Polynomial Chaos and Scaling Limits of Disordered Systems

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Coworkers

Joint work with Nikos Zygouras (Warwick) and Rongfeng Sun (NUS)
Summary

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Very general framework, illustrated by 3 concrete examples:

1. Disordered pinning models (Pinning)
2. Directed polymer in random environment (DPRE)
3. Random-field Ising model (Ising)
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Very general framework, illustrated by 3 concrete examples:

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2. Directed polymer in random environment (DPRE)
3. Random-field Ising model (Ising)

Inspired by recent work of Alberts, Quastel and Khanin on DPRE.
Outline

1. Disordered Systems and their Scaling Limits

2. Main Results (I): Partition Function

3. Main Results (II): Continuum Disordered Pinning Model

4. Further Developments
General Framework

Lattice $\Omega \subseteq \mathbb{R}^d$  
“spins” $\sigma = (\sigma_x)_{x \in \Omega} \in \{0, 1\}^\Omega$ or $\{-1, +1\}^\Omega$
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- Disorder $(\omega_x)_{x \in \mathbb{Z}^d}$ i.i.d. random variables, independent of $\sigma$

$\mathbb{E}[\omega_x] = 0 \quad \mathbb{V}ar[\omega_x] = 1 \quad \mathbb{E}[e^{t\omega_x}] < \infty$ for small $|t|$
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Disordered law

Random Gibbs measure on spin configurations $\sigma$, indexed by disorder $\omega$

\[ P^\omega_{\Omega, \lambda, h}(\sigma) \propto \exp \left( \sum_{x \in \Omega} (\lambda \omega_x + h)\sigma_x \right) P^\text{ref}_\Omega(\sigma) \]
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Disordered law

Random Gibbs measure on spin configurations $\sigma$, indexed by disorder $\omega$

$$P^\omega_{\Omega, \lambda, h}(\sigma) := \frac{1}{Z^\omega_{\Omega, \lambda, h}} \exp \left( \sum_{x \in \Omega} (\lambda \omega_x + h) \sigma_x \right) P^\text{ref}_\Omega(\sigma)$$

Partition function $Z^\omega_{\Omega, \lambda, h} = \mathbb{E}^\text{ref}_\Omega [e^{\sum_{x \in \Omega} (\lambda \omega_x + h) \sigma_x}]$
1. Disordered pinning model

Reference law: renewal process \( \tau = \{0 = \tau_0 < \tau_1 < \tau_2 < \ldots \} \subseteq \mathbb{N}_0 \)

\[ P^{\text{ref}}((\tau_{i+1} - \tau_i) = n) \sim \frac{C}{n^{1+\alpha}}, \quad \text{tail exponent } \alpha \in (0, 1) \]
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Lattice \( \Omega := \{1, \ldots, N\} \)

Disordered law: disordered pinning model

\[
P_{\Omega, \lambda, h}^\omega(\tau) = \frac{1}{Z_{\Omega, \lambda, h}^\omega} e^{\sum_{n=1}^{N}(\lambda\omega_n + h)1_{\{n \in \tau\}}} P_{\text{ref}}^\tau(\tau)
\]
2. Directed polymer in random environment

Reference law: symmetric random walk
\(X = (X_n)_{n \geq 0}\) on \(\mathbb{Z}\), in the domain of attraction of a stable Lévy process with index \(\alpha \in (0, 2]\)

\[
\text{Var}^{\text{ref}}(X_1) < \infty \quad \text{if} \quad \alpha = 2
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P^{\text{ref}}(|X_1| > x) \sim \frac{C}{x^\alpha} \quad \text{if} \quad \alpha \in (0, 2)
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“spins” \[ \sigma_{n,x} := 1_{\{X_n = x\}} \in \{0, 1\} \text{ (long-range correlations)} \]

Lattice \( \Omega := \{1, \ldots, N\} \times \mathbb{Z} \)

Disordered law: directed polymer in random environment

\[ P_{\Omega, \lambda, \omega}(X) = \frac{1}{Z_{\Omega, \lambda}^\omega} \cdot e^{\sum_{n=1}^N \lambda \omega_{n,x} 1_{\{X_n = x\}}} \cdot P^{\text{ref}}(X) \]
3. Random field Ising model

Reference law: critical 2d Ising model with “+” boundary conditions

Lattice $\Omega := \{-N, \ldots, N\} \times \{-N, \ldots, N\}$

$$P_{\Omega}^{\text{ref}}(\sigma) \propto \exp \left( \beta_c \sum_{x \sim y \in \Omega} \sigma_x \sigma_y \right)$$

where $\sigma_x = \pm 1$, $\beta_c = \frac{1}{2} \log(1 + \sqrt{2})$
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Disordered law: **random field Ising model**

$$P^\omega_{\Omega, \lambda, h}(\sigma) = \frac{1}{Z^\omega_{\Omega, \lambda, h}} e^{\sum_{x \in \Omega} (\lambda \omega_x + h) \sigma_x} P^\text{ref}_\Omega(\sigma)$$
Continuum limit?

Common feature: reference law $P^\text{ref}_\Omega$ admits a non-trivial continuum limit
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Common feature: reference law $P_{\Omega}^{\text{ref}}$ admits a non-trivial continuum limit

Fix $\Omega \subset \mathbb{R}^d$ bounded open with smooth boundary, and consider the lattice

$$\Omega_\delta := \Omega \cap (\delta \mathbb{Z})^d$$

i.e. rescale space by a factor $\delta > 0$ (in the examples $\delta = \frac{1}{N}$)
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Under $P_{\Omega_\delta}^{\text{ref}}$, for a suitable $\gamma > 0$, the rescaled spins $(\delta^{-\gamma} \sigma_x)_{x \in \Omega_\delta}$ converge in law to a (distribution-valued) continuum field $(\sigma_x)_{x \in \Omega}$
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- Ising model: recently proved by [Camia, Garban, Newman '12]
- Pinning model: renewal processes $\tau \sim \text{regenerative set } \tau$
- DPRE: random walk $X \sim \text{Lévy process } X$
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- DPRE: random walk $X \rightsquigarrow$ Lévy process $X$

Does the disordered model $P_{\Omega,\lambda,h}^{\omega}$ admit a non-trivial continuum limit?
A direct approach?

Recall the definition of the (discrete) disordered law:

\[ P_{\Omega, \lambda, h}(d\sigma) \propto \exp \left( \sum_{x \in \Omega} (\lambda \omega_x + h) \sigma_x \right) \frac{P_{\text{ref}}(d\sigma)}{P_{\Omega, \lambda, h}(d\sigma)} \]

Can we guess the \textit{continuum disordered law}?
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- replace discrete disorder \((\omega_x)_{x \in \Omega_\delta}\) by White noise \((dW_x)_{x \in \Omega}\)
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This leads to a candidate continuum model

\[ \mathcal{P}_{\Omega, \lambda, h}(d\sigma) \propto \exp \left( \int_{\Omega} (\lambda dW_x + h) \sigma_x \right) \mathcal{P}_{\Omega}^{\text{ref}}(d\sigma) \]
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This expression makes no sense, because \(\sigma_x\) is distribution-valued
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Difficulty is substantial: \(\mathcal{P}^\omega_{\Omega, \lambda, h}\) can be singular w.r.t. \(\mathcal{P}^{\text{ref}}_{\Omega}\)!
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1. Disordered Systems and their Scaling Limits

2. Main Results (I): Partition Function

3. Main Results (II): Continuum Disordered Pinning Model

4. Further Developments
The partition function

The disordered system $\mathbf{P}_{\Omega_\delta, \lambda, h}$ is a difficult object (a random probability)

Let us be less ambitious and focus on the partition function

$$Z^{\omega}_{\Omega_\delta, \lambda, h} = \mathbb{E}^{\text{ref}} \left[ \exp \left( \sum_{x \in \Omega_\delta} (\lambda \omega_x + h) \sigma_x \right) \right]$$

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Does the partition function $Z^{\omega}_{\Omega, \lambda, h}$ has a non-trivial limit in law as $\delta \downarrow 0$, letting $\lambda, h \to 0$ at suitable rates? (Continuum and weak disorder regime)
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The answer is positive, with an explicit limit. But why should we care?
The partition function

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- $Z_{\Omega, \lambda, h}^\omega$ encodes large-scale properties (free energy, phase transitions)
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- $Z^\omega_{\Omega_\delta, \lambda, h}$ encodes large-scale properties (free energy, phase transitions)
- Dream: scaling limit of $Z^\omega_{\Omega_\delta, \lambda, h} \rightsquigarrow$ scaling limit of $P^\omega_{\Omega_\delta, \lambda, h}$ ???
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The disordered system $P_{\Omega_\delta,\lambda,h}$ is a difficult object (a random probability).

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YES, for Pinning and DPRE (and hopefully for Ising too)
Assumptions

\( k \)-point function \( E_{\Omega_\delta}^{\text{ref}}[\sigma_{x_1} \cdots \sigma_{x_k}] \) defined on \( (\Omega_\delta)^k \) \( \sim \) extended on \( \Omega^k \)
Assumptions

\( k \)-point function \( E^{\text{ref}}_{\Omega_{\delta}}[\sigma_{x_1} \cdots \sigma_{x_k}] \) defined on \((\Omega_{\delta})^k \) \( \rightsquigarrow \) extended on \( \Omega^k \)

Key assumption on the reference law

The \( k \)-point functions of \( P^{\text{ref}}_{\Omega_{\delta}} \) converge in \( L^2 \) under polynomial rescaling

\[ \exists \gamma > 0 : E^{\text{ref}}_{\Omega_{\delta}}[\sigma_{x_1} \cdots \sigma_{x_k}](\delta \gamma^k) \rightarrow \psi(k)_{\Omega}(x_1, \ldots, x_k) \]

\( \forall k \in \mathbb{N} \).

Pointwise convergence in (\( \star \)) leads to \( \psi(k)_{\Omega}(x_1, \ldots, x_k) \approx |x_i - x_j| - \gamma^k \).

\( L^2 \) convergence then requires that \( \gamma < d^2 \).
**Assumptions**

A key assumption on the reference law is that the $k$-point functions of $P_{\Omega_\delta}^{\text{ref}}$ converge in $L^2$ under polynomial rescaling.

\[ \exists \gamma > 0 : \quad \mathbb{E}_{\Omega_\delta}^{\text{ref}}[\sigma_{x_1} \cdots \sigma_{x_k}] \xrightarrow{L^2(\Omega^k)} \psi^{(k)}_{\Omega}(x_1, \ldots, x_k) \quad (\star) \]

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Assumptions

$k$-point function $\mathbb{E}^{\text{ref}}_{\Omega_{\delta}}[\sigma_{x_1} \cdots \sigma_{x_k}]$ defined on $(\Omega_{\delta})^k \sim \text{extended on } \Omega^k$

Key assumption on the reference law

The $k$-point functions of $\mathbb{P}^{\text{ref}}_{\Omega_{\delta}}$ converge in $L^2$ under polynomial rescaling

$$\exists \gamma > 0 : \quad \frac{\mathbb{E}^{\text{ref}}_{\Omega_{\delta}}[\sigma_{x_1} \cdots \sigma_{x_k}]}{(\delta \gamma)^k} \underset{\delta \downarrow 0}{\longrightarrow} \psi^{(k)}_{\Omega}(x_1, \ldots, x_k) \quad (\star)$$

$\forall k \in \mathbb{N}$. Furthermore $\sum_{k \in \mathbb{N}} \|\psi^{(k)}_{\Omega}\|^2_{L^2(\Omega^k)} < \infty$
Assumptions

**k-point function** \( E_{\Omega_\delta}^{\text{ref}}[\sigma_{x_1} \cdots \sigma_{x_k}] \) defined on \((\Omega_\delta)^k \) \( \rightsquigarrow \) extended on \( \Omega^k \)

**Key assumption on the reference law**

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\exists \gamma > 0 : \quad \frac{E_{\Omega_\delta}^{\text{ref}}[\sigma_{x_1} \cdots \sigma_{x_k}]}{(\delta^\gamma)^k} \xrightarrow{\delta \downarrow 0} \psi_{\Omega}^{(k)}(x_1, \ldots, x_k) \quad (\star)
\]

\( \forall k \in \mathbb{N} \). Furthermore \( \sum_{k \in \mathbb{N}} \| \psi_{\Omega}^{(k)} \|_{L^2(\Omega^k)}^2 < \infty \)

Pointwise convergence in \((\star)\) leads to \( \psi_{\Omega}^{(k)}(x_1, \ldots, x_k) \approx |x_i - x_j|^{-\gamma} \)
Assumptions

$k$-point function \( E_{\Omega_\delta}^{\text{ref}}[\sigma_{x_1} \cdots \sigma_{x_k}] \) defined on \((\Omega_\delta)^k \) \( \rightsquigarrow \) extended on \( \Omega^k \)

Key assumption on the reference law

The $k$-point functions of \( P_{\Omega_\delta}^{\text{ref}} \) converge in $L^2$ under polynomial rescaling

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\exists \gamma > 0 : \quad \frac{E_{\Omega_\delta}^{\text{ref}}[\sigma_{x_1} \cdots \sigma_{x_k}]}{(\delta^\gamma)^k} \overset{\text{in } L^2(\Omega^k)}{\underset{\delta \downarrow 0}{\rightarrow}} \psi^{(k)}_{\Omega}(x_1, \ldots, x_k) \quad (*)
\]

\[\forall k \in \mathbb{N}. \quad \text{Furthermore } \sum_{k \in \mathbb{N}} \| \psi^{(k)}_{\Omega} \|_{L^2(\Omega^k)}^2 < \infty\]

Pointwise convergence in \((*)\) leads to \( \psi^{(k)}_{\Omega}(x_1, \ldots, x_k) \approx |x_i - x_j|^{-\gamma} \)

$L^2$ convergence then requires that \( \gamma < \frac{d}{2} \)
Main result (I): partition function

**Theorem [C., Sun, Zygouras '13]**

Assume that $\mathbf{P}_{\Omega_{\delta}}^{\text{ref}}$ satisfies (⋆) with exponent $\gamma$ (and dimension $d$)
Main result (I): partition function

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Assume that $\mathbf{P}_{\Omega_\delta}^{\text{ref}}$ satisfies $(\star)$ with exponent $\gamma$ (and dimension $d$)

- Case $\sigma_x \in \{0, 1\}$. Fix $\hat{\lambda} > 0$, $\hat{h} \in \mathbb{R}$ and scale $\lambda, h \to 0$ as

\[
\lambda := \hat{\lambda} \delta^{d/2-\gamma} \quad h := \hat{h} \delta^{d-\gamma} - \frac{1}{2} \lambda^2
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Theorem [C., Sun, Zygouras '13]

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  \lambda := \hat{\lambda} \delta^{d/2-\gamma} \quad h := \hat{h} \delta^{d-\gamma} - \frac{1}{2} \lambda^2
  \]

Then $Z_{\Omega, \delta, \lambda, h}^{\omega} \overset{\delta \downarrow 0}{\longrightarrow} Z_{\Omega, \hat{\lambda}, \hat{h}}^{W}$ with $W(dx) := \text{white noise on } \mathbb{R}^d$ and

\[
Z_{\Omega; \hat{\lambda}, \hat{h}}^{W} := \sum_{k=0}^{\infty} \frac{1}{k!} \int \cdots \int_{\Omega^k} \psi^{(k)}(x_1, \ldots, x_k) \prod_{i=1}^{k} (\hat{\lambda} W(dx_i) + \hat{h} dx_i)
\]

Wiener chaos expansion (converging in $L^{2-}$)
Main result (I): partition function

**Theorem [C., Sun, Zygouras '13]**

Assume that $P_{\Omega_\delta}^{\text{ref}}$ satisfies $(\star)$ with exponent $\gamma$ (and dimension $d$)

- **Case $\sigma_x \in \{0, 1\}$.** Fix $\hat{\lambda} > 0$, $\hat{h} \in \mathbb{R}$ and scale $\lambda, h \to 0$ as
  \[
  \lambda := \hat{\lambda} \delta^{d/2-\gamma} \\
  h := \hat{h} \delta^{d-\gamma} - \frac{1}{2} \lambda^2
  \]

  Then $Z_{\Omega_\delta, \lambda, h}^{\omega} \overset{\delta \downarrow 0}{\to} Z_{\Omega; \hat{\lambda}, \hat{h}}^W$ with $W(dx) := \text{white noise on } \mathbb{R}^d$ and

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  \]

  Wiener chaos expansion (converging in $L^{2-}$)

- **Case $\sigma_x \in \{-1, 1\}$.** The same, up to minor modifications (cf. below)
Motivating models: Pinning and DPRE

- **Pinning.** Dimension $d = 1$, exponent $\gamma = 1 - \alpha$,

\[
\psi^{(n)}_{\Omega}(x_1, \ldots, x_n) = \frac{c^n}{x_1^{1-\alpha}(x_2 - x_1)^{1-\alpha} \cdots (x_n - x_{n-1})^{1-\alpha}}
\]

(pointwise conv.) by renewal theory
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  (pointwise conv.) by renewal theory

- **DPRE.** Effective dimension $d = 1 + 1/\alpha$, exponent $\gamma = 1/\alpha$,

  $$\psi^{(k)}_{\Omega}(t_1, x_1, \ldots, t_k, x_k) = g_{t_1}(x_1)g_{t_2-t_1}(x_2 - x_1)\cdots g_{t_k-t_{k-1}}(x_k - x_{k-1})$$

  (pointwise conv.) with $g_t(x)$ density of $\alpha$-stable Lévy process $X_t$
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(pointwise conv.) with $g_t(x)$ density of $\alpha$-stable Lévy process $X_t$

For assumption ($\star$), $L^2$ convergence ($\gamma < \frac{d}{2}$) forces

$$
\alpha \in \left(\frac{1}{2}, 1\right) \quad \text{[Pinning]} \quad \quad \alpha \in (1, 2) \quad \text{[DPRE]}
$$
Motivating models: Pinning and DPRE

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(pointwise conv.) with \( g_t(x) \) density of \( \alpha \)-stable Lévy process \( X_t \)

For assumption (\( \star \)), \( L^2 \) convergence (\( \gamma < \frac{d}{2} \)) forces

\[
\alpha \in \left( \frac{1}{2}, 1 \right) \quad \text{[Pinning]} \quad \quad \alpha \in (1, 2] \quad \text{[DPRE]}
\]

These restrictions are not technical, but substantial (physical)!
Motivating models: Ising

Pointwise convergence of $k$-point function, with exponent $\gamma = \frac{1}{8}$, toward

$$\psi^{(k)}_{\Omega}(x_1, \ldots, x_k)$$

conformally covariant,

was proved in [Chelkak, Hongler, Izyurov ’12].
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This convergence holds in $L^2(\Omega^k)$, for bounded open $\Omega \subseteq \mathbb{R}^2$ with piecewise smooth boundary (we provide a uniform domination).
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Recall that we consider random field 2d Ising model at the critical point, with external field $(\lambda \omega_x + h)_{x \in \Omega_\delta}$
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Recall that we consider random field 2d Ising model at the critical point, with external field $(\lambda \omega_x + h)_{x \in \Omega}$.

We fix continuous functions $\lambda : \overline{\Omega} \to (0, \infty)$ and $h : \overline{\Omega} \to \mathbb{R}$ and set

$$\lambda = \lambda(x) \delta^{7/8} \quad h = h(x) \delta^{15/8}$$
Motivating models: Ising

**Theorem [C., Sun, Zygouras ’13]**

As $\delta \downarrow 0$ one has the convergence in law

$$e^{-\frac{1}{2} \|\hat{\lambda}\|_2^2} \delta^{-1/4} Z_{\Omega_\delta, \lambda, h} \xrightarrow{\text{law}} Z_W^{\Omega; \hat{\lambda}, \hat{h}}$$

where $W(dx)$ is white noise on $\mathbb{R}^d$ and

$$Z_W^{\Omega; \hat{\lambda}, \hat{h}} := \sum_{k=0}^{\infty} \frac{1}{k!} \int \cdots \int_{\Omega^k} \psi_{\Omega}^{(k)}(x_1, \ldots, x_k) \prod_{i=1}^{k} (\hat{\lambda}(x_i) W(dx_i) + \hat{h}(x_i) dx_i)$$
**Motivating models: Ising**

**Theorem [C., Sun, Zygouras ’13]**

As $\delta \downarrow 0$ one has the convergence in law

$$e^{-\frac{1}{2} \|\hat{\lambda}\|_2^2} \delta^{-1/4} Z^{\omega}_{\Omega, \lambda, h} \Longrightarrow Z^W_{\Omega; \hat{\lambda}, \hat{h}}$$

where $W(dx)$ is white noise on $\mathbb{R}^d$ and

$$Z^W_{\Omega; \hat{\lambda}, \hat{h}} := \sum_{k=0}^{\infty} \frac{1}{k!} \int \cdots \int_{\Omega^k} \psi^{(k)}(x_1, \ldots, x_k) \prod_{i=1}^{k} (\hat{\lambda}(x_i) W(dx_i) + \hat{h}(x_i) dx_i)$$

**Conformal covariance:** if $\phi : \tilde{\Omega} \rightarrow \Omega$ is a conformal map,

$$Z^W_{\Omega; \hat{\lambda}, \hat{h}} \overset{\text{dist.}}{=} Z^W_{\tilde{\Omega}; \tilde{\lambda}, \tilde{h}}$$

where $\tilde{\lambda}(x) := |\phi'(x)|^{7/8} \hat{\lambda}(\phi(x))$ and $\tilde{h}(x) := |\phi'(x)|^{15/8} \hat{h}(\phi(x))$
Sketch of the proof

1. Linearization. Since $\sigma_x \in \{0, 1\}$, every function of $\sigma_x$ is linear

$$Z^\omega_{\Omega\delta, \lambda, h} = E^{\text{ref}}_{\Omega\delta} \left[ \prod_{x \in \Omega\delta} e^{(\lambda \omega_x + h)\sigma_x} \right] = E^{\text{ref}}_{\Omega\delta} \left[ \prod_{x \in \Omega\delta} (1 + \epsilon_x \sigma_x) \right]$$

where $\epsilon_x := e^{\lambda \omega_x + h} - 1$. 

Sketch of the proof

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$$\mathbb{E}[\epsilon_x] \simeq h + \frac{1}{2} \lambda^2 =: h' \quad \forall \text{Var}[\epsilon_x] \simeq \lambda^2$$
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2. High-temperature expansion. By a binomial expansion of the product

$$Z_{\Omega_{\delta}, \lambda, h}^\omega = \sum_{k=0}^{|\Omega_{\delta}|} \frac{1}{k!} \sum_{(x_1, \ldots, x_k) \in (\Omega_{\delta})^k} \mathbb{E}_{\Omega_{\delta}}^{\text{ref}} \left[ \sigma_{x_1} \cdots \sigma_{x_k} \right] \epsilon_{x_1} \cdots \epsilon_{x_k}$$
Sketch of the proof

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\]

Partition function is a multilinear polynomial of the random variables \( \epsilon_x \), with coefficient given by the \( k \)-point functions of \( P^{ref} \)
Sketch of the proof

3. Lindeberg principle, extending [Mossel, ODonnell, Oleszkiewicz ’10]

The law of a multilinear polynomial is insensitive toward the distribution of the \( \epsilon_x \) (keeping same mean and variance) \( \sim \) independent Gaussians

\[
\epsilon_x \sim \mathcal{N}(h', \lambda)
\]
Sketch of the proof

3. Lindeberg principle, extending [Mossel, ODonnell, Oleszkiewicz '10]

The law of a multilinear polynomial is insensitive toward the distribution of the $\epsilon_x$ (keeping same mean and variance) $\leadsto$ independent Gaussians

$$\epsilon_x \leadsto \mathcal{N}(h', \lambda) \sim \lambda \delta^{-d/2} W(\Delta_x) + h' \delta^{-d} \text{Leb}(\Delta_x)$$

white noise $W$ integrated on cell $\Delta_x := (x - \frac{\delta}{2}, x + \frac{\delta}{2})^d$
3. Lindeberg principle, extending [Mossel, ODonnell, Oleszkiewicz ’10]

The law of a multilinear polynomial is insensitive toward the distribution of the $\epsilon_x$ (keeping same mean and variance) $\rightsquigarrow$ independent Gaussians

$$
\epsilon_x \rightsquigarrow N(h', \lambda) \sim \lambda \delta^{-d/2} W(\Delta_x) + h' \delta^{-d} \text{Leb}(\Delta_x)
$$

white noise $W$ integrated on cell $\Delta_x := (x - \frac{\delta}{2}, x + \frac{\delta}{2})^d$

Since the $k$-point function is piecewise constant on cells $\Delta_x$, we get

$$
Z_{\Omega, \delta, \lambda, h} \sim \sum_{k=0}^{\infty} \frac{1}{k!} \int \cdots \int_{\Omega^k} \mathbb{E}_{\Omega, \delta}^{\text{ref}} [\sigma_{x_1} \cdots \sigma_{x_k}] \prod_{i=1}^{k} \left( \lambda \delta^{-\frac{d}{2}} W(dx_i) + h' \delta^{-d} dx_i \right)
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Since the $k$-point function is piecewise constant on cells $\Delta_x$, we get

$$Z_{\Omega, \delta, \lambda, h} \simeq \sum_{k=0}^{\infty} \frac{1}{k!} \int \cdots \int_{\Omega^k} E^{\text{ref}}_{\Omega, \delta} [\sigma_{x_1} \cdots \sigma_{x_k}] \prod_{i=1}^{k} (\lambda \delta^{-\frac{d}{2}} W(dx_i) + h' \delta^{-d} dx_i)$$

4. Wiener chaos expansion. Plugging the assumption

$$E^{\text{ref}}_{\Omega, \delta} [\sigma_{x_1} \cdots \sigma_{x_k}] \simeq (\delta^\gamma)^k \psi^{(k)}_{\Omega}(x_1, \ldots, x_k)$$

yields a Wiener chaos expansion with $\hat{\lambda} = \lambda \delta^{\gamma - \frac{d}{2}}$ and $\hat{h} = h' \delta^{\gamma - d}$
Outline

1. Disordered Systems and their Scaling Limits

2. Main Results (I): Partition Function

3. Main Results (II): Continuum Disordered Pinning Model

4. Further Developments
Back to pinning models

\[ 0 = \tau_0 \quad \tau_1 \quad \tau_2 \quad \tau_3 \quad \tau_4 \quad \tau_5 \quad \tau_6 \]

\[ \tau = \{\tau_0 < \tau_1 < \tau_2 < \ldots\} \text{ random element of } E := \{\text{closed subsets of } \mathbb{R}\} \]
Back to pinning models

\[ 0 = \tau_0 \leq \tau_1 \leq \tau_2 \leq \tau_3 \leq \tau_4 \leq \tau_5 \leq \tau_6 \]

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Rescaled set \( (\delta \tau, P^{\text{ref}}) \xrightarrow{\delta \downarrow 0} (\tau, \mathcal{P}^{\text{ref}}) \) \( \alpha \)-stable regenerative set
Back to pinning models

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Rescaled set \((\delta \tau, \mathbb{P}^{\text{ref}}) \xrightarrow{\delta \downarrow 0} (\tau, \mathbb{P}^{\text{ref}})\) \(\alpha\)-stable regenerative set

What happens for the disordered model \(\mathbb{P}_{\Omega, \lambda, h}\)? \((\Omega = (0, 1))\)
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Rescaled set \((\delta \tau, P^{\text{ref}}) \xrightarrow{\delta \downarrow 0} (\tau, P^{\text{ref}})\) \(\alpha\)-stable regenerative set

What happens for the disordered model \(P^{\omega}_{\Omega, \lambda, h}\)? \((\Omega = (0, 1))\)

Restrict \(\alpha \in (\frac{1}{2}, 1)\). Fix \(\hat{\lambda} > 0, \hat{h} \in \mathbb{R}\) and set

\[ \lambda := \hat{\lambda} \delta^{\alpha - \frac{1}{2}} \quad h := \hat{h} \delta^\alpha - \frac{1}{2} \lambda^2 \]
Continuum Disordered Pinning Model \ [C., Sun, Zygouras '14]

\[ E := \{\text{closed subsets of } \mathbb{R}\} \text{ equipped with the Hausdorff distance} \]
Continuum Disordered Pinning Model [C., Sun, Zygouras '14]

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**Theorem (existence and universality of the CDPM)**

As \( \delta \downarrow 0 \), the rescaled discrete set \((\delta \tau, P^\omega_{\Omega, \lambda, h})\) converges in distribution on \( E \) to a universal random closed set \((\tau, P^W_{\Omega, \hat{\lambda}, \hat{h}})\), called CDPM.
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**Theorem (a.s. properties)**

The CDPM has any a.s. property of the \( \alpha \)-stable regenerative set \( P_{\text{ref}} \)

\[ A \subseteq E, \quad P_{\text{ref}}(A) = 1 \quad \implies \quad P_{\Omega, \hat{\lambda}, \hat{h}}(A) = 1, \quad P(\text{d}W)\text{-a.s.} \]

Example: \( A = \{ A \subseteq \mathbb{R} : \text{Hausdorff dim. of } A = \alpha \} \)
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**Theorem (singularity)**

The CDPM \(P_{\Omega, \hat{\lambda}, \hat{h}}^W\) law is singular w.r.t. \(P_{\text{ref}}\) for \(P\)-a.e. \(W\)
Construction strategy

Macroscopic observables (finite-dimensional distributions) expressed using partition functions with suitable boundary conditions

\[ P_{\omega}^{\Omega_{\delta}, \lambda, h} (\ldots) = \frac{Z_{0,x}^{\text{cond}}}{Z_{0,N}} \frac{C}{(y-x)^{1+\alpha}} Z_{y,N} \]
Construction strategy

Macroscopic observables (finite-dimensional distributions) expressed using partition functions with suitable boundary conditions

\[ P^{\omega}_{\Omega_\delta, \lambda, h}(\ldots) = \frac{Z_{0,x}^{\text{cond}} \frac{C}{(y-x)^{1+\alpha}} Z_{y,N}}{Z_{0,N}} \]

Scaling limit (at the process level) of \((Z_{x,y}^{\text{cond}}, Z_{x,y})_{0 \leq x < y \leq N} \xrightarrow{\sim} \)
Definition of CDPM via “finite-dimensional distributions”

The same can be done for DPRE, cf. [Alberts, Khanin, Quastel ’12]
Continuum random field Ising model?

Analogous procedure for Ising?

Need joint scaling limit of partition functions for “many” domains and boundary conditions
Continuum random field Ising model?

Analogous procedure for Ising?

Need joint scaling limit of partition functions for "many" domains and boundary conditions

Possible alternative approach: define continuum disordered law $\mathcal{P}_{\Omega, \hat{\lambda}, \hat{h}}^W$ assigning its $k$-point function $\mathcal{E}_{\Omega, \hat{\lambda}, \hat{h}}^W[\sigma_{x_1} \cdots \sigma_{x_k}]$?

A generalization of our theorem about the scaling limit of partition functions yields the corresponding scaling limit of correlations:

$$E_{\Omega_\delta, \lambda, h}[\sigma_{x_1} \cdots \sigma_{x_k}] \xrightarrow{d} \mathcal{E}_{\Omega, \hat{\lambda}, \hat{h}}^W[\sigma_{x_1} \cdots \sigma_{x_k}] \coloneqq \text{Wiener chaos expansion}$$
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1. Disordered Systems and their Scaling Limits

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Disorder relevance vs. irrelevance

Why the restriction $\alpha > \frac{1}{2}$ for pinning?

[And $\alpha \in (1, 2]$ for DPRE]
Disorder relevance vs. irrelevance

Why the restriction $\alpha > \frac{1}{2}$ for pinning? [And $\alpha \in (1, 2]$ for DPRE]

- The regime $\alpha < \frac{1}{2}$ is disorder-irrelevant for pinning models

  If $\lambda > 0$ is small, the disordered model $P_{\Omega, \lambda, h}$ has same properties (e.g. critical exponents) as the non-disordered model ($\lambda = 0$)
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Conj.: scaling limit of $P_{\Omega,\lambda,h}$ is non-disordered [Proved for DPRE]
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  Our results fit this picture nicely: even though $\lambda \to 0$ as $\delta \downarrow 0$, disordered survives in the scaling limit
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Conj.: scaling limit of $\mathbb{P}_{\Omega, \lambda, h}^\omega$ is non-disordered [Proved for DPRE]

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  For any $\lambda > 0$, the disordered model $\mathbb{P}_{\Omega, \lambda, h}^\omega$ has different properties (e.g. critical exponents) than the non-disordered model ($\lambda = 0$)

  Our results fit this picture nicely: even though $\lambda \to 0$ as $\delta \downarrow 0$, disordered survives in the scaling limit

Our restriction involving $L^2$ convergence of $k$-point function ($\gamma < \frac{d}{2}$) matches with Harris criterion $\nu < \frac{2}{d}$ for disorder relevance

($\nu$ correlation length exponent $\sim \nu = \frac{1}{d-\gamma}$)
Continuum free energy and critical exponents

Continuum partition function $Z_{\Omega, \hat{\lambda}, \hat{h}}^W \rightsquigarrow$ continuum free energy

$$F(\hat{\lambda}, \hat{h}) := \lim_{\Omega \uparrow \mathbb{R}^d} \frac{1}{\text{Leb}(\Omega)} \log Z_{\Omega, \hat{\lambda}, \hat{h}}^W$$
Continuum free energy and critical exponents

Continuum partition function $Z_{\Omega, \lambda, \hat{h}}^W \rightsquigarrow$ continuum free energy

$$F(\lambda, \hat{h}) := \lim_{\Omega \uparrow \mathbb{R}^d} \frac{1}{\text{Leb}(\Omega)} \log Z_{\Omega, \lambda, \hat{h}}^W$$

Discrete free energy

$$F(\lambda, h) := \lim_{\Omega \uparrow \mathbb{Z}^d} \frac{1}{|\Omega|} \log Z_{\Omega, \lambda, h}^W$$
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Interchanging of limits (Ising)

$$\lim_{\delta \downarrow 0} \frac{F(\hat{\lambda} \delta^{\frac{7}{8}}, \hat{h} \delta^{\frac{15}{8}})}{\delta^2} = F(\hat{\lambda}, \hat{h})$$
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Continuum partition function

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Interchanging of limits (Ising)

\[ \lim_{\delta \downarrow 0} \frac{F(\lambda \delta^{\frac{7}{8}}, h \delta^{\frac{15}{8}})}{\delta^2} = F(\lambda, \hat{h}) \]

Conjecture

\[ \lim_{h \downarrow 0} \frac{\langle \sigma_0 \rangle}{h^{\frac{7}{15}}} = \frac{\partial F}{\partial h}(\lambda, 1) \] refining [Camia, Garban, Newman '12]
Thanks