COMPLEX ANALYSIS

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1 **Textbook**

As textbook we use for this course the first 4 chapters of the book by Lars Hørmander:
An introduction to Complex Analysis in Several Variables (Third Edition)
Elsevier (Amsterdam)

2 Prerequisites of the course

All students, who want to participate successfully in this course should be well acquainted with Calculus in one and several real variables, in particular, with the ideas around "Stokes theorem" and differential forms. As a preparation we recommend the book by Klaus Jähnig, Vector Analysis (Springer) or (on a much higher level) Serge Lang: Real Analysis (addison-Wesley Publ. Company).

3 Prerequisites from Complex Analysis in One complex variable, a first quick crash course

Complex Analysis in One Complex Variable is the basis of Complex Analysis. It is the easy case. In order to understand the special features of Complex Analysis in Several Variables, we will have to present at first the basics of the theory in one Complex Variable, i.e. the case n = 1.

We will touch the following topics:

_ C1 functions are "holomorphic"
_ The open mapping theorem and the
_ The identity theorem
_ The Cauchy theory
_ Subharmonic functions

4 Cauchy integrals on polydiscs

Polydics are direct products of discs. The theory of holomorphic functions on them can mostly be understood by dealing with one variable after the other. The main topics of this first main section of the course arise from considering

1. Cauchy integrals on polydiscs and the resulting Cauchy inequalities

and by studying

2. the inhomogeneous Cauchy-Riemann equations in a polydisc

This leads in particular to the

Theorem 4.1 (The Poincaré lemma for the @-operator on polydiscs) Let D be an open polydisc and let f 2 C1(p;q+1)(D) satisfy the condition @f = 0. If D0 is relatively compact in D we can find a u 2 C1(p;q)(D0) with @u = f in D0.

A side-aspect of this will lead us to considering power series and Reinhardt domains.

5 The Cartan-Thullen theory of holomorphic convexity

With this chapter we enter into the heart of Complex Analysis showing us the first very specific features of the theory in several (n > 1) complex variables.

We come to
domains of holomorphy: their interior and exterior characteristic properties.
The notion of hulls and their properties. We discover the basic properties of plurisubharmonicity and pseudoconvexity.

This leads to the

Theorem 5.1 (First main theorem: Cartan-Thullen) Domains of holomorphy are pseudoconvex.

For a long time it has been an outstanding problem of Complex Analysis, which even today is still unsolved in some of its tricky versions, allthough it has been solved by K.Oka and H. Grauert in the form, how we can ask it at the moment:

Question 5.2 (The Levi Problem) Is any peudoconvex domain in Cn a domain of holomorphy?

It will be the major goal of this course to work out a solution to the Levi problem by mean of the L2-theory of the @-operator.

6 A first consideration of Runge domains in Cn

Domains of holomorphy D with the property that the set of polynomials on them are dense in their algebra of holomorphic functions enjoy a particularly strong global convexity condition. They are much more easy to deal with under the point of view of the Levi problem than arbitrary pseudoconvex domains in Cn. In particular the equations @u = f can be globally and much more easily solved on them. This will be done in this section.

7 L2-estimates and existence theorems for the @ operator

7.1 Linear, unbounded, densely defined operators between Hilbert spaces

Definition 7.1 1. The Hilbert spaces L2
(p;q)(; ') and their norms and L2
(p;q)(; loc)
2. The spaces D(p;q)() as dense subspaces

Definition 7.2 1. Linear, closed, densely defined operators T : H1 ! H2 between Hilbert spaces
2. Their Ranges and Domains
3. The Hilbert space adjoint T_ of T : H1 ! H2

Lemma 7.3 The identity T__ = T and other relations between T and T_

We refer all readers who want to study this tool of functional analysis more deeply to the excellent presentation in the book "Functional Analysis" by Walter Rudin (McGraw-Hill Book Company, New York 1973)

An abstract, basic, functional-analytic existence theorem as a consequence of the Hahn-Banach theorem:

Lemma 7.4 F = RT if and only if
kfkh2 _ CkT_fkH18f 2 F \ DT_ (7.1)

From this we obtain a kind of dual statement, which will be the basis for proving aproximation theorems

Lemma 7.5 Let T : H1 ! H2 be a closed densely defined operator and F _ H2 a closed subspace with RT _ F. Let (7.1) hold true. Then for every v 2 H1, v 2 ker(T)? one can find f 2 DT_ such that T_f = v and
kfkH2 _ CkvkH1 (7.2)

In order to be able to apply these abstract lemmas in a more concrete situation, we need criteria for the density of D(p;q+1) in certain subspaces with respect to the graph norm given by
f 7! kfk'2 + kT_fk'1 + kSfk'3

The exact situation will be descrbed in

Lemma 7.6 Density of D(p;q+1)() with respect to the graph norm

Another important tool is the study of

Lemma 7.7 (existence of smoothings)

7.2 A first existence theorem for @ on pseudoconvex open sets in Cn

We are now able by some rather carefull analysis to solve the @-equations in the sense of distribution theory on any open pseudoconvex set in Cn. Our main theorem will be

Theorem 7.8 (Solving @u = f for all (p; q + 1) forms) Let be a pseudoconvex
open set in Cn. Then the equation @u = f has (in the sense of dstribution theory) a
solution u 2 L2
(p;q)(; loc) for every f 2 L2
(p;q+1)(; loc) such that @f = 0
Next, we need interior regularity properties in the sense of the Sobolev spacesWs
(p;q+1)(; loc).
Our main result will be
Theorem 7.9 (regularity in Sobolev norms) Let be a pseudoconvex open set in
Cn and let 0 _ s _ 1. Then the equation @u = f has a solution u 2 Ws+1
(p;q)(; loc) for
every f 2 Ws
(p;q+1)(; loc) such that @f = 0. Every solution of the equation @u = f has
this property when q = 0.

8 The solution of the Levi problem for pseudoconvex domains in Cn

It is obvious, that each open set in C1 is pseudoconvex and,in fact, also a domain of
holomorphy. This important observation will give us the chance, to prove by induction
over n, that each pseudoconvex domain in Cn for n arbitrary is a domain of holomorphy.
This means, that we want to prove by induction over n, that one has:
Theorem 8.1 An open set in Cn is a domain of holomorphy if it is peudoconvex.
This is the claimed inverse of the Cartan-Thullen theorem. We will prove it by
solving the @-equation for all bidegrees (p; q). We formulate this in:
Theorem 8.2 Let be an open set in Cn, such that the equation @u = f has a solution
u 2 C1(0;q)() for every f 2 C1(0;q+1)() such that @f = 0 (q = 0; : : : ; n □ 2). Then is a
domain of holomorphy.

9 Approximation theorems

As we indicated already earlier the dualization of Lemma 7.4 with inequality 7.1 leads
to an approximation result. The first technical statement of this is
Lemma 9.1 Let p be a strictly plurisubharmonic C1 function in such that
Kc = fz : z 2 ; p(z) _ cg __ for every c 2 R (9.1)
Then every function analytic in a neighborhood of K0 can be approximated in L2-norm
over K0 by functions in O().
We can reformulate this statement as
Theorem 9.2 (Aproximation on plurisubharmonically convex hulls) Let be a
pseudoconvex open set in Cn and K a compact subset of such that ^K P
 = K. Then
every function which is analytic in a neighborhood of K can be approximated uniformly
on K by functions in O()
The final form, into which we want to bring our approximation results, will be
Theorem 9.3 Let 1 _ 2 be domains of holomorphy. Then the following conditions
are equivalent:
1. Every function in O(1) can be approximated by functions in O(2) uniformly on
every compact subset of 1 (1 is then called a Runge domain relative to 2)
2. For every compact set K _ 1 we have ^K2 = ^K1
3. For every compact set K _ 1 we have ^K2 \ 1 = ^K1
4. For every compact subset K _ 1 we have ^K2 \ 1 __ 1

10 Supplement (If time allows:) Existence theorems in L2-spaces

Estimates of weighted norms have deep consequences in the L2-theory. If time allows

we will enter into this domain by proving a rather strong such estimate based on the following technical
Lemma 10.1 Let be a pseudoconvex open set in Cn and let ' be a real valued function in C2() such that
c
n X1
jwj j2 _
n Xj;k=1
@2'
@zj@_zk
wj _ wk (10.1)
where c is a positive continuous function in . If g 2 L2
(p;q+1)(; ') and @g = 0 it follows
that one can find a u 2 L2
(p;q)(; ') with @u = g and
Z juj2 e□'d_ _ 2 Z jgj2 e□'=cd_ (10.2)
provided that the right hand side is finite.
5
We will deduce from this:
Theorem 10.2 Let be a pseudoconvex open set in Cn and ' any plurisubharmonic function in . For every g 2 L2
(p;q+1)(; ') with @g = 0 there is a solution u 2
L2
(p;q)(; loc) of the equation @u = g such that
Z juj2 e□'(1 + jzj2)□2d_ _ Z jgj2 e□'d_ (10.3)
Remark 10.3 The essential new feature in this theorem is, that there is no smoothness condition on '.

Lectures in English