ON THE EQUIVARIANT HOPF THEOREM

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ABSTRACT. Let $G$ be a finite group, $X$ a compact locally smooth $G$-manifold and $S$ an orthogonal $G$-sphere. The purpose of the paper is to compute, given a $G$-map $f : X \to S$ and under suitable assumptions, the number of distinct $G$-homotopy classes of maps $f' : X \to S$ such that $\deg(f'^H) = \deg(f^H)$ for every subgroup $H \subset G$, i.e. to count the number of $G$-homotopy classes in $[X, S]_G$ with the same stable equivariant degree $d_G$. To achieve this result, an unstable equivariant degree $\tilde{d}_G$ is introduced, with the property that, under the same assumptions, the number of $G$-homotopy classes of $G$-maps $f : X \to S$ having the same degree $\tilde{d}_G(f)$ is finite, and computable in terms of the orientation behavior of the Weyl groups $W_G H$ of the isotropy groups of $X$. This gives an equivariant analogue of the Hopf classification theorem. As a consequence, we find conditions under which the stable degree $d_G$ classifies $G$-maps $X \to S$ up to $G$-homotopy and we give some counter-examples.

1. Introduction

The well-known theorem of Hopf [Ho27] states that two maps $f_1, f_2 : X \to S$ from a compact manifold $X$ to a sphere $S$ of the same dimension are homotopic if and only if they have the same Brouwer degree. The purpose of the paper is to give an equivariant version of this theorem, in the perspective of the classification problem: if $G$ is a finite group, $X$ a compact locally smooth $G$-manifold and $S$ an orthogonal $G$-sphere such that $\dim X^H \leq \dim S^H$ for every subgroup $H \subset G$, then under which hypotheses two $G$-maps $f_1, f_2 : X \to S$ are $G$-homotopic if and only if the degrees of the restrictions $\deg(f_1^H) = \deg(f_2^H)$ coincide for every subgroup $H \subset G$? Furthermore, whenever this equivalence does not hold, we want to compute the number of distinct $G$-homotopy classes of maps $f_2 : X \to S$ such that $\deg(f_1^H) = \deg(f_2^H)$ for every subgroup $H \subset G$. This means computing the number of $G$-homotopy classes in $[X, S]_G$ with the same stable equivariant degree $d_G$ (see [Do83, tD87]).

In order to use equivariant obstruction theory to classify $G$-maps, we introduce an unstable equivariant degree $\tilde{d}_G$, defined simply as a

\[ \tilde{d}_G(f) = \sum_{H \subset G} \frac{\text{orient}(W_G H)}{|W_G H|} \deg(f^H), \]

where $\text{orient}(W_G H)$ is the orientation behavior of the Weyl group $W_G H$.

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The collection of Brouwer degrees of maps restricted to suitable subspaces of the fixed manifolds $X^H$. For the sake of simplicity, because for 0-dimensional spaces the Brouwer degree does not classify maps, we say that the degree $d_G$ classifies $G$-maps (up to $G$-homotopy) whenever two maps $f_1, f_2 : X \to S$, which coincide on the 0-dimensional equivariant strata $X_H$ of $X$, are $G$-homotopic if and only if their degrees $d_G(f_1)$ and $d_G(f_2)$ coincide. An analogous definition can be carried out for the unstable degree $\tilde{d}_G$. More precisely, we want to know under which extent the unstable equivariant degree can classify $G$-maps, and under which conditions the stable degree is equivalent to the unstable one.

First results specifically in this area were obtained by Rubinsztein [Rub76], Hauschild [Ha77], tom Dieck [tKP70, tD79] and Petrie [tP78], mainly using equivariant obstruction theory, with obstructions in Bredon - Illman equivariant cohomology [Wh78, tD79, tD87]. Further results were obtained by Tornehave [Tor82], Dold [Do83], Dancer [Da84], Lück [Lu87], Ulrich [Ul88] and others. More recent results have been proved by Peschke [Pe95], Kusshkuley and Balanov [BK95, BK96], Ize, Massabó and Vignoli [IV93, IMV89], Geba, Krawcewicz and Wu [GKW92], and many others, in a framework closer to nonlinear analysis. Deeper algebraic results and more general theories, as well as reviews and results about stable/unstable homotopy invariants can be found in [GM95, Ma96], by Greenlees and May.

In this paper, as already noted above, we consider the following settings: $G$ is a finite group, $X$ is a compact locally smooth manifold, and $S$ an orthogonal sphere. Actually the results could be extended to more general cases such as $G$-maps from $G$-CW-complexes or $G$-complexes on a compact Lie group $G$, to a homotopy $G$-sphere. We decided to work under these more restrictive hypotheses because the proofs and the statements of the theorems are much simpler, and on the other hand the interesting phenomena occurring are the same.

The main theorem of the paper is Theorem 5: it allows to compute the number of $G$-homotopy classes of maps $X \to S$ having the same unstable degree $\tilde{d}_G$, thus allowing –as a corollary– the same computation for the stable degree $d_G$. The important points are the following: first, this approach works only if some assumptions on $X$ and $S$ hold: not only $\dim X^H \leq S^H$ must be true for every subgroup $H \subset G$ but also a kind of “embeddability” of $X$ in $S$ must hold, denoted by $X \prec S$ (see definition 1). This hypothesis is fulfilled if $X$ has the same isotropy groups if $S$, or if $X$ has just one isotropy type, or in some other situations, but not in general. This is the first main limitation to this approach. Then, the notion of concordant and discordant chambers is needed (definition 5). In some of the known results, it was assumed that for every isotropy $H$ the space $X^H$ is orientable and that the action of the Weyl group $W_GH$ on the orientation of $X^H$ is the same as the action of $W_GH$ on $S^H$. The interesting point is that the orientation
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plays an important role, and there are just two cases: every chamber in $X/G$ contributes to the equivariant cohomology groups (in which the obstructions lie) with a factor isomorphic to $\mathbb{Z}$ whenever the chamber is $S$-orientable (see definition 4 or with a factor isomorphic to $\mathbb{Z}_2$ otherwise (see Lemma 6).

The third point of some interest is that there is a class of finite groups (named 2-split groups: see definition 2) that behave particularly well for the classification problem: if the group is 2-split and $X^H$ is orientable for every $H$, then the stable and unstable degrees are equivalent. This is related to the problem of computing the number of components in the stratum $X_H/W_G^H$: if $G$ is 2-split this is the same as the number of components of $X^H/W_G^H$.

The proofs have been carried out by using classical tools in algebraic topology: equivariant obstruction theory and Bredon-Illman cohomology, (equivariant) degree theory and induction on the isotropy types. The hypotheses needed have been weakened as much as possible: at the end of the paper a list of examples is given, that show how most of the assumptions made are necessary.

The paper has the following structure: after this introduction, in section 2 the basic notation is recalled, and a main Lemma 1 proved. In section 3 a characterization of 2-split groups is stated, and Corollary 3 proved. In section 4 the definition of the unstable equivariant degree $\tilde{d}_G$ is carried out in detail; its role and relationship with the stable equivariant degree $d_G$ is exploited in section 5. In section 6, after proving the key Lemma 6, the main Theorem 5 is proved, with two immediate Corollaries 6 and 7. The consequences for self-maps of $G$-spheres are proved in section 7, with Theorem 8. Finally, in section 8 some examples are given. Their aim is to illustrate some weak points of this approach, or to show that some assumptions cannot be weakened further, or simply to show how computations can be carried out.

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2. Preliminaries

Let $G$ be a finite group, $X$ a compact locally smooth $G$-manifold and $S$ an orthogonal $G$-sphere. In order to apply induction over isotropy types and equivariant obstruction theory, it is needed that $X \prec S$, according to the following definition.

Definition 1. We write $X \prec S$ if these two conditions are fulfilled:

1. For each $H$ in $\text{Iso}(X)$, $\dim X^H \leq \dim S^H$.
2. If $H$ and $K$ are isotropy subgroups for $X$ such that $H \subsetneq K$, then $S^K \subsetneq S^H$. 


In case \( X^H \) has components of different dimensions, \( \dim X^H \) denotes the maximal dimension.

This relation is a partial order in the set of orthogonal \( G \)-spheres, different than the usual one given by inclusion; if \( V \) is a component of the orthogonal \( G \)-representation \( U \) then \( S(V) \prec S(U) \), but it can be that \( S(V) \prec S(U) \) while \( V \) is not a component of \( U \).

Let us now recall some notation. If \( x \in X \), then we denote by \( G_x \) the isotropy group of \( x \), i.e. \( G_x = \{ g \in G \mid gx = x \} \). If \( H \) is a subgroup of \( G \), then let \( (H) \) denote the conjugacy class of \( H \) in \( G \), or, equivalently, the orbit type of \( G/H \). The set of fixed points of a subgroup \( H \) of \( G \) is \( X^H = \{ x \in X \mid G_x \supseteq H \} = \{ x \in X \mid Hx = x \} \); the singular set, the \( H \)-stratum and the \( (H) \)-stratum are respectively denoted by \( X^H_s = \{ x \in X \mid G_x \supsetneq H \} \), \( X_H = \{ x \in X \mid G_x = H \} = X^H \setminus X^H_s \), \( X_{(H)} = \{ x \in X \mid (G_x) = (H) \} = G(X^H \setminus X^H_s) \). The components of \( X_{(H)}/G \) are called chambers of \( X/G \) with isotropy type \( (H) \).

If \( (H) \) and \( (K) \) are two conjugacy classes of subgroups of \( G \), the symbol \( (H) \leq (K) \) or \( (H) < (K) \) means that \( H \) is subconjugated in \( K \). The set of isotropy types \( \text{iso}(X) \) is a poset with respect to subconjugation. For each conjugacy class \( (H) \) in \( \text{iso}(X) \) choose a representative \( H \) of \( (H) \), and let \( \text{Iso}(X) \) denote the set of such representatives. The normalizer of \( H \) in \( G \) is denoted by \( N_G H \); \( W_G H = N_G H / H \) is the Weyl group of \( H \) in \( G \).

If \( Y \) is a space, then \([X,Y]\) is the set of the (free) homotopy classes of maps \( f : X \to Y \); if \( A \) is a subspace of \( Y \) and \( f_0 \) a given map, then \([X,Y]^A\) denotes the set of maps which coincide with \( f_0 \) on \( A \), up to homotopy relative to \( A \). If \( X \) is a \( G \)-space, then \([X,Y]_G\) is the set of the free \( G \)-homotopy classes of \( G \)-maps from \( X \) to \( Y \). If \( A \subseteq X \) is a \( G \)-subspace and \( f_0 \) a given \( G \)-map, then \([X,Y]_G^A\) denotes the set of the \( G \)-homotopy classes relative to \( A \) of \( G \)-maps of \( X \) which coincide with \( f_0 \) on \( A \). Even if \( f_0 \) does not explicitly appear in the symbol, it is easy to deduce it from the context. For every subgroup \( H \), an equivariant map \( f : X \to Y \) induces a self-map \( f^H : X^H \to Y^H \) by restriction.

Throughout the paper by cohomology groups we mean singular cohomology groups with coefficients in \( \mathbb{Z} \), whenever not explicitly different; the equivariant cohomology groups we are considering are Bredon-Illman cohomology groups. Definitions and details can be found in [tD87, Wh78, Ha77, Do72, GM95, Ma96].

**Lemma 1.** Let \( H \in \text{Iso}(X) \), \( n = \dim S^H > 0 \), \( W = W_G H \) and \( M = \pi_n(S^H) \) as a \( \mathbb{Z}W \)-module. Let \( \overline{C} \) be a chamber in \( X_{(H)}/G \) of dimension \( n \) and \( C \) its pre-image in \( X_H \) under the projection \( p_H : X_H \to X_{(H)}/G \).
Let $P$ a base-point in $C$. Then the following diagram is commutative,

$$
\begin{array}{c}
H^{n-1}(X^H \setminus WP, X^H \setminus C; M) \xrightarrow{\partial} H^n(X^H \setminus WP; M) \xrightarrow{i^*} H^n(X^H, X^H \setminus C; M) \\
\uparrow N' \downarrow N \\
H^{n-1}(X^H \setminus WP, X^H \setminus C; M) \xrightarrow{\delta_W} H^n(X^H, X^H \setminus W; M) \xrightarrow{i_W^*} H^n(X^H, X^H \setminus C; M)
\end{array}
$$

where the horizontal rows are pieces of the exact sequences of the triple $(X^H, X^H \setminus WP, X^H \setminus C)$ with non-equivariant and Bredon-Illman equivariant cohomology respectively, and the vertical homomorphisms are norm homomorphisms ($N$, $N'$ and $N''$) or forgetting homomorphisms ($\Delta$ and $i_H$). Moreover, $N, N', N'', i^*$ and $i^*_W$ are surjective.

Proof. The diagram commutes because of the naturality of the norm and forgetting homomorphisms. To see that $i^*$ is onto, consider the piece of the exact sequence of the triple

$$
\cdots \longrightarrow H^n(X^H, X^H \setminus WP; M) \xrightarrow{i^*} H^n(X^H, X^H \setminus C; M) \xrightarrow{\partial} H^n(X^H \setminus WP, X^H \setminus C; M) \longrightarrow \cdots
$$

By Poincaré duality (with coefficients in $\mathbb{Z}$ if $C$ is orientable, otherwise with coefficients in $\mathbb{Z}_2$ and using the universal coefficient theorem) the latter group is 0

$$H^n(X^H \setminus WP, X^H \setminus C; M) \cong H_0(C, WP) \cong 0,$$

thus $i^*$ is onto.

The norm homomorphisms $N$ and $N''$ are onto because the dimension of $C$ is $n$ (and thus they are onto already at a cochain level: this holds for every top-dimensional cohomology group). To see that $i^*_W$ is onto, it is enough to use as for $i^*$ the exact sequence of the triad in equivariant cohomology; because the norm homomorphism

$$0 \cong H^n(X^H \setminus WP, X^H \setminus C; M) \longrightarrow H^n_W(X^H \setminus WP, X^H \setminus C; M)$$

is onto, the equivariant cohomology group is trivial, and thus $i^*_W$ is onto.

Note that the cohomology group $H^n(X^H, X^H \setminus WP; M)$ is isomorphic to $ZW$, while $H^n_W(X^H, X^H \setminus WP; M)$ is isomorphic to $Z$. This means that $N$, up to a change of the basis in $ZW$, sends generators $w \in W$ in $1 \in \mathbb{Z}$, and that $\Delta$ is the inclusion defined by $\Delta(1) = \sum w \in ZW$.

The next step is to show that $N'$ is onto. Consider the following diagram.

$$
\begin{array}{c}
\hom(C_n(X^H \setminus WP, X^H \setminus C), M) \xrightarrow{\partial''} \hom(C_{n-1}(X^H \setminus WP, X^H \setminus C), M) \xrightarrow{\partial''} \cdots \\
\downarrow N_n \uparrow N' \\
\hom_W(C_n(X^H \setminus WP, X^H \setminus C), M) \xrightarrow{\partial''_W} \hom_W(C_{n-1}(X^H \setminus WP, X^H \setminus C), M) \xrightarrow{\partial''_W} \cdots
\end{array}
$$
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The vertical homomorphisms \( N_n \) and \( N_{n-1} \) are the norm homomorphisms defined on the \( n \) and \((n-1)\)-dimensional cochain groups. They are surjective because all the groups involved are free \( \mathbb{Z}W \)-modules. The boundary homomorphisms \( \partial \) and \( \partial_W \) are onto, as a consequence of the fact that the cohomology groups are trivial

\[
H^n(X^H \setminus WP, \mathbb{Z}W ; M) \cong H^n_W(X^H \setminus WP, X^H \setminus C; M) \cong 0
\]

and the dimension of \( C \) is equal to \( n \). To make things simpler, let us define

\[
A = \text{hom}(C_n(X^H \setminus WP, X^H \setminus C), M)
\]
\[
B = \text{hom}(C_{n-1}(X^H \setminus WP, X^H \setminus C), M)
\]

and \( \partial = \partial^n : B \to A \). The \( \mathbb{Z}W \)-modules \( A \) and \( B \) are free, \( \partial \) is a \( \mathbb{Z}W \)-homomorphism, and the following diagram is commutative (the modules \( A_W \), \( B_W \) and \( \ker \partial_W \) are the submodules fixed by \( W \)).

\[
\begin{array}{cccc}
A & \xrightarrow{\sigma} & B & \xleftarrow{\kappa(\partial)} \\
\downarrow{N_n} & & \downarrow{N_{n-1}} & \\
A_W & \xleftarrow{\partial_W} & B_W & \xleftarrow{\ker(\partial_W)}
\end{array}
\]

Let \( b \in \ker \partial_W \). Then there is \( \tilde{b} \in B \) such that \( N_{n-1}(\tilde{b}) = b \). Let \( y \) be the element of \( B \) equal to \( y = \tilde{b} - \sigma(\partial(\tilde{b})) \), where \( \sigma \) is any equivariant section of \( \partial \) (at least one \( W \)-section exists, because \( A \) is \( \mathbb{Z}W \)-free). It is easy to see that \( y \in \ker \partial \) and that \( N_{n-1}(y) = b \). This means that \( N_{n-1} \) maps \( \ker \partial \) onto \( \ker \partial_W \). At a cochain level, this means that \( N_{n-1} \) maps the group of \((n-1)\)-cocycles onto the group of \((n-1)\)-equivariant cocycles, and this implies that \( N' \) is onto. The proof is complete.

The following Corollary is an immediate consequence of Lemma 1.

**Corollary 1.** Under the same notation of Lemma 1, the equivariant cohomology group \( H^n_W(X^H, X^H \setminus C; M) \) is isomorphic to \( H^n_W(X^H, X^H \setminus WP; M) \) mod. the image of \( \ker i^* \) under the norm homomorphism \( N \)

\[
H^n_W(X^H, X^H \setminus C; M) \cong \frac{H^n_W(X^H, X^H \setminus WP; M)}{N(ker i^*)} \cong \mathbb{Z} \frac{N(ker i^*)}{N(ker i^*)}.
\]

3. **A characterization of 2-split groups**

**Definition 2.** A finite group \( G \) is said to be 2-split if it is the direct product of its 2-Sylow subgroup and the 2-Sylow complement.

The following is a characterization of 2-split groups.

**Proposition 2.** Let \( G \) be a finite group. Then \( G \) is 2-split if and only if for every real nontrivial irreducible linear representation \( V \) of \( G \), and any self-normalizing isotropy subgroup \( H \) of \( G \), \( \dim V^H \neq 1 \).
Proof. See [Fe00], Theorem 1.1, page 353.

In the next Corollary, \( W = W_GH \) is the Weyl group of the isotropy group \( H \), \( n = \dim S^H > 0 \), \( M \) is the \( \mathbb{Z} W \)-module \( \pi_n(S^H) \cong \mathbb{Z} \), and \( (H^n(X^H, X^H_s) \otimes M)^W \) denotes the subgroup of the cohomology group \( H^n(X^H, X^H_s) \otimes M \cong H^n(X^H, X^H_s) \) fixed by \( W \), where the action on \( H^n(X^H, X^H_s) \) is the natural action induced by the action of \( W \) on \( X^H \).

**Corollary 3.** If \( G \) is a 2-split finite group, \( X \) a locally smooth \( G \)-manifold such that \( X \prec S \) and \( X^H \) is orientable then the homomorphism \( j \) defined by the following composition

\[
(H^n(X^H, X^H_s) \otimes M)^W \xrightarrow{j} H^n(X^H) \rightarrow H^n(X^H)
\]

is mono.

Proof. Let \( X^H_{s_0} \) denote the singular set of \( X^H \) with respect to the action of \( W \) on \( X^H \). It is contained in \( X^H_s \), and it could be contained properly (see [Fe99, Fe00]). Consider a point \( x \in X^H_s \setminus X^H_{s_0} \). Let \( G_x \) be its isotropy group. Because of the Slice Theorem for smooth \( G \)-manifolds, there is a real representation \( V \) of \( G_x \) and a \( G \)-neighborhood \( U \) of \( x \) in \( M \) \( G \)-diffeomorphic to the tube \( G \times_{G_x} V \). The codimension of \( V^{G_x} \) in \( V^H \) is the same as the codimension of a neighborhood of \( X^H_s \) in \( X^H \); if the codimension of \( V^{G_x} \) in \( V^H \) is 1, then the \( G_x \)-representation \( V \) is the sum of the trivial representation with a certain multiplicity, plus a non-trivial real representation \( V' \). It is easy to see that \( V'H \) has dimension 1. Moreover, because \( x \not\in X^H_{s_0} \), its isotropy with respect to the action of \( W_GH \) is trivial, and thus \( N_{G}H \cap G_x = H \), that is \( H \) is self-normalizing in \( G_x \). But \( G_x \) is a 2-split group, because it is a subgroup of the 2-split group \( G \), and therefore \( V' \) cannot have a self-normalizing isotropy subgroup which fixes a one-dimensional subspace. Hence it is not possible that the codimension of \( V^{G_x} \) in \( V^H \) is 1, and because \( x \) was arbitrary the codimension of \( X^H_s \setminus X^H_{s_0} \) in \( X^H \) is at least 2 (see also [Fe00], Lemma 3.3). As a consequence, \( H^n(X^H, X^H_s) \cong H^n(X^H, X^H_{s_0}) \), and the number of chambers in \( X(H)/G \) is the same as the number of components of \( X^H/W_GH \).

Now consider \( H^n(X^H, X^H_{s_0}) \). By Poincaré duality, because \( X^H \) is orientable, it is isomorphic to \( H_0(X^H \setminus X^H_{s_0}) \cong H_0(X^H) \). If \( C \) is a component of \( X^H \setminus X^H_{s_0} \), let \( W_0 \) be the subgroup of the elements of \( W \) which send \( C \) to \( C \). It is easy to see that

\[
H^n(X^H, X^H_{s_0}) \cong H_0(X^H) \otimes H^n(X^H)
\]
as $ZW$-modules. This implies that if the orientation behavior of $W_0$ on $C$ coincides with its orientation behavior on $S^H$ (that is on $M$), then

$$\left( H^n(X^H, X^H \setminus C) \otimes M \right)^{W_0} = Z,$$

otherwise

$$\left( H^n(X^H, X^H \setminus C) \otimes M \right)^{W_0} = 0.$$

Let us consider the first case, and let $W_1$ be the subgroup of $W$ generated by the elements $w$ such that $(X^H)^w$ has codimension 1 in $X^H$. If $w_1$ is such a generator, then $w_1^2 = 1$, and because we are assuming $X \preceq S$, $(S^H)^{w_1}$ is a sphere of codimension 1 in $S^H$. Thus $w_1$ has the same orientation behavior on $X^H$ and $S^H$. The Weyl group $W$ is generated by $W_0$ and $W_1$, because its action on the components of $WC$ is transitive, thus in this case,

$$H^n(X^H, X^H \setminus WC) \otimes M \cong H_0(WC) \otimes H^n(X^H) \otimes M \cong H_0(WC)$$

as $ZW$-modules, and the claim is now equivalent to saying that the composition

$$H_0(WC)^W \cong (ZW/W_0)^W \xrightarrow{\epsilon} H_0(WC) \cong ZW/W_0 \xrightarrow{\epsilon} Z$$

is mono, where $\epsilon$ is the augmentation homomorphism; this is clearly true.

In the second case,

$$\left( H^n(X^H, X^H \setminus WC) \otimes M \right)^W \subset \left( H^n(X^H, X^H \setminus WC) \otimes M \right)^{W_0} = 0,$$

so that $j$ is mono. The proof now is complete.

Remark 2. The assumption that $G$ is 2-split is necessary, as it can be seen in the proof of Theorem 8, or in examples 3 and 4. Also the assumption that $X^H$ is orientable is necessary, as it can be seen in examples 2, 6 and 7.

Remark 3. The submodule fixed by $W$ in the cohomology group

$$\left( H^n(X^H, X^H_s) \otimes M \right)^W \subset H^n(X^H, X^H_s)$$

is the same as the submodule

$$\left( \sum_{w \in W} \epsilon(w)w \right) \cdot H^n(X^H, X^H_s),$$

where $\epsilon : W \to \{+1, -1\}$ is the character of the $ZW$-module $M$. In the last section we will give examples showing that in general the fact that the stable degree $d_G$ classifies $G$-maps is not equivalent to the injectivity of $j$ restricted to $(H^n(X^H, X^H_s) \otimes M)^W$: in example 1
\((H^n(X^H, X^H_s) \otimes M)^W = 0\), so that \(j\) is injective, but \(d_G\) does not classify \(G\)-maps. In example 5 the cohomology submodule fixed by \(W\) is not trivial and \(j\) is injective, nevertheless \(d_G\) does not classify \(G\)-maps. Finally, in example 8 \(j\) is not injective, in spite of the fact that the equivariant degree \(d_G\) classifies \(G\)-maps.

4. Definition of the unstable equivariant degree \(\tilde{d}_G(f)\)

Let \(X\) be a compact locally smooth \(G\)-manifold and \(S\) an orthogonal \(G\)-sphere such that \(X \prec S\). Let \(\mathcal{J}(H)\) denote the set of components of \(X^H\), and \(\mathcal{J}\) the union of all the sets \(\mathcal{J}(H)\), with \(H\) in \(\text{Iso}(X)\). That is, \(\mathcal{J}\) is the set of all the components \(C\) of some \(X^H\), with \(H \in \text{Iso}(X)\). The isotropy subgroup of \(C\) will be denoted by \(H_C\) or simply by \(H\) whenever possible. For each \(C\) in \(\mathcal{J}\) let \(n_C\) denote the dimension of the sphere \(S^H_C\); let \(R_C\) be defined as \(R_C = H^{n_C}(X^H_C, X^H_C \setminus C)\).

This means that \(R_C\) is the group of integers \(\mathbb{Z}\) if \(X^H\) is orientable along \(C\) and the dimension of \(C\) is equal to \(n_C\), or the group \(\mathbb{Z}_2\) if \(X^H\) is not orientable along \(C\) and the dimension of \(C\) is equal to \(n_C\), and \(\{0\}\) otherwise. Finally, let \(\tilde{A}(X)\) denote the direct sum \(\tilde{A}(X) = \bigoplus_{C \in \mathcal{J}} R(C)\).

It is not difficult to see that
\[
\tilde{A}(X) = \bigoplus_{C \in \mathcal{J}} H^{n_C}(X^H_C, X^H_C \setminus C) \cong \bigoplus_{H \in \text{Iso}(X)} H^{n_H}(X^H, X^H_s).
\]

Now, for each \(H\) in \(\text{Iso}X\), choose a point \(y_H \in S^H\) with minimal isotropy type (that is, \(H \subset G_{y_H}\) and \(G_{y_H}\) is minimal). Because of the assumption \(X \prec S\), it is also \(\dim(X^H) \leq \dim(S^H)\), thus \(S^H\) cannot be empty, i.e. there exists such a point \(y_H\). It needs not to be unique, but its isotropy subgroup is unique: it is the intersection of all the isotropy subgroups for \(S\) which contain \(H\) (the poset of isotropy subgroups is closed with respect to intersections). Moreover, choose an orientation for \(S^H\) and for the orientable components of \(X^H\).

**Definition 3.** The unstable equivariant degree of the \(G\)-map \(f\) (over the points \(\{y_H\} \subset S\) is the element \(\tilde{d}_G(f) \in \tilde{A}(X)\) such that for each \(C \in \mathcal{J}\)

\[
\tilde{d}_G(f)(C) = \begin{cases} 
\deg(f^H_C|C, y_{HC}) & \text{if } \dim C = \dim S^H_C \\
0 & \text{if } \dim C < \dim S^H_C.
\end{cases}
\]

where \(\deg(f^H_C|C, y_{HC})\) denotes the degree of the map \(f^H_C\) restricted to the (open) subset \(C\) over the point \(y_{HC}\).
If $X^H$ is orientable along $C$, then the degree $\deg(f^H|C, y_H)$ is an integer, otherwise an integer mod. 2.

To see that $\tilde{d}_G(f)$ is well-defined, we only need to show that $(f^H)^{-1}y_H$ is compact in $C$ whenever $\dim C = \dim S^H$. This is true because if $x$ is a point in the closure of $C$ in $X^H$ and not in $C$, its isotropy $G_x$ is strictly greater than $H$. Thus, because $X \preceq S$, $S^H \supseteq S^{G_x}$. On the other hand, $y_H$ has minimal isotropy, therefore $y_H \notin S^{G_x}$; this means that $x$ cannot be in $(f^H)^{-1}y_H$, and therefore (because the closure of $C$ in $X^H$ is compact) that $(f^H)^{-1}y_H$ is compact.

By the same argument it is possible to prove that if $f_0$ and $f_1$ are two $G$-homotopic maps $f_0 \sim f_1 : X \to S$, then

$$\tilde{d}_G(f_1) = \tilde{d}_G(f_2),$$

that is, the unstable degree defines a function $\tilde{d}_G : [X, S]_G \to \tilde{A}(X)$.

5. Stable and unstable equivariant degree

We recall the definition of stable equivariant degree [Do83, Ul88, tD87, GM95]: let $A(X)$ be the ring $A(X) = \bigoplus_{H \in \text{Iso}(X)} H^{n_H}(X^H)$, and $\pi : \tilde{A}(X) \to A(X)$ the projection induced by

$$\bigoplus_{H \in \text{Iso}(X)} H^{n_H}(X^H, X^H) \to \bigoplus_{H \in \text{Iso}(X)} H^{n_H}(X^H).$$

The symbol $d_G$ denotes the (stable) equivariant degree, defined by $d_G(f)(H) = \deg(f^H, y_H)$ for all $H \in \text{Iso}(X)$. A trivial consequence of the properties of the degree is that $\pi \tilde{d}_G(f) = d_G(f)$, i.e. the following diagram is commutative

$$\begin{array}{ccc}
\tilde{A}(X) & \xrightarrow{\tilde{d}_G} & \tilde{A}(X) \\
\downarrow{\pi} & & \downarrow{\pi} \\
[X, S]_G & \xrightarrow{d_G} & A(X).
\end{array}$$

Remark 4. It is possible that $\pi$ is not an isomorphism, whenever restricted to the image of $\tilde{d}_G$, even if the top-dimensional component of $X_H/W_GH$ is connected for each $H$: it suffices to find examples in which $X_H$ is orientable but $X^H$ not; see example 3.

**Proposition 4.** Let $G$ be a 2-split group, $X$ a locally smooth $G$-manifold such that $X^H$ is orientable for every $H \in \text{Iso}(X)$, and $S$ an orthogonal $G$-sphere with $X \preceq S$. Then two $G$-maps $f_1, f_2 : X \to S$ have the
same unstable equivariant degree \( \tilde{d}_G \) if and only if they have the same stable equivariant degree \( d_G \).

**Proof.** Let \( f_1 \) and \( f_2 \) be two \( G \)-maps \( f_1, f_2 : X \rightarrow S \). Of course if \( \tilde{d}_G(f_1) = \tilde{d}_G(f_2) \) then \( d_G(f_1) = d_G(f_2) \). On the other hand, let \( f_1 \) and \( f_2 \) be two \( G \)-maps with \( d_G(f_1) = d_G(f_2) \). We can consider \( \tilde{d}_G(f_1) \) and \( \tilde{d}_G(f_2) \) as elements of \( \bigoplus_{H \in \text{Iso}(X)} H^{n_H}(X^H, X^H_s) \) and, because of Corollary 3, the conclusion follows once it is proved that

\[
\tilde{d}_G(f_1) - \tilde{d}_G(f_2) \in \left( H^n(X^H, X^H_s) \otimes M \right)^W,
\]

where \( M \) and \( W \) are as in Corollary 3. Let \( H \in \text{Iso}(X) \), \( y_H \in S^H \) the point chosen to define the unstable degree, \( n_H = \dim S^H \) and \( C \) a component of \( X_H \). While the degree \( \deg(f_1^H|C, y_H) \) depends upon the choice made about the point \( y_H \in S^H \), the difference \( \deg(f_1^H|C, y_H) - \deg(f_2^H|C, y_H) \) does not. As an immediate consequence, for every \( w \in W \),

\[
\deg(f_1^H|C, y_H) - \deg(f_2^H|C, y_H) = \deg(w : S^H) \cdot \deg(w : X^H) \cdot \left( \deg(f_1^H|wC, y_H) - \deg(f_2^H|wC, y_H) \right),
\]

where \( \deg(w : S^H) \) and \( \deg(w : X^H) \) denote the orientation behavior of \( w \) on \( S^H \) and \( X^H \) respectively. But this exactly means that

\[
\tilde{d}_G(f_1) - \tilde{d}_G(f_2) \in \left( H^{n_H}(X^H, X^H_s) \otimes M \right)^W,
\]

that is the thesis. \( \square \)

**Remark 5.** In general, if \( S \) is a \( G \)-sphere, \( X \prec S \) a compact locally smooth \( G \)-manifold, \( H \in \text{Iso}(X) \) an isotropy group and \( y_H \in S^H \) the point chosen as above in definition 3 and \( n = \dim S^H \), then it is possible to define (in a unique way, up to homotopy) a map (not necessarily \( WG_H \)-equivariant)

\[
F : X^H \times \partial I \cup X^H_s \times I \rightarrow S^H
\]

such that \( F(X^H) \subset S^H \setminus \{y_H\} \) and \( F(-, 0) = f_1, F(-, 1) = f_2 \). As in [Wh78], page 240, let \( (f_1, f_2)_H^* \) denote the composition

\[
\begin{array}{ccc}
H^n(S^H) & \xrightarrow{F^*} & H^n(X^H \times \partial I \cup X^H_s \times I) \\
\downarrow{\delta^*} & & \downarrow{(i^* \times -)^{-1}} \\
H^{n+1}(X^H \times I, X^H \times \partial I \cup X^H_s \times I) & \xrightarrow{(i^* \times -)^{-1}} & H^n(X^H, X^H_s).
\end{array}
\]

With an abuse of notation, again we can consider \( \tilde{d}_G(f_i) \) as elements of the direct sum cohomology groups \( H^n(X^H, X^H_s) \), simply by choosing
coherent orientations on the components of $X_H$. Thus it is not difficult to see that, if $\iota(S^H)$ denotes the standard generator of $H^n(S^H)$,

$$\bar{d}_G(f_2) - \bar{d}_G(f_1) = \sum_{H \in \text{Iso}(X)} (f_1, f_2)_H \iota(S^H).$$

6. Homotopy classification of $G$-maps

**Definition 4.** Let $H \in \text{Iso}(X)$, and let $p_H : X_H \to X_H/W_GH$ be the projection onto the quotient. A chamber $\overline{C}$ of $X/G$ with isotropy type $(H)$ (i.e. a component of $X(\iota(H))/G$) is said to be $S$-orientable if $H^{n_H}(X^H, X^H \setminus C) \cong H^{n_H}(S^H)$ as $\mathbb{Z}W_0$-modules, for a component $C$ of $p_H^{-1}\overline{C}$, where $W_0$ denotes the subgroup of the Weyl group given by the elements $w \in W_GH$ such that $wC = C$.

The definition does not depend upon the choice of the component $C$ in $p_H^{-1}\overline{C}$, nor upon the choice of the representative $H$ in $(H)$. Equivalently, a chamber $\overline{C}$ of $X/G$ with isotropy type $(H)$ is $S$-orientable whenever there is an orientable component $C$ in $X_H$, covering $\overline{C}$, such that an element of $W_GH$ which sends $C$ to $C$ reverses the orientation of $C$ if and only if it reverses the orientation of $S^H$.

**Definition 5.** We say that a chamber $\overline{C}$ of $X/G$ of isotropy type $(H)$ is discordant with $S$ if one of the following two cases holds

1. $p_H^{-1}\overline{C}$ is orientable but $\overline{C}$ is not $S$-orientable;
2. $p_H^{-1}\overline{C}$ is not orientable and there is a component $C$ of $p_H^{-1}\overline{C}$ such that there is an even number of elements $w \in W_GH$ such that $wC = C$.

If $\overline{C}$ is not discordant with $S$, then it is said concordant with $S$.

In other words, $\overline{C}$ is concordant with $S$ if $\overline{C}$ is $S$-orientable, or if $p_H^{-1}\overline{C}$ is not orientable and the subgroup $W_0 = \{w \in W \mid wC = C\}$ has odd order, with $C$ component of $p_H^{-1}\overline{C}$.

Furthermore, we say that a chamber $\overline{C}$ of isotropy type $(H)$ is full-dimensional whenever $\dim(\overline{C}) > 0$ and $\dim(\overline{C}) = \dim(S^H)$.

**Lemma 6.** Let $H \in \text{Iso}(X)$ with $n = n_H = \dim(S^H) > 0$ and $W = W_GH$. Then the equivariant Borel-Ilman cohomology group is

$$H^n_W(X^H, X^H_s; \pi_n(S^H)) \cong \mathbb{Z}^\varphi \oplus \mathbb{Z}_2^\tau,$$

where $\varphi$ is the number of full-dimensional $S$-orientable chambers of $X/G$ with isotropy type $(H)$, and $\tau$ the number of the other full-dimensional chambers of isotropy type $(H)$. Moreover, the kernel of the natural forgetting homomorphism

$$t_H : H^n_W(X^H, X^H_s; \pi_n(S^H)) \to H^n(X^H, X^H_s),$$

is an elementary Abelian 2-group with $\kappa$ generators, where $\kappa$ is the number of full-dimensional chambers of $X/G$ discordant with $S$, with isotropy type $(H)$. 

Proof. Let $\overline{C}$ be a chamber in $X(H)/G$ and $C$ a component of $p_H^{-1}\overline{C}$. Let $W$ denote the Weyl group $W_GH$ and $n = n_H$. The conclusion follows if we prove that

$$H^n_W(\mathcal{X}H, \mathcal{X}H \setminus WC; \pi_n(S^H))$$

is isomorphic to $\mathbb{Z}$ whenever $\overline{C}$ is a full-dimensional $S$-orientable chamber, and to $\mathbb{Z}_2$ if it is full-dimensional but not $S$-orientable; moreover, we must prove that if $\overline{C}$ is full-dimensional, then the kernel of the homomorphism

$$t : H^n_W(\mathcal{X}H, \mathcal{X}H \setminus WC; \pi_n(S^H)) \rightarrow H^n(\mathcal{X}H, \mathcal{X}H \setminus WC)$$

is trivial whenever $\overline{C}$ in concordant with $S$ and is equal to $\mathbb{Z}_2$ whenever $\overline{C}$ is discordant with $S$.

The following cases are possible: $C$ not orientable, and $C$ orientable. Let $W_0$ denote the subgroup of $W$ of the elements $w \in W$ such that $wC = C$, $M$ the $\mathbb{Z}W$-module $\pi_n(S^H)$ and $P$ a point in $C$. Because of Lemma 1, the following diagram is commutative,

$$
\begin{array}{ccc}
H^n(X^H, X^H \setminus WP; M) & \xrightarrow{i^*} & H^n(X^H, X^H \setminus WC; M) \\
\downarrow N & & \downarrow t_H \\
H^n_W(X^H, X^H \setminus WP; M) & \xrightarrow{i_W^*} & H^n_W(X^H, X^H \setminus WC; M)
\end{array}
$$

and the kernel of $i_W^*$ is the image of ker $i^*$ under $N$.

If $C$ is orientable, then the main diagram is specified into the following one

$$
\begin{array}{ccc}
\mathbb{Z}W & \xrightarrow{i^*} & \mathbb{Z}W/W_0 \\
\downarrow N & & \downarrow t_H \\
\mathbb{Z} & \xrightarrow{i_W^*} & H^n_W(X^H, X^H \setminus WC; M)
\end{array}
$$

where $N$ sends any free generator $w \in W$ in $N(w) \in \mathbb{Z}$ (up to choosing suitable orientations); by applying the definition of Bredon-Illman equivariant cohomology, it is not difficult to prove that $N(w)$ is equal to $-1$ if $w$ reverses the orientation in $S^H$ and to $+1$ if $w$ preserves the orientation.

On the other hand $i^*$ is determined by the local orientation behavior of the elements of $W_0$ on $X^H$ as follows: once an orientation is chosen in every component of $WC$, the image of a free generator $w \in W$ under $i^*$ is equal to $\epsilon(w)wW_0 \in \mathbb{Z}W/W_0$, where $\epsilon(w)$ is $+1$ whenever $w$ preserves the chosen orientation in $WC = p_H^{-1}\overline{C}$ and $-1$ otherwise.

Thus the kernel ker $i^*$ is generated by those elements in $\mathbb{Z}W$ that can be written as $(1 - \epsilon(w_0)w_0)w$ with $w_0 \in W_0$ and $w \in W$ (a system of free generators can be obtained by imposing that $w$ belongs to a suitable transversal set for $W_0$).
The kernel of $i^*_W$ is generated by the images of such generators under $N$, that is by the set of integers

$$(1 - \epsilon(w_0)N(w_0)) \cdot N(w) = \pm (1 - \epsilon(w_0)N(w_0))$$

with $w_0 \in W_0$ and $w \in W$.

If $\overline{C}$ is $S$-orientable, then by definition for every $w_0 \in W_0 \epsilon(w_0) = N(w_0)$, therefore $\ker i^*_W = 0$ and the forgetting homomorphism $t_H : H^n_W(X^H, X^H \setminus WC; M) \cong \mathbb{Z}W/W_0$ is mono, as claimed.

If $\overline{C}$ is not $S$-orientable, then $\ker i^*_W = 2\mathbb{Z}$ and therefore $t_H : H^n_W(X^H, X^H \setminus WC; M) \cong \mathbb{Z}_2 \rightarrow H^n(X^H, X^H \setminus WC; M) \cong \mathbb{Z}W/W_0$ is the trivial homomorphism, with kernel $\mathbb{Z}_2$. This was the claim in this case.

Now consider the case $C$ not orientable. The main diagram is the following.

$$\begin{array}{ccc}
\mathbb{Z}W & \xrightarrow{i^*} & \mathbb{Z}_2W/W_0 \\
\downarrow N & & \downarrow t_H \\
\mathbb{Z} & \xrightarrow{i^*_W} & H^n_W(X^H, X^H \setminus WC; M)
\end{array}$$

The projection $N$ is defined as before, but $i^*$ is now simply defined by $i^*(w) = wW_0 \in \mathbb{Z}_2W/W_0$. This means that the kernel of $i^*$ is generated by all those elements that can be written as $(1 \pm w_0)w$, with $w_0 \in W_0$ and $w \in W$. Therefore the kernel of $i^*_W$ is $2\mathbb{Z}$, and $t_H$ is defined by

$$t_H : 1 \in \mathbb{Z}_2H^n_W(X^H, X^H \setminus WC; M) \mapsto \sum_{w \in W} w \in \mathbb{Z}_2W/W_0 \cong H^n(X^H, X^H \setminus WC; M).$$

This implies that if the order $|W_0|$ is odd then $t_H$ is mono and if $|W_0|$ is even then $t_H$ is the trivial homomorphism. There are no other cases, so the proof is complete.

We come now to the main point: under which hypotheses can the degree $\tilde{d}_G$ classify equivariant maps $X \to S$? Furthermore, how many homotopy classes of $G$-maps do have the same unstable degree of a given $G$-map $f : X \to S$?

**Theorem 5.** Let $X$ be a compact locally smooth $G$-manifold and $S$ a $G$-sphere with $X \prec S$. Let $\kappa(X,S)$ denote the number of full-dimensional chambers of $X/G$ discordant with $S$. For every homotopy class $[f] \in [X,S]_G$ there are $2^{\kappa(X,S)}$ homotopy classes $[f']$ (including $[f]$) with the same unstable degree $\tilde{d}_G(f') = \tilde{d}_G(f)$ which coincide with $f$ on the 0-dimensional equivariant strata.
Proof. Let \( f : X \to S \) be a \( G \)-map, and let \( K \subset [X,S]_G \) be the set of all the classes of \( G \)-maps \([f']\) with the same unstable degree \( \tilde{d}_G(f') = \tilde{d}_G(f) \).

We can assume that the isotropy subgroups are indexed as \( \text{Iso}(X) = \{H_i\}_{i=1..N} \), such that \((H_i) \leq (H_j) \implies i \geq j \) (\( N \) is the cardinality of \( \text{Iso}(X) \)). For \( i = 1 \ldots N \) let
\[
X_i := \bigcup_{j=1}^{i} X_{(H_i)},
\]
and let \( X_0 := \emptyset \).

Because of the choice of the total ordering, for each \( i \leq j \), \( X_j^{H_i} = X_j^{H_i} \); therefore \( X_i^{H_i} = X_j^{H_i} \), and \( X_{i-1}^{H_i} = X_{s}^{H_i} \).

The restriction map \( f \mapsto f^{H_i} \) induces an isomorphism
\[
[X_i,S]_G^{X_{i-1}} \cong [X_j^{H_i},S]_{W_{G,H_i}}^{X_{j}^{H_i}}.
\]
For details see [tD87], Proposition I.7.4, page 52.

For each \( i = 1..N \) let \( K_i \subset [X_i,S]_G \) be the set of classes of \( G \)-maps obtained by restricting maps in \( K = K_N \) to \( X_i \), that is the image of \( K \) under the projection
\[
[X,S]_G \to [X_i,S]_G.
\]
We want to show by induction that the cardinality of \( K_i \) is \( 2^\kappa \), where \( \kappa \) is the number of full-dimensional chambers of \( X/G \) discordant with \( S \) with isotropy in \( \{(H_1),\ldots,(H_i)\} \).

If \( i = 1 \), then \( H = H_1 \) is maximal, thus by equivariant obstruction theory
\[
[X_i,S]_G \cong [X^{H_1},S^{H_1}]_W \cong H^n_W (X^{H_1};\pi_n S^{H_1}),
\]
where \( W = W_{G,H} \) and the latter bijection is given by the map
\[
[f^{H_1}] \in [X^{H_1},S^{H_1}]_W \mapsto \gamma_W (f^H, f^{H_1}) \in H^n_W (X^{H_1};\pi_n S^{H_1})
\]
and \( \gamma_W (f^H, f^{H_1}) \) is the class of the equivariant difference cochain of \( f^H \) and \( f^{H_1} \) (see [Wh78, tD87] and remark 5).

By definition of \( \gamma_W \), \([f^{H_1}]\) is in \( K_1 \) if and only if \( \gamma_W (f^H, f^{H_1}) \) belongs to the kernel of the natural forgetting homomorphism
\[
t_H : H^n_W (X^{H_1};\pi_n S^{H_1}) \to H^n (X^H).
\]
By Lemma 6, the kernel is an elementary Abelian 2-group with \( k \) generators, where \( k \) is the number of full-dimensional chambers of \( X/G \) discordant with \( S \), with isotropy type equal to \((H_1)\). This proves the assertion in the case \( i = 1 \).

Now consider \( H = H_i \) with \( i \geq 2 \), and the projection \( \tau_i \)
\[
\tau_i : K_i \to K_{i-1}
\]
induced by the restriction. By equivariant obstruction theory, the pre-
image under $r_i$ of any element of $K_{i-1}$ has as many elements as the
kernel of the natural homomorphism

$$t_H : H^n_W(X^H, X^H_s, \pi_n(S^H)) \to H^n(X^H, X^H_s).$$

By Lemma 6, the kernel of $t_H$ is an elementary Abelian 2-group with
$k$ generators, where $k$ is the number of full-dimensional chambers of
$X/G$ discordant with $S$, with isotropy type equal to $(H)$. Thus $K_i$ has
$2^k \cdot |K_{i-1}|$ elements. If we assume that the proposition holds for
$K_{i-1}$, this implies that $K_i$ has $2^k \cdot 2^h = 2^{h+k}$ elements, where $h$ is the number
of full-dimensional chambers of $X/G$ discordant with $S$, with isotropy
type in $\{(H_1), \ldots, (H_{i-1})\}$; this completes the proof. □

The next Corollary follows immediately from Theorem 5

Corollary 6. Let $G$ be a finite group, $X$ a compact locally smooth $G$-
manifold and $S$ an orthogonal $G$-sphere such that $X \preceq S$. Then the
unstable equivariant degree $\tilde{d}_G$ classifies $G$-maps in $[X,S]_G$ if and only
if every full-dimensional chamber in $X/G$ is concordant with $S$.

Corollary 7. Let $G$ be a finite group, $X$ a compact locally smooth $G$-
manifold and $S$ an orthogonal $G$-sphere such that $X \preceq S$. If the group
$G$ is 2-split and for every $H \in \text{Iso}(X)$ the space $X^H$ is orientable, then
the stable degree $d_G$ classifies $G$-maps in $[X,S]_G$ if and only if every
full-dimensional chamber in $X/G$ is discordant with $S$. More precisely,
for every $G$-homotopy class $[f] \in [X,S]_G$ there are $2^{\kappa(X,S)}$ $G$-homotopy
classes $[f']$ (including $[f]$ and coinciding with $f$ on the 0-dimensional
equivariant strata) with the same stable degree $d_G(f') = d_G(f)$, where
$\kappa(X,S)$ is the number of full-dimensional chambers of $X/G$ discordant
with $S$.

Proof. As a consequence of Proposition 4, two maps have the same
stable equivariant degree if and only if they have the same unstable
equivariant degree. This fact and Theorem 5 together imply that for every
homotopy class $[f] \in [X,S]_G$ there are $2^{\kappa(X,S)}$ homotopy classes
$[f']$ (including $[f]$) with the same stable degree $d_G(f') = d_G(f)$, where
$\kappa(X,S)$ is the number of chambers of $X/G$ discordant with $S$ with
non-zero dimension. □

Remark 7. Note that if $G$ is 2-split and $X^H$ is connected, then $X_{(H)}/G$
is connected, i.e. there is just one chamber of isotropy type $(H)$. So in
this case $\kappa(X,S)$ can not be greater than the number of isotropy types
in $X$.

7. EQuivariant self-maps of representation spheres

Theorem 8. Let $G$ be a finite group. If $G$ is 2-split, then the stable
degree $d_G$ classifies equivariant self-maps of every linear $G$-sphere. If
If $G$ is not 2-split, then there exists a $G$-sphere $S$ such that there are infinitely many $G$-homotopy classes of self-maps of $S$ with the same stable degree $d_G(f)$.

Proof. If $X$ is a linear $G$-sphere, then for every $H \in \text{Iso}(X)$ the subspace $X^H$ is orientable, and clearly for self-maps every chamber in $X/G$ is concordant with $S = X$. Thus, as a consequence of Corollary 7, if $G$ is 2-split then the unstable degree $d_G$ classifies equivariant self-maps of $X$.

If $G$ is not 2-split, then applying Proposition 2 it is possible to find a nontrivial (irreducible) orthogonal representation $V$ of $G$ such that there is a self-normalizing isotropy subgroup $H$ of $G$ with $\dim V^H = 1$. Let $X$ be the unit sphere in $V \oplus \mathbb{R}$, where $\mathbb{R}$ denotes the trivial real representation. The sphere fixed by $H$ is a circle, with two points of isotropy $G$. The Weyl group is trivial, because $H$ is self-normalizing, thus $X_{(H)}/G \cong X_H/W_GH = X_H$ has at least two chambers. Given a $G$-self-map $f_0 : X \to X$, it is therefore easy to define infinitely many $G$-self-maps $f : X \to X$ with the same stable degree $d_G(f) = d_G(f_0)$ but with different unstable degrees as follows (see also [Fe00]). Let $f$ coincide with $f_0$ on the 0-dimensional fixed subspaces $X^K$. Then, because the Weyl group $W_GH$ of $H$ is trivial, the equivariant cohomology group coincides with the ordinary cohomology group $H^n(X^H, X^H_s)$, thus the map

$$[f] \in [X^H, X^H_s] \xrightarrow{X^H \mapsto (f^H, f_0^H)^*\iota(X^H)} H^n(X^H, X^H_s)$$

gives a bijection between $W_GH$-extensions of $f^H|X^H_s$ (up to homotopy) and cohomology classes in $H^n(X^H, X^H_s)$. Two such maps have the same degree $d(f^H) = d(f_0^H)$ if and only if $(f^H, f_0^H)^*\iota(X^H)$ belongs to the kernel of the homomorphism

$$H^n_W(X^H, X^H_s) \to X^n(X^H).$$

Because $X_H$ has at least two components, such a kernel is an infinite subgroup. Now, every such an extension $f^H$ can be further extended to a $G$-map $f$ with the same stable degree of $f_0$. By definition of the unstable degree (see remark 5), this procedure gives the wanted infinite family of $G$-maps. \qed

Remark 8. If $X$ is a complex $G$-sphere, then the codimension of every singular set $X^H_s$ in $X^H$ is at least 2, thus there is at most one chamber in $X_{(H)}/G$ with positive dimension, and this means that the stable degree classifies $G$-self-maps.
8. Examples

**Example 1.** Let $G$ be the group of order 2, with generator $g$, $X$ the unit circle in the complex plane $\mathbb{C}$ with the $G$-action given by $gz = -z$; let $S$ denote the unit circle in $\mathbb{C}$ with the $G$-action given by $gz = \overline{z}$, where $\overline{z}$ denotes the complex conjugate of $z$. The set $\text{Iso}(X)$ consists only of the trivial subgroup, hence $X \prec S$. Moreover, $G$ is 2-split and $X$ is orientable. Let $H = \{1\}$ be the trivial subgroup of $G$. The orbit space $X/G = X_H/G$ is connected, thus there is just one chamber in $X/G$. Because the orientation behavior of $g$ on $X$ is different than the behavior of $g$ in $S$, this chamber is discordant with $S$. Therefore the number $\kappa(X, S)$ of (full-dimensional) chambers in $X/G$ discordant with $S$ is 1.

Thus, applying Corollary 7, for every homotopy class $[f]$ in $[X, S]_G$ there are $2^{\kappa(X, S)} = 2$ $G$-homotopy classes $[f']$ (including $[f]$) with the same stable degree $d_G(f)$. On the other hand if $p$ is the degree of a $G$-map $f : X \to S$, then by equivariance it must be $\deg(f \circ g) = \deg(g \circ f)$, that is, $\deg f = -\deg f \implies \deg f = 0$. Hence there are exactly 2 distinct $G$-homotopy classes of equivariant maps in $[X, S]_G$, and for both the stable degree is 0.

It is possible to get the same result simply by computing the equivariant cohomology group using Lemma 6

$$H^1_W(X, X_s; \pi_1(S^1)) = H^1_W(X; \pi_1(S^1)) \cong \mathbb{Z}_2,$$

where $W = W_GH = G$ is the Weyl group of the trivial subgroup and $\pi_1(S^1)$ is endowed with the non-trivial action of $G$. Moreover, a more careful sight of the equivariant obstruction shows that representatives of the two classes in $[X, S]_G$ are indeed given by the two constant maps $f_1 : z \mapsto 1$ and $f_2 : z \mapsto -1$.

**Example 2.** Let $G$ be the group of order 2 generated by the element $g$, $X$ the real projective plane $\mathbb{RP}^2$ with homogeneous coordinates $(x_0 : x_1 : x_2)$ on which $g$ acts by sending $(x_0 : x_1 : x_2)$ to $(-x_0 : x_1 : x_2)$, and $S$ the 2-dimensional unit sphere in $\mathbb{R}^3$; let $g$ act on $S$ by reflection along a plane through the origin $0 \in \mathbb{R}^3$. The set of isotropies $\text{Iso}(X)$ has two elements: $G$ and the trivial subgroup $H = \{1\}$. The fixed subspace $X^G$ of $X$ is the projective line with equation $x_0 = 0$ in $X$, topologically a circle, plus the point $P = (1 : 0 : 0)$. The space fixed by $G$ in $S$ is a maximal circle $S^G$, and $X \prec S$. There is one chamber in $X/G$ with isotropy type $G$ and dimension 1, and one with isotropy $H$ and dimension 2. The Weyl group $W_GG$ is trivial, thus the one-dimensional chamber in $X^G/G$ is concordant with $S$. The stratum $X_H = X \setminus X^G$ is connected, thus the subgroup $W_0$ of the Weyl group defined in Definition 4 is the whole $G$, and, under the same notation,
its unique component is \( C = X_H \). Because
\[
H^2(X, X \ \setminus \ \{C\}) = H^2(X, X) \cong \mathbb{Z},
\]
it remains to check whether \( X_H/G \) is \( S \)-orientable or not. The action of \( g \) on \( X_H \) preserves the orientation, while its action on \( S \) reverses the orientation. This means that the chamber \( C/G \) is discordant with \( S \).

We can now apply Theorem 5: the number \( \kappa(X, S) \) of full-dimensional chambers discordant with \( S \) is 1. Thus if \( f : X \rightarrow S \) is a \( G \)-map, there is another \( G \)-homotopy class (distinct from \([f]\)) with the same unstable degree \( \tilde{d}_G(f) \). Clearly this implies that the unstable degree does not classify \( G \)-maps.

The set of \( G \)-maps \([X, S]_G\) is in bijection with
\[
H^1(X^G; \pi_1(S^G)) \times H^2_W(X, X^G; \pi_2(S)) \cong \mathbb{Z} \times \mathbb{Z}_2
\]
via the usual bijection of obstruction theory (and induction over orbit types). Furthermore, the image of
\[
t_H : H^2_W(X, X^G; \pi_2(S)) \rightarrow H^2(X, X^G; \pi_2(S))
\]
is 0: this means that the unstable degrees of two maps \( f_1 \) and \( f_2 \) coincide whenever the degrees of \( f_1^G \) and \( f_2^G \) are the same, i.e. that the degree of \( f : X \rightarrow S \) is uniquely determined by the degree of the restriction \( f^G : X^G \rightarrow S^G \). Actually in this case it is a trivial consequence of the fact that the degree of a \( G \)-map \( f : X \rightarrow S \) is always 0 mod 2).

**Example 3.** Let \( G \) denote the dihedral group of 6 elements, or equivalently the symmetric group \( S_3 \). Let \( X \) be the three-dimensional real projective space \( \mathbb{RP}^3 \) with homogeneous coordinates \((x_0 : x_1 : x_2 : x_3)\). Let \( G \) act on \( X \) by permuting the last three coordinates \( x_1, x_2 \) and \( x_3 \). We denote the elements of \( G \) as permutations, in the disjoint cyclic notation. The subgroups (up to conjugacy) of \( G \) are the trivial subgroup \( H_1 \), the subgroup \( H_2 \) of order 2 generated by the permutation \((1 2)\), the subgroup of order 3 generated by \((1 2 3)\) and \( G \) itself. The fixed space \( X^{H_2} \) is the disjoint union of the point \((0 : 1 : -1 : 0)\) and the projective plane of equation \( x_1 = x_2 \). The space \( X^{H_3} \) and the space \( X^G \) are both equal to the projective line through the points \((0 : 1 : 1 : 1)\) and \((1 : 0 : 0 : 0)\). Thus the set \( \text{Iso}(X) \) consists of the elements \( H_1 = \{1\} \), \( H_2 \) and \( G \). Consider now the action of \( G \) on \( \mathbb{R}^4 \) given by the same permutations on the Euclidean coordinates \((x_0, x_1, x_2, x_3)\); let \( S \) be the 3-dimensional sphere in \( \mathbb{R}^4 \). The space fixed by \( G \) on \( S \) is a circle and the space \( X^{H_2} \) is a 2-sphere; Because the sets \( \text{Iso}(X) \) and \( \text{Iso}(S) \) coincide, \( X \prec S \).

Now consider the chambers in \( X/G \). The chamber with isotropy type \( G \) is concordant with \( S \), because the Weyl group \( W_GG \) is trivial and \( X_G = X^G \) is orientable. The stratum with isotropy type \((H_2)\)
has a 0-dimensional chamber and a full-dimensional chamber, the pre-image of which in $X_{H_2}$ is homeomorphic to a disc. Furthermore, $H_2$ has trivial normalizer. As a consequence, this chamber is concordant with $S$. The chamber corresponding to the trivial isotropy type $(H_1)$ is concordant with $S$: in fact, $X_{H_1}$ is homeomorphic to $\mathbb{R}P^3$ minus three points and three planes with 1-dimensional intersection. Hence it has 3 components, and $G$ acts transitively on them. The subgroup $W_0$ of $G$ fixing one component is conjugate to $H_2$, and it acts reversing the orientation. Thus also this chamber is concordant with $S$.

By Corollary 6, the unstable degree $d_G$ classifies equivariant maps in $[X,S]$. Moreover, it is now possible to define, for every $i \in \mathbb{Z}$, $G$-maps $f_i : X \to S$ such that $\deg(f^G_i) = 0$, $\deg(f_{H_2}, y_{H_2}) = i$ (simply extending $f^G$ to $X^{H_2} = \mathbb{R}P^2 \cup \{x\} \supset \mathbb{R}P^3 = X^G$) and $\deg(f) = 0$, so that $f_i$ and $f_j$ are $G$-homotopic if and only if $i = j$. But, on the other hand, $d_G(f_i) = d_G(f_j)$ if and only if $i = j$ mod 2. We have just seen an example of a space for which the stable degree does not classifies maps, despite the fact that $W_G H$ acts transitively on the components of $X_H$ for every isotropy $H$ and that every full-dimensional chamber in $X/G$ is concordant with $S$.

**Example 4.** Let $G = S_n$ be the $n$-th symmetric group, acting on $\mathbb{R}^n$ by permuting the coordinates $(x_1, x_2, \ldots, x_n)$ and let $X = S$ be the unit sphere in $\mathbb{R}^n$. Because $X^H$ is orientable for every $H \subset G$ and every chamber is concordant with $S$, the unstable degree $\tilde{d}_G$ classifies equivariant self-maps of $X$. Now, the fixed space $X^G$ is the set of two points with $\forall i, j$, $x_i = x_j$. Let $H$ be the isotropy group of the point $(0,0,\ldots,1) \in X$. It is isomorphic to $S_{n-1} \subset S_n = G$, the subgroup of permutations fixing $n$, and it is self-normalizing in $G$, if $n \geq 3$. For every $j \in \mathbb{Z}$ define a $G$-map $f_j : X \to X$ as follows. First, it is the antipodal map on $X^G$. The complement of $X^G$ in $X^H$ is the disjoint union of two open sets $X_{H^+}$ and $X_{H^-}$. Then, define $f^G_j$ on $X^H$ by extending $f^G_j$ in a way such that $\deg(f^G_j | X_{H^+}, y_{H}) = j + 1$ and $\deg(f^G_j | X_{H^-}, y_{H}) = -j$, where $y_{H}$ is the base-point for the isotropy $H$ chosen in $S^H = X^H$. Now proceed by induction over isotropy type: if $K$ is an isotropy subgroup, extend the map $f_j$ defined already on $X^K_{i}$ to the whole $X^K$ in a $W_K K$-equivariant way. It is easy to see that it is always possible to do this letting $f^K$ have no fixed points in $X_K$, and such that the fixed points of $f^K$ with isotropy type $H$ have the same index as the corresponding fixed points of $f^H$.

This implies that the fixed point index of $f^K_j$ in $X^K$ is 0 for every $K \in \text{Iso}(X)$, thus that the degree $\deg(f^K_j) = (-1)^{\dim X^K + 1}$. So $d_G(f_i) = d_G(f_j)$ for every $i, j \in \mathbb{Z}$, while $\tilde{d}_G(f_i) = \tilde{d}_G(f_j)$ if and only if $i = j$.

**Example 5.** Let $K$ denote the Klein bottle, that is the quotient space of the complex plane $\mathbb{C}$ under the action of the group generated by the
two isometries
\[
\begin{aligned}
A & : z \mapsto z + i \\
B & : z \mapsto \bar{z} + 1 + i,
\end{aligned}
\]
and \( G = \mathbb{Z}_2 \) the cyclic group of order 2 generated by the map \( w : [z] \mapsto [z + i/2] \). The action is well defined, because \( wA = Aw \) and \( wB = ABw \), and it is free; hence \( \text{Iso}(X) \) consists of the trivial subgroup only. Let \( S \) be the 2-sphere endowed with the trivial action.

The unique chamber in \( K/G \) is discordant with \( S \), because \( K \) is not orientable and the Weyl group of the trivial subgroup of \( G \) (itself) has order two. Thus, because of Theorem 5, there is another \( G \)-map with the same degree of the constant map \( c \in [K, S]_G \). Furthermore, the equivariant cohomology group \( H^G(K, \mathbb{Z}) \) is isomorphic to \( \mathbb{Z}_2 \); therefore there are just two classes of \( G \)-maps, both with the same unstable degree (which coincides with the stable one).

In this example we have seen that it is possible that \( X_H \) is non-orientable but \( X_H/W_GH \) is discordant with \( S \); furthermore, that the kernel of
\[
j| (H^n(X^H, X^H \otimes M)^W \to H^n(X^H)
\]
can be trivial even when \( d_G \) does not classify \( G \)-maps (see remark 3).

**Example 6.** Let us consider the same Klein bottle as in the previous example, but with a different action of \( G = \mathbb{Z}_2 \). In this case, let \( w : K \to K \) be defined by \( wz = z + 1 \). Because \( w^2 = B^2 \), \( wA = Aw \) and \( wB = Bw \), the action is well-defined. The fixed subspace \( K^G \) is the union of the two circles \( C_1 = \{ z = \bar{z} \} \) and \( C_2 = i/2 + C_1 \). Let \( S \) be a 2-sphere, with the action of \( G \) given by the reflection along a plane through its center.

The chamber of isotropy type \( G \) is concordant with \( S \), because the Weyl group of \( G \) is trivial. The chamber of trivial isotropy type has as a pre-image \( K_1 = K \setminus K^G \), which is orientable and connected. The element \( w \) preserves the orientation on \( K_1 \), while reverses the orientation in \( S \). This means that \( X_1/G \) is discordant with \( S \). Again, by simply looking at the equivariant cohomology of \( K \), we see that there are two distinct \( G \)-classes of maps which are homotopic to a constant whenever restricted to \( X^G \), such that their stable and unstable degrees coincide.

**Example 7.** Let again \( K \) denote the Klein bottle, and \( w : K \to K \) be the involution defined by \( z \mapsto \bar{z} + i/2 \). The fixed subspace \( K^G \) is a single circle, and the free part \( K_1 = K \setminus K^G \) is the disjoint union of two open Möbius bands. Let \( S \) be the 2-sphere with a reflection along a plane as action of \( w \in G \). The chamber with isotropy \( G \) is concordant with \( S \); on the other hand the components \( K_{1+} \) and \( K_{1-} \)
of \( K_1 \) are not fixed by \( w \), thus also the chamber \( K_1/G \) is concordant with \( S \). By Corollary 6, the unstable equivariant degree classifies \( G \)-maps. Moreover, it is easy to define two \( G \)-maps with \( f^G \) constant and \( \deg(f|_{K_1^+}) = \deg(f|_{K_1^-}) = 1 \mod 2 \) or \( \deg(f|_{K_1^+}) = \deg(f|_{K_1^-}) = 0 \mod 2 \). In both cases the degree of \( f : K \to S \) is zero, so that the stable equivariant degree does not classifies \( G \)-maps \( K \to S \).

**Example 8.** Let \( G \) be the group of order \( 2 \), \( X \) the unit circle in the complex plane \( \mathbb{C} \), with \( G \)-action given by conjugation \( g : z \mapsto z^2 \), and \( S \) the unit circle with trivial \( G \)-action. Even if the dimension hypothesis is fulfilled, \( X \not\prec S \). It is easy to see that the unstable degree \( \tilde{d}_G \) is not well-defined. Furthermore, the homomorphism \( j : H^1(X, X_s)^G \to H^1(X) \) is not injective, but there is just one \( G \)-homotopy class of \( G \)-maps in \([X, S]^G\), so that the stable equivariant degree \( d_G \) actually classifies \( G \)-maps.

**References**


ON THE EQUIVARIANT HOPF THEOREM


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