On some equations arising in Nonlinear Optics

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Plan of the talk

1. Axi-symmetric TE-modes.


Maxwell’s equations in a charge-free medium

\[
\partial_t B = -c \nabla \wedge E, \quad \partial_t D = c \nabla \wedge H, \\
\nabla \cdot B = 0, \quad \nabla \cdot D = 0.
\]

**Definition.** A field \( F : \mathbb{R}^4 \to \mathbb{R}^3 \) is *monochromatic* if it has the form

\[
F(x, t) = F_1(x) \cos(\omega t) + F_2(x) \sin(\omega t),
\]

where \( \omega \in \mathbb{R} \), \( x \in \mathbb{R}^3 \), \( F_1, F_2 : \mathbb{R}^3 \to \mathbb{R}^3 \).

Assumptions in the study of wave propagation in optical fibers:

\( B = H \) (non-magnetic medium)

\[
D(x, t) = \mathcal{E}(\omega, x, \frac{1}{2}(|E_1(x)|^2 + |E_2(x)|^2)) E(x, t),
\]

\[
E(x, t) = E_1(x) \cos(\omega t) + E_2(x) \sin(\omega t),
\]

where \( \mathcal{E} : J \times \mathbb{R}^3 \times [0, +\infty) \to (0, +\infty) \) is the dielectric response.
When the medium is axi-symmetric (optical fiber), we use cylindrical coordinates \((r, \theta, z)\):

\[
\mathbf{i}_r = (\cos \theta, \sin \theta, 0), \quad \mathbf{i}_\theta = (-\sin \theta, \cos \theta, 0), \\
\mathbf{i}_z = (0, 0, 1).
\]

Medium axi-symmetric \(\rightarrow\)

\[
\mathcal{E}(\omega, x, s) = \varepsilon(\omega, r, s), \quad \varepsilon(\omega, \cdot) \in C([0, \infty)^2),
\]

\[
\begin{cases}
0 < a(\omega) \leq \varepsilon(\omega, r, s) \leq b(\omega) < \infty, \\
\varepsilon(\omega, r, s) \xrightarrow{\text{unif. w.r.t. } r} \varepsilon(\omega, r, 0) \quad \text{as } s \to 0.
\end{cases}
\]

Self-focusing medium \(\rightarrow\)

\((\text{AS})\) \(\varepsilon(\omega, r, s)\) is non-decreasing in \(s\) \((\omega, r\) fixed).
A travelling wave propagating in \( x_3 \)-direction is a field \( F : \mathbb{R}^4 \to \mathbb{R}^3 \) of the type

\[
F(x, t) = w(x - t\xi), \quad \xi = (0, 0, s), \quad w : \mathbb{R}^3 \to \mathbb{R}^3,
\]

where \( s > 0 \) is the speed of propagation.

**Definition.** A field \( F : \mathbb{R}^4 \to \mathbb{R}^3 \) is axi-symmetric if \( F(\Gamma_\theta x, t) = \Gamma_\theta F(x, t) \) for all rotations \( \Gamma_\theta \) around \( x_3 \)-axis.

A *TE-mode* is a solution to Maxwell’s equations in which the electric field is an axi-symmetric, monochromatic travelling wave transverse to the direction of propagation:

\[
E(x, t) = v(r) \cos(kz - \omega t) i_\theta \quad \text{where} \\
v : [0, +\infty) \to \mathbb{R} \quad \text{amplitude} \\
k > 0 \quad \text{wave number,} \quad s = \omega/k \quad \text{wave-speed.}
\]
Existence of TE-modes $\longrightarrow$ solve 2nd order ODE

$\begin{cases}
v'' + \frac{1}{r}v' - \frac{1}{r^2}v + \left(\frac{\omega}{c}\right)^2 \varepsilon(\omega, r, \frac{v^2(r)}{2}) v - k^2 v = 0 \\
v \in C^2(0, \infty), \quad v(0) = 0.
\end{cases}$

If $v$ solves (1) $\Rightarrow$ a complete solution to Maxwell’s equations is

\[
D(x, t) = \varepsilon(\omega, r, \frac{1}{2}v^2(r)) v(r) \cos(kz - \omega t) \mathbf{i}_\theta, \\
B(x, t) = H(x, t) = \frac{c}{\omega} \left[ \frac{1}{r} [rv(r)]' \mathbf{i}_z \sin(kz - \omega t) \\
- kv(r) \mathbf{i}_r \cos(kz - \omega t) - k \right].
\]

**Guided TE-mode:** TE-mode satisfying the following guidance conditions: electro-magnetic energy is finite and all fields decay to 0 as distance from the axis of symmetry goes to $\infty$, i.e.

1. $\int_0^\infty (v^2 + |v'|^2) r \, dr < \infty$,
2. $v(r) \to 0$ and $v'(r) \to 0$ as $r \to \infty$. 


Fix $\omega$, and search for $(k, v)$ such that (1) and (i-ii) are satisfied. Set

$$g(r, s) = (\omega/c)^2 \varepsilon(\omega, r, s^2/2), \quad r \geq 0, \ s \in \mathbb{R},$$

$$u(r) = \sqrt{r} v(r),$$

$$\downarrow$$

(2) \quad \begin{cases}
  u'' - \frac{3}{4r^2} u + g(r, \frac{u(r)}{\sqrt{r}}) u - k^2 u = 0 \\
  u \in H = \{u \in H^1(0, \infty) : u(0) = 0\}.
\end{cases}$$
Positive solutions are critical points of a functional $\phi_k : H \to \mathbb{R}$

$$\phi_k(u) = \frac{1}{2} \int_0^\infty \left[ |u'(r)|^2 + \frac{3}{4r^2} u^2(r) + k^2 u^2(r) \right] dr$$

$$- \int_0^\infty G(r, u(r)) \, dr$$

where $G(r, s) = \int_0^s g(r, r^{-1/2} \tau^+) \, d\tau$.

Note that

$$\frac{1}{2} A(\omega) s^2 \leq G(r, s) \leq \frac{1}{2} B(\omega) s^2,$$

$$A(\omega) = (\omega/c)^2 a(\omega), \quad B(\omega) = (\omega/c)^2 b(\omega).$$

Set

$$\Lambda_0 = \inf_{u \in H, \|u\|_{L^2} = 1} \int_0^\infty |u'|^2 + \left( \frac{3}{4r^2} - g_0(r) \right) u^2$$

$$\Lambda_\infty = \inf_{u \in H, \|u\|_{L^2} = 1} \int_0^\infty |u'|^2 + \left( \frac{3}{4r^2} - g_\infty(r) \right) u^2$$

$$g_0(r) = g(r, 0), \quad g_\infty(r) = \lim_{s \to \infty} g(r, s).$$
Theorem [Stuart-Zhou]. If the dielectric response satisfies (A) and (AS), $\Lambda_\infty < \Lambda_0$, then for any $k > 0$ such that

$$\Lambda_\infty < -k^2 < \Lambda_0$$

there exists a positive solution $u \in H \setminus \{0\}$ to (2). The corresponding $v$ satisfies

$$v \in C^2([0, \infty)), \ v(0) = v''(0) = 0, \ v > 0 \text{ in } (0, \infty)$$

and guidance conditions.

Remarks

- (A) and (AS) are satisfied by linear dielectric response

$$\varepsilon(\omega, r, s) = \varepsilon_0(\omega, r),$$

in this case $\Lambda_\infty = \Lambda_0$.

- In a homogeneous medium

$$\varepsilon(\omega, r, s) = \varepsilon(\omega, 0, s),$$

thus for a homogeneous self-focusing material, $\Lambda_\infty = \Lambda_0$ iff the dielectric response is linear.
• The theorem improves [Stuart-Zhou, Mathematical Meth. Appl. Sci. (1996)], where only homogeneous media (i.e. $g(r, s) = g(s)$) are treated (and extra technical assumptions are required).

• The proof is based on Mountain-Pass Theorem. Since $0 < A \leq g \leq B < \infty$, $\phi_k$ has a quadratic growth $\Rightarrow$ difficulty in proving boundedness in $H$ of Palais-Smale sequences.
\[ i u_t + u_{xx} - \omega u + 4|u|^2u \]
\[ = i\varepsilon \left[ \frac{E_0 \Gamma_{\text{gain}}}{E_0 \| u \|^2} (\tau u_{xx} + u) - \Gamma_{\text{loss}} u + \beta|u|^2u \right], \]
where
\[ u(x, t) \in \mathbb{C}, \quad \| u \|^2 = \int_{-\infty}^{+\infty} |u|^2, \]
\[ \langle u, v \rangle_{L^2(\mathbb{R})} = \int_{-\infty}^{+\infty} u\bar{v}, \quad \Gamma_{\text{gain}}, \tau, \varepsilon \geq 0. \]

- Model for cavity lasers with bandwidth-limited saturated gain and intensity-dependent loss or gain provided by saturable absorbers in the cavity.
- Left-hand side = focusing NLSE which models propagation of pulses in ideal nonlinear optical fibers.
- Nonlocal term = bandwidth-limited gain with bandwidth \( \frac{1}{\sqrt{\tau}} \) whose saturation energy is \( E_0 \).
- \(-i\varepsilon\Gamma_{\text{loss}}u = \) loss in the fiber.
- \( \beta|u|^2u \) models a saturable absorber that introduces intensity-dependent loss (\( \beta < 0 \)) or gain (\( \beta > 0 \)).
ε = 0 ⇒ NLS \( iu_t + u_{xx} - \omega u + 4|u|^2u = 0 \)

Solitons \( \phi_0(x; \omega) = \sqrt{\frac{\omega}{2}} \text{sech}(\sqrt{\omega}x) \).

[Kapitula, Kutz, and Sandstede]:

1. Persistence.
2. Stability and pattern which bifurcate at essential spectrum.

1. **Persistence.** Consider equation

\[ iu_t + u_{xx} - \omega u + 4|u|^2u = i\varepsilon G(u). \tag{3} \]

**Proposition.** If

\[ g(\omega) = \left\langle \phi_0(\cdot, \omega), G(\phi_0(\cdot, \omega)) \right\rangle_{L^2(\mathbb{R})} \]

has a simple root \( \omega_* \Rightarrow \) there are unique functions \( \omega(\varepsilon) \) and \( \phi(\varepsilon) \) with \( \omega(0) = \omega_* \) and \( \phi(0) = \phi_0(\omega_*) \), such that (3) has the unique steady state \( \phi(\varepsilon) \) near \( \phi_0(\omega_*) \) for \( \varepsilon \sim 0 \) (uniqueness up to translations and gauge symmetries). If \( g(\omega) \neq 0 \) then the steady state does not persist for \( \varepsilon \neq 0 \).

**Proof:** by Liapunov-Schmidt reduction.
In our case

\[ i u_t + u_{xx} - \omega u + 4|u|^2 u = i \varepsilon \left[ \frac{E_0 \Gamma_{\text{gain}}}{E_0 + \|u\|^2} (\tau u_{xx} + u) - \Gamma_{\text{loss}} u + \beta |u|^2 u \right], \]

\[ G(u) = \frac{E_0 \Gamma_{\text{gain}}}{E_0 + \|u\|^2} (\tau u_{xx} + u) - \Gamma_{\text{loss}} u + \beta |u|^2 u, \]

\[ g(\omega) = \frac{\sqrt{\omega}}{3(\sqrt{\omega + E_0})} \left[ \beta \omega \frac{3}{2} + E_0 (\beta - \tau \Gamma_{\text{gain}}) \omega \right. \]
\[ \left. - 3\Gamma_{\text{loss}} \sqrt{\omega} + 3E_0 (\Gamma_{\text{gain}} - \Gamma_{\text{loss}}) \right]. \]
If $\Gamma_{\text{loss}} > \Gamma_{\text{gain}}$ then $g(\omega)$ has a unique positive root $\omega \quad \forall \beta > 0$.

If $\Gamma_{\text{loss}} < \Gamma_{\text{gain}}$ then

$\exists \beta^* > 0$ such that

for $\beta \leq 0$ $g(\omega)$ has one positive root,

for $0 < \beta < \beta^*$ $g(\omega)$ has two positive roots,

for $\beta > \beta^*$ $g(\omega)$ has no positive roots.

For $\varepsilon > 0$ the soliton is given by

\[
\begin{align*}
\phi(x) &= \sqrt{\frac{\omega}{2}} \sech(\sqrt{\omega}x) [1 + i\varepsilon A_1 \ln(\sech(\sqrt{\omega}x))] \\
&\quad + O(\varepsilon^2), \\
A_1 &= \frac{\tau \Gamma_{\text{gain}} E_0 \omega - \Gamma_{\text{loss}} \sqrt{\omega} + E_0 (\Gamma_{\text{gain}} - \Gamma_{\text{loss}})}{2\omega(E_0 + \sqrt{\omega})}.
\end{align*}
\]
Numerical simulations → the stable stationary pulse becomes time-periodic at a certain parameter threshold.

Goal: study stability of the pulses and the nature of bifurcation that generates time-periodic waves → study the spectrum of linearization at a wave $\phi$.

For $\varepsilon = 0$:
0 is the only point spectrum; the rest is essential spectrum included in the imaginary axis.

When $\varepsilon > 0$: there is at most a pair of eigenvalues that can move off the essential spectrum emerging from the edge of the essential spectrum located at $\pm i\omega$. 
(a) If isolated eigenvalues cross the imaginary axis → standard steady-state bifurcations (e.g. pitchforks, Hopf, ...)

(b) If the essential spectrum crosses the imaginary axis → it can destabilize the wave. This essential instability can generate stable time periodic waves.

Question: is there any additional point spectrum which can move into the right half-plane prior to the essential spectrum?
\[ \text{Re } \lambda_{\text{ess}} \leq \varepsilon \left( \frac{\Gamma_{\text{gain}} E_0}{E_0 + \|\phi\|^2} - \Gamma_{\text{loss}} \right), \]

\[ \|\phi\|^2 \sim \sqrt{\omega}. \]

- Hence \( \Sigma_{\text{ess}} \subset \{ \text{Re } \lambda < 0 \} \) if

\[ \Gamma_{\text{loss}} > \frac{E_0 \Gamma_{\text{gain}}}{E_0 + \sqrt{\omega}} \] (stable essential spectrum).

- Equality in (*) and \( g(\omega) = 0 \) when

\[ \beta = \beta^* = \tau \Gamma_{\text{loss}} > 0 \quad \text{and} \]

\[ \sqrt{\omega} = \sqrt{\omega^*} = \frac{E_0 (\Gamma_{\text{gain}} - \Gamma_{\text{loss}})}{\Gamma_{\text{loss}}}. \]

In this case the marginal essential spectrum touches the imaginary axis.
Linearized operator maps

\[ H_{\text{even}}^1 \rightarrow H_{\text{even}}^1, \quad H_{\text{odd}}^1 \rightarrow H_{\text{odd}}^1 \]

⇒ compute the point spectrum by restricting to \( H_{\text{even}}^1 / H_{\text{odd}}^1 \) and compute the spectrum of restriction.

\( \lambda = 0 \) remains to be an eigenvalue

\[
\lambda_{\text{even}} = \varepsilon \frac{4E_0}{(E_0 + \sqrt{\omega})^2} \left[ \frac{1}{6} \tau \Gamma_{\text{gain}} \omega^{3/2} + \frac{\Gamma_{\text{loss}}}{E_0} \omega \right.
\]

\[ + \left( 2\Gamma_{\text{loss}} - \frac{3\Gamma_{\text{gain}}}{2} \right) \sqrt{\omega} + E_0 (\Gamma_{\text{loss}} - \Gamma_{\text{gain}}) \]

\[
\lambda_{\text{odd}} = -\varepsilon \frac{4\tau \Gamma_{\text{gain}} E_0 \omega}{2(E_0 + \sqrt{\omega})} < 0.
\]

• Point spectrum near the origin is stable iff

\[
\frac{1}{6} \tau \Gamma_{\text{gain}} \omega^{3/2} + \frac{\Gamma_{\text{loss}}}{E_0} \omega + \left( 2\Gamma_{\text{loss}} - \frac{3\Gamma_{\text{gain}}}{2} \right) \sqrt{\omega}
\]

\[ + E_0 (\Gamma_{\text{loss}} - \Gamma_{\text{gain}}) < 0. \]

• Necessary criterion for stability of point spectrum near \( \lambda = 0 \) is \( \Gamma_{\text{gain}} > \Gamma_{\text{loss}} \)

(if \( \Gamma_{\text{gain}} < \Gamma_{\text{loss}} \) then \( \lambda_{\text{even}} \) is positive).
Evans function consider the linear nonlocal problem

\[ \frac{d u}{d x}(x) = A(x, \lambda)u(x) + \varepsilon h(x)\langle g, u \rangle_{L^2} \]

rewritten as differential-algebraic equation

\[ \frac{d u}{d x}(x) = A(x, \lambda)u(x) + ah(x) \quad (4) \]
\[ a = \varepsilon \langle g, u \rangle_{L^2}. \quad (5) \]

Construction of \( E(\lambda) \):

- find all bounded solutions \( U^\pm \) of \( u' = A(x, \lambda)u \) on \( \mathbf{R}^\pm \)
- find particular solutions \( V^\pm \) of (4) on \( \mathbf{R}^\pm \)
- define \( E(\lambda) \) as the determinant of the matrix

\[
\begin{bmatrix}
U^+(0) - U^-(0) & V^+(0) - V^-(0) \\
-\varepsilon(\langle g, U^+ \rangle + \langle g, U^- \rangle) & 1 - \varepsilon(\langle g, V^+ \rangle + \langle g, V^- \rangle)
\end{bmatrix}
\]
1. \(E(\lambda) = 0\) for \(\lambda \notin \Sigma_{ess}\) iff \(\lambda\) is eigenvalue
2. order of roots = algebraic multiplicity of eigenvalues
3. \(E(\lambda)\) is analytic for \(\lambda \notin \Sigma_{ess}\)
4. \(E(\lambda)\) can be extended across \(\Sigma_{ess}\).

Edge bifurcation:

- \(\varepsilon = 0\): Evans function of NLS: \(E(\lambda) = 0\) iff \(\lambda = 0\) or \(\lambda = \pm i\omega\)
- \(\varepsilon > 0\): perturbation analysis of Evans function near \(\lambda = \pm i\omega\).

\[
E_{\text{NLS}}(\gamma) = 4\sqrt{2}\omega\gamma + O(\gamma^2) \quad \lambda = i(\omega - \gamma^2)
\]

near \(\lambda = i\omega\),

\[
E_{\text{bp}}(\gamma, \varepsilon) = 4\sqrt{2}\omega\gamma + i\varepsilon \frac{4\sqrt{2}\beta\omega}{3} + O(\gamma^2 + \varepsilon^2),
\]

\[
\lambda_{\text{bp}}(\varepsilon) = i\omega + \varepsilon \left( \frac{E_0\Gamma_{\text{gain}}}{E_0 + \sqrt{\omega}} - \Gamma_{\text{loss}} \right).
\]
For $\varepsilon > 0$ the root of $E_{bp}$ near $\gamma = 0$ is

$$\gamma = -i\varepsilon \frac{\beta \sqrt{\omega}}{3} + O(\varepsilon^2)$$

corresponding to

$$\lambda = \lambda_{bp}(\varepsilon) + i\varepsilon^2 \frac{\beta^2 \omega}{9} + O(\varepsilon^2).$$

This root $\lambda$ is eigenvalue $\Leftrightarrow$ Re $\gamma > 0$

$\beta < 0$ and Re $\gamma > 0$ $\Rightarrow$ $\lambda$ lies to the right of $\Sigma_{ess}$

$\beta > 0$ and Re $\gamma > 0$ $\Rightarrow$ $\lambda$ lies to the left of $\Sigma_{ess}$

$\Rightarrow$ such $\lambda$ cannot move into the right half-plane until part of the spectrum is already destabilized

$\Rightarrow$ no Hopf instability.

**Conclusion:** the stable time-periodic waves, if they bifurcate at all, are created by an essential instability and not by a Hopf instability.
**Dissipative solitons:** a rough classification of non-linear systems with infinite degrees of freedom admitting soliton solutions:

![Dissipative systems diagram]

Hamiltonian systems can be considered a subclass of dissipative ones.
In Hamiltonian systems solitons are the result of a single balance and comprise one or few parameter families (diffraction spreads the beam while nonlinearity focuses it and makes it narrower)

→ the balance between the two opposite effects results in stationary solutions, which are usually a one-parameter family
In dissipative systems the soliton solutions are the result of a double balance and in general are isolated. In order to have stationary solutions gain and loss must also be balanced

→ this additional balance imposes a second constraint
→ we get solutions which are fixed.
(CGLE) describes pulse generation in laser systems with saturable absorber. The addition of a 4-th order spectral filtering term to (CGLE) (to make the model more realistic and describe more involved pulse generation effects) leads to (CSHE)

\[
i\psi_t + \frac{1}{2}\psi_{xx} + |\psi|^2\psi + \nu|\psi|^4\psi = i\delta\psi + i\varepsilon|\psi|^2\psi + i\mu|\psi|^4\psi + i\beta\psi_{xx} + i\gamma\psi_{xxxx}
\]

\(t =\) propagation distance
\(x =\) retarded time (in a frame moving with pulse)
\(i\varepsilon|\psi|^2\psi =\) nonlinear gain
\(i\delta\psi =\) difference between linear gain-loss spectral filtering: \(i\beta\psi_{xx}, i\gamma\psi_{xxxx}\).

- some families of exact solutions of (CSHE) can be obtained analytically
- (CSHE) can mainly be analyze by computer simulations.
Energy balance equation

(CSHE) describes a nonconservative system and so has not conserved quantities. The energy

\[ Q = \int_{-\infty}^{+\infty} |\psi|^2 \, dx \]

is not conserved. We have the energy balance equation instead

\[ \frac{dQ}{dt} = 2 \int_{-\infty}^{+\infty} \left[ \delta |\psi|^2 - \beta \left| \frac{\partial \psi}{\partial x} \right|^2 + \gamma \left| \frac{\partial^2 \psi}{\partial x^2} \right|^2 \right. \]

\[ + \varepsilon |\psi|^4 + \mu |\psi|^2 \left. \right] \, dx. \]

For arbitrary solutions \( Q \) is not conserved \( \longrightarrow \) the right-hand side is nonzero.

The first three terms on the right determine the spectrally dependent linear losses and they have to be balanced with the nonlinear gain given by the other terms.
- Various type of solitons can exist simultaneously. Simultaneous existence in the experiment (fiber laser): Grelu and Akhmediev.

- Numerical simulations ($\varepsilon$-parameter): the system has a larger number of solitons than (CGLE), which at a certain value of parameter become pulsating.