Programmi Corso Estivo Perugia : 31 luglio – 2 settembre 2011

ALGEBRA

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Programma :

ALGEBRA: INTRODUCTION TO LIE ALGEBRAS AND REPRESENTATION THEORY

Course philosophy

The idea of the course is to provide background and develop basic skills in representation theory and Lie algebras. As a preparation we start with the classical representation theory of finite groups.

Our main goal is to study two of the major pieces of the XX century Mathematics: the Cartan—Killing classification of the simple complex Lie algebras and the Cartan—Weyl classification of their finite-dimensional representations in terms of highest weights. These theories form the backbone of some central parts of Mathematics and served as models for innumerous generalisations (classification of simple algebraic groups, and their representations, classification of finite simple groups, and their representations, superalgebras, Kac—Moody algebras, infinite dimensional representations, quantum groups, etc., etc., etc.)

It is our intention to provide large portions of the proofs (maybe skipping some topological parts), develop some basic computational techniques, and convey the spirit of some most immediate applications of these results.

Time permitting we could further develop the course in the direction of representations of classical Lie groups, representations of simple algebraic groups, modular representations of finite groups, construction of Chevalley groups, or whatever other *related* subject that would be of interest to the students.

Textbooks

There are a number of excellent textbooks, some of which have equally excellent sets of exercises. Below I list two of my favourites:

1. W.Fulton, J.Harris, Representation theory: a first course. — Springer-Verlag, Berlin et al., 1991, or any later edition.
2. J.E.Humphreys, Introduction to Lie algebras and representation theory. — Springer-Verlag, Berlin et al., 1978, or any later edition.

I will also make available to the students my own notes for large parts of the course.

Below I sketch a possible detailed lecture plan for 4-5 weeks. However, if during the first week we discover that the preparation of students allows a faster pace, other topics may be covered as well.

I. Representations of finite groups

1. Linear representations, examples
2. Invariant subspaces, sub-representations and factor-representations
3. Direct sums of representations, irreducible and indecomposable representations
4. Averaging over a finite group, Maschke's theorem
5. Inner products, unitary representations
6. Schur's lemma, representations of abelian groups
7. Characters of finite groups, class functions
8. Tensor product of modules
9. Operations on characters, characters of $U\oplus V$ и $U\otimes V$
10. First orthogonality relation
11. Multiplicity, Krull—Schmidt theorem
12. Decomposition of regular representation
13. The number of irreducible representations of a finite group
14. Young diagrams, representations of S_n
15. Representations of direct products
16. Second orthogonality relation
17. Character table, examples
18. Algebraic integers
19. Integrality properties of characters
20. Degrees of irreducible representations
21. Induced representations
22. Induced characters
23. Frobenius reciprocity
24. Characters of semi-direct product, Mackey's theorem

II. Lie Algebras

1. Lie Algebras, first examples
2. Classical Lie algebras and classical groups
3. Alternative algebras, Cayley—Dickson algebra, construction of G_2
4. Jordan algebras, Albert algebra, construction of F_4
5. Homomorphisms, subalgebras, ideals, characteristic ideals
6. Solvable Lie algebras
7. Nilpotent Lie algebras
8. Classification of Lie algebras of dimension ≤ 3
9. Radical, simple and semi-simple Lie algebras
10. Simplicity of sl_{l+1}
11. Representations of Lie algebras, basic constructions
12. Irreducibility and indecomposability, Weyl theorem
13. Simplicity of classical Lie algebras
14. Cartan criterion of semi-simplicity
15. Universal enveloping algebra
16. Graded and filtered algebras
17. Poincare—Birkhoff—Witt theorem and corollaries

III. Root systems and Weyl groups

1. Abstract root systems, Weyl group, first examples

2. Fundamental root systems, order and ordering
3. Height of a root, integrality
4. Length of a Weyl group element
5. System of fundamental root associated to a Weyl chamber
6. Mutual position of two roots
7. Dynkin diagrams and Coxeter graphs
8. Classification of root systems
9. Construction of classical root systems
10. Construction of G_2 and F_4
11. Construction of E_l in Minkowski space
12. Coxeter groups and real groups generated by reflections
13. Tits form and absence of cycles in the Coxeter graph
14. Fundamental inequality and its corollaries
15. Contraction of vertices
16. Inequality for the length of tails
17. Root lattice and weight lattice
18. Fundamental weights

IV. Classification of simple Lie algebras and their representations

1. Cartan subalgebras
2. Root systems with respect to a Cartan subalgebra
3. Representations of Lie algebra sl_2, construction and irreducibility
4. Highest weight and highest weight vector
5. Killing form and root subspaces
6. Fundamental sl_2 in a semisimple Lie algebra
7. Root system of a semisimple Lie algebra with respect to a Cartan subalgebra
8. Existence and uniqueness theorems, an outline of the proof
9. Structure constants in a Weyl base
10. Existence of E_l after Frenkel—Kac
11. Serre's theorem
12. Cartan—Killing classification of simple Lie algebras
13. Modules with highest weight
14. Classification of finite-dimensional representations
15. Weyl character formula

Office hours

I will be happy to answer any questions related to the course (or otherwise) during the office hours. Outside of Russian I feel most comfortable with Italian, German and English (more or less in this order, depending on the subject).