

Cortona Summer Course 2009: Degenerations, Projective Varieties and Interpolation Theory

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Prerequisites

The course will require some knowledge of the following subjects, for which we recommend one of the following books [24, 25, 26, 43]. When necessary, we will recall the basics.

Affine and projective varieties. Morphisms and rational maps. Smooth and singular points of a variety. Zariski tangent space and tangent cones. Linear systems and maps to projective spaces. Blow-ups. Basics on intersection theory: in particular, intersection of divisors on a smooth variety.

Multilinear algebra and related projective varieties: Veronese varieties, Segre varieties, Grassmannians.

Basics on cohomology. Flatness and semicontinuity.

Basics on curves and surfaces. Rational surfaces: minimal rational surfaces. Complete intersections and scrolls. The Riemann–Roch theorem.

Basics on vector bundles.

Programme

In the course we will treat the following three topics: Projective varieties, Degenerations, Interpolation. Next we give some details about the programme, with some references which we may follow during the course. It is not necessary any specific knowledge about them, but the participants may wish to have them at hand during the course.

Projective varieties

Joins of varieties and secant varieties. Defective varieties. Terracini’s lemma [39, 40, 45]. Interpolation and defectivity [12].

Weakly defective varieties. Curves are never (weakly) defective. Defectivity and tangential projections [6]. Second fundamental form. Dual varieties and contact loci. Osculating spaces [39, 40]. Grassmann defectivity [9, 13].

Classification of defective and weakly defective surfaces [6]. The concept of secant order. Classification of surfaces with finite secant order greater than 1 [10]. Classification of defective threefolds [7, 8]. QEL varieties [28, 29, 42].

Defectivity and non-defectivity of classical varieties (Veronese, Segre, Grassmannians) [5]. The Waring problem for polynomials and tensors, rank and border rank for tensors [12, 38, 30].

The technique of degeneration of projections. OADP varieties. Classification of OADP curves, surfaces and threefolds [41, 15]. Tangent cones to secant varieties. Varieties with minimal secant degree [20].

Fulton–Hansen connectedness theorem. Zak’s theorems on tangencies and on linear normality. Severi varieties and their classification. Scorza varieties [39, 40, 45]. Extension of Zak’s theorems to higher secant varieties [11].

Degenerations

Degenerations of varieties. Degenerations of curves. Stable curves, degenerations of curves of genus zero and one [35].

Degenerations of surfaces. Normal crossings degenerations. The triple point formula [35, 37]. Zappatic singularities and variations on the triple points formula [3, 4].

Invariants of the singular fiber and the general fiber. Monodromy. Clemens–Schmidt. Degenerations of $K3$ and Abelian surfaces [3, 4].

Constructing degenerations via blow-ups. Collisions of zero-dimensional schemes. Various examples of generic and non generic collisions [18].

Introduction to toric varieties [22]. Toric degenerations [27, 36]. Interpretation of the blow-up method in terms of toric geometry [14]. Relations with tropical geometry [23, 34, 44].

Degenerations of secant varieties [14]. Draisma’s theorem [21]. Computation of the degree of some secant varieties via toric degeneration.

Interpolation

Introduction to interpolation theory. Presentation of the main results on the subject. SHGH-conjecture and Nagata’s conjecture [12]. Basics on Seshadri constants and relations with the Nagata conjecture [31].

Degeneration techniques for interpolation. The Lorentz–Lorentz method [32, 33]. The blow-up-and-twist method [16, 17, 18, 19]. The Horace method and the differential Horace method reinterpreted. Aligned ideals.

The double points case [6, 14]. The Alexander–Hirschowitz Theorem [1, 2]. The planar case. The general case.

The blow-up-and-twist method applied to the homogeneous case with ten points [19].

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