SELF-EQUIVALENCES OF DIHEDRAL SPHERES

DAVIDE L. FERRARIO

ABSTRACT. Let $G$ be a finite group. The group of homotopy self-equivalences $\mathcal{E}_G(X)$ of an orthogonal $G$-sphere $X$ is related to the Burnside ring $A(G)$ of $G$ via the stabilization map $I : \mathcal{E}_G(X) \subset [X, X]_G \to \{X, X\}_G = A(G)$ from the set of $G$-homotopy classes of self-equivalences of $X$ to the ring of stable $G$-homotopy classes of self-maps of $X$ (that is, the 0-th dimensional $G$-homotopy group of $S^0$, which is isomorphic to the Burnside ring). As a consequence of the properties of $I$, $\mathcal{E}_G(X)$ is equal to an extension of a subgroup of the group of units in $A(G)$ with the kernel of $I$. The aim of the paper is to give examples of a family of equivariant (dihedral) spheres with the property that the kernel of $I$ is infinite and with many generators, so that $\mathcal{E}_G(X)$ itself is infinite with many generators.

1. Introduction

Let $S^n$, with $n > 0$, denote the sphere of dimension $n$. The canonical stabilization map $I$ sending homotopy classes of self-maps to stable homotopy classes is a bijection from $[S^n, S^n]$ to $\{S^n, S^n\}$. Moreover, the group of self homotopy equivalences $\mathcal{E}(S^n)$ of $S^n$ is the finite cyclic group of order 2. These facts, even if trivial, are at the basis of many of the ideas in homology and homotopy theory. On the other hand, if we consider equivariant spheres in the category of $G$-spaces and $G$-maps (with $G$ finite group) these properties do not hold. If $S$ is a $G$-sphere, it can be that the canonical stabilization map $I : [S, S]_G \to \{S, S\}_G = A(G)$ from the monoid of equivariant homotopy classes of self-maps of $S$ to the ring of stable homotopy classes of self-maps (which is isomorphic to the 0-th dimensional equivariant homotopy group of $S^0$ and to the Burnside ring of the group $G$) is not a bijection. It is trivial to see that it can be not onto, and it was first found by Rubinszstein [Rub76] that it can be not mono.

This idea can be applied to the problem of homotopy self-equivalences of equivariant spheres, in the framework of the classical results relating equivariant stable homotopy theory to ordinary stable homotopy theory (dating back to the works of G. Segal, A. Kosinski, T. tomDieck and H. Hauschild): it is easy to see that $I$ sends the group of equivariant self homotopy equivalences $\mathcal{E}_G(S) \subset [S, S]_G$ to the group of units $A(G)^*$ of the Burnside ring (which is a finite abelian 2-group). As before, it is trivial to see that the image $I\mathcal{E}_G(S)$ can be properly contained in $A(G)^*$, and that $I$ can be non-injective even if restricted to $\mathcal{E}_G(S)$. The purpose of this paper is to provide some examples of equivariant spheres with the property that the kernel of $I : \mathcal{E}_G(S) \to A(G)^*$ is an infinite group (actually, we produce a family of spheres $X_k$, with $k = 2 \ldots \infty$, equivariant with respect to a suitable group $G_k$ depending upon $k$, such that the kernel of $I$ is a torsion free solvable group with a number of generators which tends to infinity with an exponential growth, while the dimension of $X_k$ tends to infinity with a linear growth).

In the last decades many results have been proved about homotopy self-equivalences: for general surveys and the state-of-the-art we refer to the well-known references [Ru97, Pi90, MR01]. In [MR01] one can find a complete and up-to-date bibliography.

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2. Preliminaries

If $G$ is a group acting on a space $X$, then let $G_x$ be the isotropy group of $x$ (that is, the stabilizer of $x$, or the fixer of $x$), $G_x = \{g \in G \mid gx = x\}$. Then the space of the points fixed by a subgroup $K \subset G$ is denoted by $X^K = \{x \in X \mid Kx = x\}$; the singular set of $X^K$ is $X^K_s = \{x \in X^K \mid G_x \neq K\}$. If $f : X \to X$ is a $G$-equivariant map and $K$ a subgroup of $G$, the restriction of $f$ to $X^K$ is denoted by $f^K : X^K \to X^K$. The isotropy type of an isotropy group $K \subset G$ is the conjugacy class if $K$ and is denoted with $(K)$. The normalizer of the subgroup $K$ of $G$ in $G$ is denoted by $N_GK$ and is equal to $\{g \in G \mid gKg^{-1} = K\}$. The Weyl group $W_GK$ is equal to the quotient $N_GK/K$. The monoid (with respect to composition of maps) of free $G$-homotopy classes of self-maps of $X$ is denoted by $[X,X]_G$; the ring of stable $G$-homotopy classes of self-maps of $X$ is denoted by $\{X,X\}_G$. If $X$ is a sphere, then this is isomorphic to the Burnside ring $A(G)$ (e.g., via the equivariant degree homomorphism).

3. Dihedral spheres

Let $n \geq 2$ be an integer; let $\zeta_n$ and $h$ be the transformations of the complex plane $\mathbb{C}$ defined by $\zeta_n(z) = e^{\frac{2\pi i}{n}}z$ and $h(z) = \overline{z}$, for every $z \in \mathbb{C}$. With an abuse of notation it is possible to write $\zeta_n^\alpha = e^{\frac{2\pi i \alpha}{n}}$.

With the symbol $D_{2n}$ we denote the dihedral group of order $2n$, generated by $\zeta_n$ and $h$. Let $G$ be equal to $D_{2n}$, $N \subset G$ the normal subgroup of $G$ generated by $\zeta_n$ and $H$ the subgroup generated by $h$.

For any integer $j$ let $V(j)$ be the $(2$-dimensional real) linear representation of $D_{2n}$ given by the action $\zeta_n \cdot z = \zeta_n^j z$ and $h \cdot z = \overline{z}$. Therefore the space fixed by the element $\zeta_n^\alpha \in G$, with $\alpha = 0 \ldots n - 1$, is

$$(V(j))^{\zeta_n^\alpha} = \begin{cases} V(j) & \text{if } \alpha j \equiv 0 \pmod{n} \\ 0 & \text{if } \alpha j \not\equiv 0 \pmod{n} \end{cases},$$

while the space fixed by the element $\zeta_n^\alpha h \in G$, with $\alpha = 0 \ldots n - 1$ is

$$(V(j))^{\zeta_n^\alpha h} = \langle e^{\alpha j \frac{2\pi i}{n}} \rangle,$$

where $\langle z \rangle$ denotes the one-dimensional subspace of the vector space $V(j)$ over $\mathbb{R}$ generated by the element $z$.

If $j_1, j_2, \ldots, j_k$ are integers, with $k \geq 2$, then let $D(j_1, j_2, \ldots, j_k)$ denote the unit sphere in $V(j_1) \oplus V(j_2) \oplus \cdots \oplus V(j_k)$, endowed with the $D_{2n}$-action. Assume that $n = p_1p_2\ldots p_k$ is the product of the $k$ first odd primes $p_i$. We denote the corresponding dihedral group with $G_k$. For every $i = 1 \ldots k$, let $j_i = \frac{n}{p_i}$, and $X_k$ the unit sphere $D(j_1, j_2, \ldots, j_k)$ for such a choice of $j_i$.

Let $\Gamma(j)$ denote the lattice of isotropy subgroups of $G$ relative to the $G$-representation $V(j)$. The lattice of isotropy subgroups relative to the $G$-representation $V(j_1, j_2, \ldots, j_k)$ is the intersection closure of all the $\Gamma(j_i)$, with $i = 1 \ldots k$, in the full lattice of subgroups of $G$. As a consequence, the minimal non-trivial isotropy subgroups of $G$ relative to $X$ are the subgroups of order $2 H_i = \langle \zeta_n^\alpha h \rangle$, for $\alpha = 0 \ldots n - 1$, conjugated to the subgroup $H$ (due to the fact that $n$ is odd). Furthermore $X^G = \emptyset$.

The space fixed by $H$ is a $(k - 1)$-dimensional sphere; it is the unit sphere in the vector space of the real parts of the $V(j_i)$, with $i = 1 \ldots k$. The Weyl group $W_GH$ is trivial. The singular set $X^H$ in $X^H$ is given by the points in $X^H$ with coordinates $(z_1, z_2, \ldots, z_k)$ such that $\prod_{i=1}^k z_i = 0$ (that is, such that at least one coordinate is zero; it is the union of the intersections of the hyperplanes $z_i = 0$ with $X^H$).
4. The kernel of $I$

Let $G$ be a finite group and $X$ be an orthogonal $G$-sphere. Consider the stabilization map $I : \mathcal{E}_G(X) \to A(G)^\ast$. If $K$ is an isotropy group in $G$ for $X$, then let $\gamma_K$ denote the number of positive-dimensional components of the space $X_{(K)}/G$ (which is the subspace of the orbit space $X/G$ of the elements with isotropy type equal to $(K)$).

The kernel size is computed as follows.

**Proposition 1.** The kernel $\mathcal{K}$ of $I : \mathcal{E}_G(X) \to A(G)^\ast$ is torsion-free, finitely generated and solvable. In fact, there is an abelian series $1 = \mathcal{K}_s \subset \mathcal{K}_{s-1} \cdots \subset \mathcal{K}_1 \subset \mathcal{K}_0 = \mathcal{K}$ such that

$$\frac{\mathcal{K}_{j-1}}{\mathcal{K}_i} \cong \begin{cases} \mathbb{Z}^{(\gamma_{K_j})-1} & \text{if } \dim X^{K_j} > 0 \\ 0 & \text{if } \dim X^{K_j} = 0, \end{cases}$$

where $s$ is the number of isotropy types in $X$.

**Proof.** The kernel of $I$ is equal to the kernel of the equivariant degree homomorphism $d_G : \mathcal{E}_G(X) \to \prod_{(K)} \mathbb{Z}^\ast$, where $d_G$ is defined by $d_G([f])(K) = \deg(f^K)$, $(K)$ ranges over the isotropy types of $X$ and $\mathbb{Z}^\ast$ is the group of units of the ring $\mathbb{Z}$.

Consider a totally ordered indexing $(K_1) \ldots K_s$ of the set of isotropy types $(K)$ of $X$ such that $(K_i) \leq (K_j) \implies i \geq j$ ($s$ is the number of isotropy types). This is the canonical ordering for induction over isotropy types (see [Di87] for details). Let $\mathcal{K}_i$ be the kernel of the composition $\mathcal{E}_G(X) \subset [X, X]_G \to [X_i, X_i]_G$ (the latter projection is given by restriction to $X_i$). For $i = 0$, let $\mathcal{K}_0 = \mathcal{K} = \ker I$. Because $X_s = X$, $\mathcal{K}_s$ is the trivial subgroup of $G$. These subgroups give rise to a chain of subgroups

$$1 = \mathcal{K}_s \subset \mathcal{K}_{s-1} \subset \cdots \subset \mathcal{K}_1 \subset \mathcal{K}_0 = \mathcal{K}.$$

For each $j = 1 \ldots s$, let $\delta_j : \mathcal{K}_{j-1} \to \mathbb{Z}^{\gamma_{K_j}}$ be defined as follows: an element of $\mathcal{K}_{j-1}$ is a $G$-homotopy class $[f]$ with $f : X \to X$. If the dimension of $X^{K_j}$ is zero, then clearly $\mathcal{K}_j = \mathcal{K}_{j-1}$. The map $f^{K_j} : X^{K_j} \to X^{K_i}$ restricted to the space $X^{K_j}$ fixed by $K_j$ has the property that its restriction to the singular set $X_{s,K_j}$ is $W_G K_j$-homotopic to the identity map (because $[X^{K_j}, X^{K_j}]_{W_G K_j}$ is isomorphic to $[X_j, X_j]_G$). So without loss of generality we can assume that $f^{K_j}$ is the identity on $X^{K_j}$. The space $X^{K_j} \setminus X_{s,K_j}$ contains exactly $\gamma_{K_j} W_G K_j$-orbits. Again without loss of generality we can deform $f^{K_j}$ equivariantly in order to obtain a map which is the identity outside a small equivariant spherical neighbourhood of such orbits, and is regular in the neighborhood. Let $B_1, \ldots, B_{\gamma_{K_j}}$ denote the $\gamma_{K_j} W_G K_j$-discs around the orbits, and let $x_1, \ldots, x_{\gamma_{K_j}}$ denote points in the interiors of $B_1, \ldots, B_{\gamma_{K_j}}$ respectively. The map $\delta_j$ is defined by

$$\delta_j([f]) = \left( \deg(f^{K_j}|_{B_1}, x_1) - 1, \ldots, \deg(f^{K_j}|_{B_{\gamma_{K_j}}}, x_{\gamma_{K_j}}) - 1 \right) \in \mathbb{Z}^{\gamma_{K_j}}.$$

Because $f^{K_j}$ is $W_G K_j$-equivariant, the degrees do not depend upon the choice of the points $x_i$ or the neighborhoods $B_i$, and clearly they do not depend on the representative in the $G$-homotopy class of $f$.

Now it is not difficult to prove that $\delta_j$ is a homomorphism (see also remark 2 or the proof in [Fe01], Corollary 3.3), and therefore the following sequence of groups and homomorphisms

$$0 \longrightarrow \mathcal{K}_j \longrightarrow \mathcal{K}_{j-1} \xrightarrow{\delta_j} \mathbb{Z}^{\gamma_{K_j}} \xrightarrow{\epsilon} \mathbb{Z} \longrightarrow 0$$

where $\epsilon$ is the augmentation homomorphism.

**Lemma 1.** The sequence is exact.
Proof the lemma. The sequence is obviously exact in $K_{j-1}$ and $\mathbb{Z}$. If $[f]$ is in $K_j$, then the map $f^{K_j}$ is $W_GK_j$-homotopic to the identity, and therefore $\delta_j([f]) = (0, \ldots, 0)$ (all the degrees are 1). On the other hand, by an easy application of obstruction and degree theory, it is possible to show that the kernel of $\delta_j$ is contained in the image of $K_j$. Thus the sequence is exact in $K_{j-1}$.

To see that the sequence is exact in $\mathbb{Z}^{\gamma K_j}$, consider the following argument. Because for every $[f] \in K_{j-1}$, every $i$ and every point $x'_i$ distant from $B_i$, $\deg(f^{K_j}|B_i, x_i) - 1 = \deg(f^{K_j}|B_i, x_i')$, we have that

$$\sum_i \left( \deg(f^{K_j}|B_i, x_i) - 1 \right) = \sum_i \deg(f^{K_j}|B_i, x_i'),$$

$$= \deg(f^{K_j}) - 1 = 0$$

if $x'$ is a point distant from all the $B_i$'s. This implies that $\text{Im} \delta_j \subset \ker \epsilon$. The converse is easy, so that the proof of the lemma is complete.

$$\square$$

Remark 2. Let $u = (u_1, \ldots, u_{\gamma K_j})$ be an element of $\ker \epsilon \subset \mathbb{Z}^{\gamma K_j}$, and define $M(u)$ as the square matrix with $q = |W_GK_j| \cdot \gamma K_j$ columns, with column entries given by

$$\begin{pmatrix} u_1 \Delta, u_1 \Delta, \ldots, u_1 \Delta, u_2 \Delta, u_2 \Delta, \ldots, u_2 \Delta, \ldots, u_{\gamma K_j} \Delta, u_{\gamma K_j} \Delta, \ldots, u_{\gamma K_j} \Delta \end{pmatrix}_{|W_GK_j}|$$

where $\Delta$ is the column with all entries equal to 1. The function $T$ defined by

$$T(u) = I + M(u),$$

where $I$ is the suitable identity matrix, is a homomorphism from $\ker \epsilon$ to $GL(q, \mathbb{Z})$ (invertible square matrices of order $q$ and integer entries), because $M(u)M(v) = 0$ for all $u, v \in \ker \epsilon$. The entries of $T(u)$ are exactly the degrees of the map $f^{K_j}$ restricted to the components of $B_i$ and on the points $W_GK_k \cdot x_i$.

Now we can go back to the chain

$$1 = K_s \subset K_{s-1} \subset \cdots \subset K_1 \subset K_0 = K.$$ 

We have seen that, if the dimension of $X^{K_j}$ is not 0, the quotient $K_{j-1}/K_j$ is isomorphic to the kernel of $\epsilon : \mathbb{Z}^{\gamma K_j} \rightarrow \mathbb{Z}$, i.e. is a torsion free abelian group of rank $\gamma K_j - 1$. As a consequence, $K$ is torsion-free, finitely generated and solvable. The number of generators of $K$ is the sum

$$\sum_{(K)} (\gamma K - 1),$$

where $(K)$ ranges over the conjugacy classes of isotropy subgroups of $G$ relative to $X$ with $\gamma K > 1$. This completes the proof of proposition 1. $\square$

5. Self homotopy equivalences of $X_k$

Now let $k \geq 2$ and go back to the equivariant sphere $X = X_k$ defined in section 3. Let $H$ be the minimal isotropy subgroup generated by $h \in G$. The complement of the singular part $X^H_s$ in $X^H$ has $2^k$ components. Because the Weyl group $W_GH$ is trivial,

$$\gamma_H = 2^k.$$ 

Thus, the following theorem is next to be proved.
Theorem 2. For every $k \geq 2$ there exists a $G$-sphere $X_k$ of dimension $2k - 1$, with $G = D_{2n}$ and $n$ equal to the product of the first $k$ distinct odd primes, such that the kernel of the stable injection

$$I : \mathcal{E}_G(X_k) \to A(G)^*$$

has a subgroup $J \subset \ker(I)$ that is a free abelian group of rank $2^k - 1$.

Proof. The group $H$ is the minimal non-trivial isotropy subgroup of $G$ relative to $X_k$. This implies that, if $s$ is the number of isotropy types in $X_k$, we can assume that $(K_{s-1}) = (H)$ and $K_s = 1$. Because of proposition 1,

$$\frac{K_{s-1}}{K_s} \cong \mathbb{Z}^{(\gamma_K, 1)}.$$

But $\gamma_{\{1\}} = 1$, because $X_k/G$ has just one component of points with trivial isotropy type (for a simple codimension argument). This means that $K_{s-1} = K_s = 1$. On the other hand

$$K_{s-2} = \frac{K_{s-2}}{K_{s-1}} \cong \mathbb{Z}^{(\gamma_K, 1)} = \mathbb{Z}^{(\gamma_{1}, 1)} = \mathbb{Z}^{(2^k - 1)}.$$

This means that $J = K_{s-2}$ is the wanted subgroup: it is a free abelian group of rank $2^k - 1$. □

6. Remarks

Remark 3. Consider the general case of the dihedral group $G = D_{2n}$. The 3-dimensional dihedral sphere $D(q_1, q_2)$ defined as above might be of some interest by itself: if $q_1$ and $q_2$ are arbitrary integers such that $(q_1, n) = (q_2, n) = 1$, then the quotient $D(j_1, j_2)/G$ is another (orbifold) sphere. It is an example of a quotient of a non-free action which is a manifold (like the rotation on the 2-disc). The partial factorization $D(q_1, q_2) \to D(q_1, q_2)/N \to D(q_1, q_2)/G$, where $N = \langle \zeta_n \rangle \subset G$ is the normal $n$-cyclic subgroup, gives rise to two maps of degrees $n$ and 2 (the first is the covering)

$$S^3 \to L(n, q_1^{-1}q_2) \to S^3,$$

where $L(n, q_1^{-1}q_2)$ is the lens space and the inverse $q_1^{-1}$ is meant as inverse mod $n$. Another feature of this equivariant sphere is that the singular set is a link of $n$ circles, with linking numbers 1.

Remark 4. Unfortunately I do not know under which conditions the kernel $\mathcal{K}$ is abelian (in case there are examples of non-abelian groups of self-equivalences of $G$-spheres). In the cases that I checked $\mathcal{K}$ was always abelian, but I was not able to prove a general proposition. It is relatively easy to compute $\gamma_K$ for any isotropy group $K$, because it suffices to have the character of the representation to obtain information on the equivariant arrangement structure of the isotropy subspaces of $X$. On the other hand this does not give any information on the structure and commutators in $\mathcal{K}$ (or in $\mathcal{E}_G(X)$). To use the algebraic tools and the known results in homotopy theory it is needed at least the $G$-CW-structure of $X$, and even if this is a basically easy task to pursue (because a dihedral sphere is a $G$-join of simplicial complexes), the end computation gets too long and maybe it might be better a computer-algebra approach (as in [Fe01]).

References


Dipartimento di Matematica del Politecnico di Milano, Piazza Leonardo da Vinci 32, 20133 Milano (I)

E-mail address: ferrario@mate.polimi.it