \gamma_n o]$. Thus $y \in [o, \gamma_n o]$, i.e., $\gamma_n o \in T_o(y)$, for all $n \geq n_m$. This shows that $\gamma_n o \to \omega$.

Let $\lambda_\Gamma$ denote the left regular representation of $\Gamma$. If $n = 1, 2, \ldots, \infty$, let $n\lambda_\Gamma$ denote the sum of $n$ copies of $\lambda_\Gamma$. If $\sigma_1$ and $\sigma_2$ are two unitary representations of $\Gamma$, we write $\sigma_1 \leq \sigma_2$ if $\sigma_1$ is equivalent to a subrepresentation of $\sigma_2$.

**Lemma 5.2.** Let $\sigma$ denote a special or cuspidal irreducible unitary representation of $G$. Then the restriction $\sigma|_\Gamma$ of $\sigma$ to $\Gamma$ satisfies $\sigma|_\Gamma \leq \infty \lambda_\Gamma$.

**Proof.** By [3, Lemmas III.2.5 and III.3.12] there is a finite complete subtree $\mathfrak{t} \subset T$ such that $|\mathfrak{t}| \geq 2$, and there is a closed subspace $M_\sigma$ of $L^2(G)$ consisting of right $U$-invariant functions and invariant under the left regular representation $\lambda_G$ of $G$, such that $\sigma$ is equivalent to the restriction of $\lambda_G$ to $M_\sigma$. Here $U = \{g \in G : gx = x \text{ for all } x \in \mathfrak{t}\}$.

Let $\{g_\alpha : \alpha \in A\}$ be a set of double coset representatives for $\Gamma \backslash G / U$. For $\gamma_1, \gamma_2 \in \Gamma$, $\gamma_1 g_\alpha U = \gamma_2 g_\alpha U$ if and only if $\gamma_2^{-1} \gamma_1 \in \Gamma_\alpha$, where $\Gamma_\alpha = \Gamma \cap g_\alpha U g_\alpha^{-1}$.
As $\Gamma$ is discrete and as $U$ is compact, $\Gamma_1$ is finite. Let $dg$ denote a Haar measure on $G$. Then for $f \in M_\sigma$ we have

$$
\int_G |f(g)|^2 \, dg = \sum_\alpha \int_{\Gamma_\alpha} |f(g)|^2 \, dg \\
= \sum_\alpha \frac{1}{|\Gamma_\alpha|} \sum_{\gamma \in \Gamma_\alpha} \int_{\gamma g_\alpha U} |f(g)|^2 \, dg \\
= \sum_\alpha \frac{m(U)}{|\Gamma_\alpha|} \sum_{\gamma \in \Gamma} |f(\gamma g_\alpha)|^2,
$$

where $m(U)$ denotes the Haar measure of $U$. Writing $f_\alpha(\gamma) = (m(U)/|\Gamma_\alpha|)^{\frac{1}{2}} f(\gamma g_\alpha)$ for $\alpha \in A$ and $\gamma \in \Gamma$, we see that $f \mapsto (f_\alpha)_{\alpha \in A}$ is an isometric embedding of $M_\sigma$ into the direct sum of $\ell^2(\Gamma)$ which intertwines $\sigma|_\Gamma$ and $n\lambda_\Gamma$.

In the next lemma, assume that $s$ is a parameter corresponding to either the principal or the complementary series (with $s \not\in \{\pm \sqrt{q}, \pm 1/\sqrt{q}\}$). In the former case, we write $\langle f, g \rangle_s$ for $\langle f, g \rangle$ defined in (1.4). If $d(o, x) = n \geq 1$, define $\xi_x = N_n 1_x - N_{n-1} 1_{x'}$, where $x'$ is the vertex on $[o, x]$ at distance 1 from $x$, and where $1_x$ is the indicator function of $\Omega_o(x)$, as before. In the complementary series case, we shall need below the following fact about the inner product $\langle \cdot, \cdot \rangle_s$ mentioned in the introduction: for $f, g \in K(\Omega)$, $\langle f, g \rangle_s = \langle f, J_s g \rangle$ where $J_s : K(\Omega) \to K(\Omega)$ is a linear map such that $J_s 1 = 1$ and $J_s \xi_x = j_n(s) \xi_x$ if $d(o, x) = n \geq 1$. Here $j_n(s) = s^{(n-1)}(qs^2 - 1)/(q - s^2) > 0$ (see [3, p. 45], where the definition of $\xi_x$ and the proof of Lemma 3.2 need correcting).

**Lemma 5.3.** Suppose that $x$ and $u$ are vertices distinct from $o$, and let $g \in G$.

(a) If

$$
\langle \pi^s(g) 1, \xi_x \rangle_s \neq 0
$$

then $x'$ lies on $[o, go]$.

(b) If

$$
\langle \pi^s(g) \xi_x, \xi_u \rangle_s \neq 0.
$$

Then either
(i) $gx' = u'$, or
(ii) $gx'$ and $u'$ lie on the geodesic $[o, go]$.

**Proof.** (a) In the principal series case, $\langle \pi^s(g) 1, \xi_x \rangle_s$ equals

$$
\langle \pi^s(g) 1, \xi_x \rangle = \int_{\Omega} \left( \frac{s}{\sqrt{q}} \right)^{-\delta(o, go, \omega)} \xi_x(\omega) \, d\nu_o(\omega),
$$

while in the complementary series case, $\langle \pi^s(g) 1, \xi_x \rangle_s$ is a multiple of this integral, because of the form of $\langle \cdot, \cdot \rangle_s$ noted above. If $x' \not\in [o, go]$, then $\delta(o, go, \omega)$ is constant on $\Omega_o(x') = \text{Supp}(\xi_x)$. Hence the integrand is a constant multiple of $\xi_x(\omega)$, and so the integral is 0. This proves (a).

(b) As in (a), $\langle \pi^s(g) \xi_x, \xi_u \rangle_s$ is a multiple of

$$
\langle \pi^s(g) \xi_x, \xi_u \rangle = \int_{\Omega} \left( \frac{s}{\sqrt{q}} \right)^{-\delta(o, go, \omega)} \xi_x(g^{-1} \omega) \xi_u(\omega) \, d\nu_o(\omega).
$$
Suppose that \( gx' \notin [o, go] \). Let \( n = d(o, go) \) and \( m = d(o, x) \). Let \( v_0 = o, \ldots, v_n = go \) be the geodesic \([o, go]\) from \( o \) to \( go \), and suppose that \( v_j \) is the last vertex common to \([o, go]\) and \([o, gx']\). Then \( g\Omega_o(x) = \Omega_o(gx) \), \( g\Omega_o(x') = \Omega_o(gx') \), \( (gx)' = gx' \) and \( d(o, gx) = m - n + 2j \). It follows that \( \xi_x(g^{-1}\omega) = q^{n-2j}\xi_{gx}(\omega) \). Also, if \( \xi_x(g^{-1}\omega) \neq 0 \), then \( g^{-1}\omega \in \Omega_o(x') \), so that \( \omega \in \Omega_o(gx') \) and \( -\delta(o, go, \omega) = n - 2j \).

Hence the integral in (5.1) equals

\[
\left( \frac{8}{\sqrt{q}} \right)^{n-2j} q^{n-2j} \int_{\Omega} \xi_{gx}(\omega)\xi_{u}(\omega) \ d\nu_o(\omega).
\]

For vertices \( u, v \neq o \), \( \int_{\Omega} \xi_{v}(\omega)\xi_{u}(\omega) \ d\nu_o(\omega) = 0 \) unless \( v' = u' \). So if also \( gx' \neq u' \), then the integral in (5.1) is zero. Also, if \( u' \notin [o, go] \) and \( gx' \neq u' \), then \( g^{-1}u' \notin [o, g^{-1}o] \) and \( g^{-1}u' \neq x' \), and as \( \langle \pi^s(g)\xi_x, \xi_u \rangle_s = \langle \xi_x, \pi^s(g^{-1})\xi_u \rangle_s \), this expression is zero for the same reasons. This completes the proof of (b).

**Lemma 5.4.** There exist \( a, b \in T \setminus \{o\} \) such that \([o, a] \cap [o, b] = \{o\} \) and \( \Gamma_a \cap \Gamma_b = \{1\} \).

**Proof.** Suppose that there are no such \( a, b \).

1. We first show that there is a \( y \in T \) and a \( g \in \Gamma_o \setminus \{1\} \) such that \( gv = v \) for all \( v \in T_o(y) \). To see this, let \( x_0, x_1 \) be distinct neighbours of \( o \). By our hypothesis, there exists \( g_1 \in \Gamma_{x_0} \cap \Gamma_{x_1} \setminus \{1\} \). Notice that \( g_1 \) must fix each vertex on the geodesic \([x_0, x_1]\), and so \( g_1 \in \Gamma_0 \). If \( g_1v = v \) for all \( v \in T_o(x_1) \), then take \( y = x_1 \) and \( g = g_1 \). Otherwise, choose \( x_2 \in T_o(x_1) \) such that \( g_1x_2 \neq x_2 \). Next choose \( g_2 \in \Gamma_{x_0} \cap \Gamma_{x_2} \setminus \{1\} \). Notice that \( g_2 \neq g_1 \) and that \( g_2 \in \Gamma_o \). We continue in this way, choosing \( x_1, x_2, \ldots \in T \) and \( g_1, g_2, \ldots \in \Gamma_0 \setminus \{1\} \) such that \( x_{i+1} \in T_o(x_i) \), \( g_1x_i = x_i \) and \( g_1x_{i+1} \neq x_{i+1} \) for each \( i \). As \( g_1 \) must fix all vertices on the geodesic \([o, x_1]\), it fixes \( x_1, \ldots, x_i \), and so \( g_1 \neq g_1, \ldots, g_{i-1} \). As \( \Gamma_o \) is finite, this procedure must stop, and we can take \( y = x_i \) and \( g = g_i \) if it stops at the \( i \)-th step.

2. Now let \( y \) and \( g \) be as in 1. As \( g \neq 1 \), there is a vertex \( c \in T \) such that \( gc \neq c \). Because of Lemma 5.1, there is a sequence \( (\gamma_n) \) of distinct elements of \( \Gamma \) such that \( \gamma_n o \in T_o(y) \) for each \( n \). Then \( \gamma_n g\gamma_n \in \Gamma_o \), as \( g \) fixes \( \gamma_n o \). Now \( \Gamma_o \) is finite, and so there must exist \( n_0 \) such that \( \gamma_n g\gamma_n = \gamma_{n_0} g\gamma_{n_0} \gamma_n \) for infinitely many \( n \). Choose such an \( n \) satisfying also \( d(c, \gamma_n o) \leq d(y, \gamma_n o) \). Now \( g \) commutes with \( \gamma = \gamma_n \gamma_n^{-1} \). So \( g\gamma c = \gamma gc \neq \gamma c \). But

\[
d(\gamma c, \gamma_n o) = d(c, \gamma_{n_0} o) \leq d(y, \gamma_n o),
\]

which implies that \( \gamma c \in T_o(y) \). This is a contradiction, as \( g \) fixes each vertex in \( T_o(y) \).

We conclude that \( a, b \) must exist with the desired properties.

In the next lemma, let \( s_1 \) and \( s_2 \) be parameters corresponding to either the principal or the complementary series (with \( s_1, s_2 \notin \{ \pm \sqrt{q}, \pm 1/\sqrt{q} \} \)). Let \( \pi = \pi^{s_1} \otimes \pi^{s_2} \), and let \( H_2 \) be the orthogonal complement in \( H_{s_1, s_2} \) of the closed linear span \( H_1 \) of \( \{\pi(g)(1 \otimes 1) : g \in G\} \).

**Lemma 5.5.** There are sequences \( x_1, x_2, \ldots \) and \( y_1, y_2, \ldots \) of vertices such that

\[
\xi_{x_i} \otimes \xi_{y_i} \in H_2 \quad \text{for each} \quad i \tag{5.2}
\]
and

$$\langle \pi(\gamma)(\xi_{x_i} \otimes \xi_{y_i}), \xi_{x_i} \otimes \xi_{y_i} \rangle_{s_1, s_2} = C_i \delta_1(\gamma) \delta_{i,j}$$

(5.3)

for all $\gamma \in \Gamma$ and $i, j = 1, 2, \ldots$, where $C_i = \langle \xi_{x_i}, \xi_{x_i} \rangle_{s_1} \langle \xi_{y_i}, \xi_{y_i} \rangle_{s_2} > 0$.

**Proof.** We shall choose certain points $x'_j$ and $y'_j$ by induction. Then any points $x_j \in T_o(x'_j)$ and $y_j \in T_o(y'_j)$ with $d(x_j, x'_j) = 1 = d(y_j, y'_j)$ will have the desired properties. Notice that we need only check (5.3) for $i \leq j$, because $\pi$ is unitary.

Let $a, b$ be as in Lemma 5.4. If $F \subset T$ is finite, we can choose $\gamma \in \Gamma$ such that $\gamma x \in T_o(a)$ for each $x \in F$. For $M = \max\{d(o, x) : x \in F\}$, we can, by Lemma 5.1, choose $\gamma \in \Gamma$ such that $\gamma o \in T_o(a)$ and $d(\gamma o, a) > M$. For $x \in F$, $d(\gamma x, \gamma o) = d(x, o) \leq M$, and so $\gamma x \in T_o(a)$.

We apply this to get $\alpha_1 \in \Gamma$ such that $\alpha_1 o \in T_o(a)$. Let $x'_1 = \alpha_1 o$. Similarly, there exists $\beta_1 \in \Gamma$ such that $\beta_1 o \in T_o(b)$. We set $y'_1 = \beta_1 o$. First notice that, by Lemma 5.3(a), if $g \in G$ then $\langle \pi(g)(1 \otimes 1), \xi_{x_1} \otimes \xi_{y_1} \rangle_{s_1, s_2} \neq 0$ implies that $x_1, y_1 \in [o, go]$, which is impossible, as $[o, a] \cap [o, b] = \{o\}$. Hence $\xi_{x_1} \otimes \xi_{y_1} \in H_2$.

By Lemma 5.3(b), if $\gamma \in \Gamma$, then $\langle \pi(\gamma)(\xi_{x_1} \otimes \xi_{y_1}), \xi_{x_1} \otimes \xi_{y_1} \rangle_{s_1, s_2} \neq 0$ implies that either (i) $\gamma x_1 = x_1$ and $\gamma y_1 = y_1$, (ii) $\gamma x_1 = x_1$ and $\gamma y_1 = y_1 \in [o, \gamma o]$, (iii) $\gamma y_1 = y_1$ and $\gamma x_1, x'_1 \in [o, \gamma o]$, or (iv) $\gamma x_1, x'_1, \gamma y_1$ and $y'_1$ all lie on $[o, \gamma o]$.

If (i) holds, then $\gamma \in \Gamma \cap \Gamma_b = \{1\}$. Also, (iv) cannot hold, because $[o, a] \cap [o, b] = \{o\}$ implies that $[o, x'_1] \cap [o, y'_1] = \{o\}$. Suppose that (ii) holds. Then $\gamma x'_1 = x'_1$ implies that $d(x'_1, o) = d(x'_1, \gamma o)$ and that the geodesic $[o, x'_1]$ branches off from the geodesic $[o, \gamma o]$ at a point which is exactly half way between $o$ and $\gamma o$. Then $y'_1 \in [o, \gamma o]$ and $[o, a] \cap [o, b] = \{o\}$ imply that $\gamma o = o = b$, contrary to the choice of $b$. So (ii) is impossible. Similarly, (iii) is impossible. So (5.3) holds for $i = j = 1$.

Now suppose that $r \geq 2$, and that $\alpha_1, \ldots, \alpha_{r-1}, \beta_1, \ldots, \beta_{r-1} \in \Gamma$ have been found so that vertices $x_i, y_i$ corresponding to $x'_i = \alpha_i \cdots \alpha_1 o$ and $y'_i = \beta_i \cdots \beta_1 o$ satisfy (5.2) and (5.3) for $i, j \leq r - 1$.

We now apply the above remark to the finite set

$$F = \{\alpha_{r-1} \cdots \alpha_1 k(o_i \cdots o_1) o : 0 \leq i \leq r - 1 \text{ and } k \in \Gamma_o\}$$

to obtain $\alpha_r \in \Gamma$ such that $\alpha_r x \in T_o(a)$ for all $x \in F$. Note that $x'_{r-1} \in F$ (take $i = 0$ and $k = 1$). Let $x'_r = \alpha_r x'_{r-1} = \alpha_0 \cdots \alpha_1 o$. Similarly, we can choose $\beta_r \in \Gamma$ such that $\beta_r \beta_{r-1} \cdots \beta_1 k(\beta_i \cdots \beta_1) o \in T_o(b)$ for $0 \leq i \leq r - 1$ and for $k \in \Gamma_o$. Let $y'_r = \beta_r y'_{r-1} = \beta_0 \cdots \beta_1 o$.

Clearly Lemma 5.3(a) implies that $\xi_{x_r} \otimes \xi_{g_r} \in H_2$, as $x'_r \in T_o(a)$ and $y'_r \in T_o(b)$.

Now suppose that $\langle \pi(\gamma)(\xi_{x_i} \otimes \xi_{y_i}), \xi_{x_i} \otimes \xi_{y_i} \rangle_{s_1, s_2} \neq 0$ for some $0 \leq i \leq r - 1$. By Lemma 5.3(b) either (i) $\gamma x'_i = x'_i$ and $\gamma y'_i = y'_i$, (ii) $\gamma x'_i = x'_i$ and $\gamma y'_i = y'_i \in [o, \gamma o]$, (iii) $\gamma y'_i = y'_i$ and $\gamma x'_i, x'_i \in [o, \gamma o]$, or (iv) $\gamma x'_i, x'_i, \gamma y'_i$ and $y'_i$ all lie on $[o, \gamma o]$. Possibility (iv) is excluded, because $x'_i \in T_o(a)$ and $y'_i \in T_o(b)$. If (ii) holds, then $\gamma x'_i = x'_i$, and so $\gamma = \alpha_r \cdots \alpha_1 k(o_i \cdots o_1) o$ for some $k \in \Gamma_o$. Hence $\gamma o \in T_o(a)$ by the choice of $\alpha_r$. So $y'_r \in [o, \gamma o]$ is impossible, because $y'_r \in T_o(b)$. So (ii) is excluded. Similarly, (iii) cannot happen. If (i) holds, then the same argument shows both that $\gamma o \in T_o(a)$ and that $\gamma o \in T_o(b)$, which is impossible.

Now suppose that $\langle \pi(\gamma)(\xi_{x_i} \otimes \xi_{y_i}), \xi_{x_i} \otimes \xi_{y_i} \rangle_{s_1, s_2} \neq 0$ for $i = r$. Arguing as in the case $r = 1$, we see that (ii)–(iv) cannot happen. If (i) holds, then $\gamma \in \Gamma_{x'_r} \cap \Gamma_{y'_r} \subset \Gamma_a \cap \Gamma_b = \{1\}$, because $a, b$ lie on the geodesic $[x'_r, y'_r]$. Thus (5.3) holds for $i, j \leq r$.

**Theorem 5.6.** Assume that $s_1, s_2$ are parameters corresponding to either the principal or the complementary series (with $s_1, s_2 \notin \{-\sqrt{q}, \pm 1/\sqrt{q}\}$). Then the restriction $(\pi_2)|_{\Gamma}$ to $\Gamma$ of the $H_2$ component $\pi_2$ of $\pi = \pi^{s_1} \otimes \pi^{s_2}$ is equivalent to $\infty \lambda_{\Gamma}$.  

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Proof. We know that $\pi_2$ is the sum of the special and cuspidal irreducible unitary representations $\sigma_k$ of $G$, with certain multiplicities. By Lemma 5.2, for each $k$ we have $(\sigma_k)|_\Gamma \leq \infty \lambda_\Gamma$. Hence $(\pi_2)|_\Gamma \leq \infty \lambda_\Gamma$. On the other hand, the construction in the last lemma shows that $\infty \lambda_\Gamma \leq (\pi_2)|_\Gamma$. By [5, Thm. 1.1], the result is proved.

Remarks. By taking restrictions, Theorem 4.1 gives a direct integral decomposition of the restriction $(\pi_1)|_\Gamma$ of the $H_1$ component $\pi_1$ of $\pi$. According to [3, Thm. II.7.1], at least in the co-compact case (but see also [3, p. 83]), each $(\pi^*)|_\Gamma$ appearing in this decomposition is irreducible, except when $s = \pm i$, when $(\pi^*)|_\Gamma$ may be the sum of 2 irreducible components.

It is easy to see that $\Gamma$ is i.c.c. (that is, each conjugacy class other than $\{1\}$ is infinite). See [2, Lemma 6.5] for a related result.
References.


1 School of Mathematics and Statistics, University of Sydney, N.S.W. 2006, Australia.

2 Dipartimento di Matematica, Università di Milano, Via C. Saldini, 50, 20133 Milano, Italy.