Techniques in the Calculus of Variations

Thesis submitted for the degree of Dottore di Ricerca

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Introduction

The classical problem of the Calculus of Variations consists in minimizing a functional of the type
\[ \int_a^b L(t, x(t), x'(t))\,dt \]
or
\[ \int_{\Omega} L(x, u(x), \nabla u(x))\,dx \]
under suitable boundary conditions.

The aim of this thesis is to discuss different techniques to obtain results related to this problem.

The first technique considered is the classical one of “taking variations”. In particular, assuming that the minimum problem admits a solution, \( \hat{x} \) or \( \hat{u} \), we discuss some necessary conditions. A basic principle of analysis is that, given a minimum point \( \xi \) belonging to the interior of the domain of a differentiable function \( F \), we obtain a necessary condition by exploring a neighborhood of \( \xi \), and, in this way, we obtain the condition \( \langle \nabla F(\xi), \delta \rangle = 0 \), yielding
\[ \nabla F(\xi) = 0. \]
In the same order of ideas, one considers an admissible variation, i.e., a smooth function \( \eta \), equal to zero at the boundary, multiplies this function by a scalar \( \lambda \) and considers the function \( \hat{x} + \lambda \eta \) or \( \hat{u} + \lambda \eta \). In principle, by deriving with respect to the parameter \( \lambda \) and passing to the limit under the integral sign (this is the difficult step), one obtains the Euler-Lagrange equations (E-L):
\[ \int_a^b \left[ (\nabla_x L(t, \hat{x}(t), \hat{x}'(t)), \eta'(t)) + (\nabla_t L(t, \hat{x}(t), \hat{x}'(t)), \eta(t)) \right] \,dt = 0 \]
or
\[ \int_{\Omega} \left[ (\nabla_u L(x, \hat{u}(x), \nabla \hat{u}(x)), \nabla \eta(x)) + L_u(x, \hat{u}(x), \nabla \hat{u}(x))\eta(x) \right] \,dx = 0 \]
for every variation \( \eta \) such that \( \eta \) equals zero at the boundary. In the variational notation, the Euler-Lagrange equations are written as
\[ \frac{d}{dt} \nabla_x L(t, \hat{x}(t), \hat{x}'(t)) = \nabla_t L(t, \hat{x}(t), \hat{x}'(t)) \]
or
\[
\text{div}\nabla_u L(x, \hat{u}(x), \nabla \hat{u}(x)) = L_u(x, \hat{u}(x), \nabla \hat{u}(x)).
\]

This classical approach, consisting in “exploring” a neighborhood of the solution through variations, is not applicable when dealing with sets which are not open. In particular, when we consider a constrained problem, we work with closed sets. In 1950 Pontriagin proposed a very general necessary condition, the Maximum Principle, applicable also in these last cases. Chapter 1 deals with these classical results.

In Chapter 4, we provide a new result on the validity of Euler-Lagrange equations under Carathéodory’s conditions. When the Lagrangian \( L \), as well as its gradients w.r.t. \( x \) and \( x' \), satisfy Carathéodory’s assumptions, a suitable condition to assume is the local Lipschitzianity of \( L \) w.r.t. \( x \) in a neighborhood of the solution. The validity of the Euler-Lagrange equations under this hypothesis was first proved by Clarke in the context of Differential Inclusions. Our result (Chapter 4), has been obtained under a weaker condition that does not imply this Lipschitzianity. We do not assume any convexity hypothesis on the Lagrangian. Moreover, no growth condition whatsoever is assumed and therefore, as far as we know, relaxation theorems cannot be applied.

The remaining part of this work deals with techniques on the domain.

One of the aims is to discuss reparametrizations and to present new results obtained applying this technique. Reparametrizations are a useful tool when working with problems involving integrals defined on an interval. In Chapter 2, given an absolutely continuous function \( x \), defined on an interval, and a functional
\[
J(x) = \int_a^b L(t, x(t), x'(t))dt,
\]
we discuss what happens to \( x \) and to \( J \) when we reparametrize the time \( t \).

Applying this technique, we obtain the new results of Chapter 5 and Chapter 6. In particular, in Chapter 5 we prove a theorem on reparametrizations of an interval onto itself which states that: under appropriate conditions on \( L \), given an absolutely continuous function \( x \), defined on an interval, and \( \epsilon > 0 \), there exists a Lipschitzian function \( x_\epsilon \), obtained from \( x \) by reparametrizing the time, such that
\[
\int_a^b L(t, x_\epsilon(t), x'_\epsilon(t))dt \leq \int_a^b L(t, x(t), x'(t))dt + \epsilon.
\]

Our result prevents the occurrence of the Lavrentiev Phenomenon for a class of functionals of the Calculus of Variations. In 1926 M. Lavrentiev provided an example of a functional whose infimum taken over the space of absolutely continuous functions was strictly lower than the infimum taken over the space of Lipschitzian functions. Alberti and Serra Cassano have proved the non-occurrence of this phenomenon in the autonomous case. The result of Chapter 5 applies to autonomous and non-autonomous problems, to problems with obstacles or with other constraints and to multidimensional rotationally invariant problems.
In Chapter 6 we introduce a new theorem for the existence of solutions to minimum time control problems. In 1959 Filippov proved a general theorem on the existence of solutions to minimum time control problems requiring convexity and compactness of the images. Our result is proved under assumptions that do not require the convexity of the images, and it applies to one of the oldest problems of the Calculus of Variations, the Brachystocrone, which is the problem of finding the path from an initial point $x^0$ to a target point $x^f$ such that, a body, subject only to gravity, starting from $x^0$ with initial velocity zero, reaches $x^f$ in minimum time.

As an application of another technique on the domain, we discuss the validity of two classical results for elliptic equations: Hopf’s Lemma and the Strong Maximum Principle. Consider a Euler-Lagrange equation of the kind

$$F(u) = \sum_{i=1}^{N} g_i(u^2_{x_i})u_{x_ix_i} = 0.$$  \hspace{1cm} (0.1)

If (0.1) is an elliptic equation and $u$ solves it in a connected, open and bounded set $\Omega \subset \mathbb{R}^N$, then the Strong Maximum Principle holds, i.e. if $u$ attains maximum or minimum in $\Omega$ then it is a constant. In Chapter 3 we give a proof of this classical result in the simple case of the Laplace equation $\Delta u = 0$. A primary technical tool in the proof of the Strong maximum Principle is Hopf’s Lemma which states that: let $u$ be such that $\Delta u \leq 0$; suppose that there exists $z \in \partial \Omega$ such that $u(z) < u(x)$, for all $x$ in $\Omega$, and that $z$ satisfies the interior ball condition at $z$ (i.e. there exists an open ball $B \subset \Omega$ with $z \in \partial B$); then

$$\frac{\partial u}{\partial \nu}(z) < 0,$$

where $\nu$ is the outer unit normal to $B$ at $z$. We prove this result by applying a comparison lemma and obtaining a contradiction. With this aim, we need to find a subsolution $v$ which has to be compared to the solution $u$ in a domain $\omega \subset \Omega$. By the radial symmetry of the Laplace equation, it is not difficult to find a subsolution among the radial functions, so that the validity of Hopf’s Lemma and (from which) of the Strong Maximum Principle follows.

If the functions $g_i$ could assume value zero, then (0.1) is non elliptic. In Chapter 7 we prove the validity of Hopf’s Lemma and the Strong Maximum Principle under hypotheses that allow $g_i$ to assume value zero. In particular we give a sufficient condition and a necessary condition for the validity of Hopf’s Lemma and a sufficient condition for the validity of the Strong Maximum Principle. As in the case of the Laplacian, to prove Hopf’s Lemma and the Strong Maximum Principle we need to find a subsolution $v$ which has to be compared to the solution $u$ in a domain $\omega \subset \Omega$ to arrive to a contradiction. Suppose that near zero $g_i \geq g_N$, when $i < N$. The possibility of finding such a subsolution seems to depend on the velocity with which $g_N$ goes to zero. In the case

$$\int_{0}^{\xi} \frac{g_N(\zeta^2/N)}{\zeta} d\zeta = +\infty,$$
we find a radial subsolution to be compared to the solution, and we are able to prove Hopf’s Lemma, from which it follows the Strong Maximum Principle. In the case
\[
\int_0^\xi \frac{g_N(\zeta^2/N)}{\zeta} d\zeta < +\infty,
\]
the condition
\[
\lim_{t \to 0^+} \frac{(g_N(t))^{3/2}}{t g_N'(t)} > 0, \tag{0.2}
\]
is necessary to have the validity of Hopf’s Lemma, indeed, in the case the limit is zero, we are able to find a counterexample to the lemma. (0.2) is also a sufficient condition for the validity of the Strong Maximum Principle. We prove this result by applying a comparison lemma, to arrive to a contradiction. The main difficulty is to find a subsolution to be compared to the solution \( u \). In this case, we can not find a radial subsolution. We are able to define a non radial subsolution on a domain \( \omega \subset \Omega \), built with a suitable symmetry, that can be used in a comparison lemma to prove the Strong Maximum Principle.

This thesis is developed into three parts. In the first part we recall those known results that we improve in the following parts, and those techniques that we apply to obtain them. In Part II we present the new result on the validity of the Euler-Lagrange equations, while Part III concerns the new results we have obtained applying techniques on the domain.
Part I

Techniques and known results
Chapter 1

Necessary conditions

1.1 The Euler-Lagrange equations

Let $\hat{x}$ be a solution to the classical problem of the Calculus of Variations consisting in minimizing the functional

$$J(x) = \int_I L(t, x(t), x'(t))dt,$$

where $I = (a, b) \subset \mathbb{R}$, on the set of those absolutely continuous functions $x : I \rightarrow \mathbb{R}^N$ satisfying the boundary conditions $x(a) = A, x(b) = B$.

Consider an admissible variation, i.e., a smooth function $\eta$, equal to zero at the boundary, multiply this functions by a scalar $\lambda$ and consider the function $\hat{x} + \lambda \eta$. By deriving with respect to the parameter $\lambda$ and passing to the limit under the integral sign, one obtains the Euler-Lagrange equations (E-L):

$$\int_a^b (\nabla_x L(t, \hat{x}(t), \dot{\hat{x}}(t)), \eta'(t)) + (\nabla_t L(t, \hat{x}(t), \dot{\hat{x}}(t)), \eta(t))dt = 0$$

for every variation $\eta$ such that $\eta$ equals zero at the boundary.

We have that

$$0 \leq \frac{1}{\lambda} \left[ \int_a^b L(t, \dot{x}(t) + \lambda \eta(t), \ddot{x}(t) + \lambda \eta'(t))dt - \int_a^b L(t, \dot{x}(t), \ddot{x}(t))dt \right] =$$

$$\int_a^b \frac{1}{\lambda} [L(t, \dot{x}(t) + \lambda \eta(t), \ddot{x}(t) + \lambda \eta'(t)) - L(t, \dot{x}(t) + \lambda \eta(t), \ddot{x}(t)) - L(t, \dot{x}(t), \ddot{x}(t))]dt +$$

$$+ \int_a^b \frac{1}{\lambda} [L(t, \dot{x}(t) + \lambda \eta(t), x'(t)) - L(t, \dot{x}(t), x'(t))]dt.$$

Pointwise, the integrands converge respectively to

$$\langle \nabla_x L(t, \dot{x}(t), \ddot{x}(t)), \eta'(t) \rangle$$

and to

$$\langle \nabla_t L(t, \dot{x}(t), \ddot{x}(t))\eta(t) \rangle.$$
Establishing the validity of the Euler-Lagrange equations consists simply in proving that one can pass to the limit under the integral sign. If one assumes that both $\dot{x}$ and $\ddot{x}_0$ are continuous on the compact interval $[a, b]$, and also that the gradients of the Lagrangean $L$ are continuous in their arguments, there is no problem for passing to the limit: in fact, in this case, there is some number $K$ that bounds from above both $kr x L(t; \dot{x}(t), \dot{x}(t))$ and $kr x_0 L(t; \dot{x}(t), \dot{x}(t))$, hence $khr x L(t; \dot{x}(t), \dot{x}(t))$ and $khr x_0 L(t; \dot{x}(t), \dot{x}(t))$. By continuity and compactness, there is a $\delta > 0$ such that $kr x L(t; y, \xi) \leq (K + 1)$ and $kr x_0 L(t; y, \xi) \leq (K + 1)$ whenever $\|y - \hat{x}(t)\| \leq \delta$ and $\|\xi - \hat{x}'(t)\| \leq \delta$. By the mean value Theorem, the integrands are the scalar products $\langle \nabla_x L, \eta \rangle$ and $\langle \nabla_x L, \nabla \eta \rangle$ computed nearby the solution: hence, by the previous remark on continuity, they are dominated by some scalar and one can pass to the limit.

The very same reasoning applies to the case where one minimizes

$$\int_\Omega L(x, u(x), \nabla u(x)) \, dx$$

under additional boundary conditions: the assumption that both $\nabla \hat{u}$ and $\hat{u}$ are continuous on the closure of $\Omega$ (at least for bounded $\Omega$) allows one to pass to the limit by dominated convergence.

However, these assumptions, that both $\dot{x}$ and $\ddot{x}'$ are continuous, are not satisfied even in very simple cases: consider, for instance, the problem of minimizing

$$\int_0^1 (x(t)x'(t))^2 \, dt$$

under the conditions $x(0) = 0$, $x(1) = 1$. Then, the solution is $x(t) = \sqrt{t}$, whose derivative is unbounded on $[0, 1]$.

### 1.2 The Maximum Principle

Around 1950, Pontriagin worked on an innovative minimum problem, that of minimizing

$$\int_a^b L(t, x(t), u(t)) \, dt$$

with the additional conditions

$$x'(t) = f(t, x(t), u(t))$$

and $u(t) \in U$. The functions $u$ are called controls; the corresponding theory, the theory of optimal control. In the special case when the differential equation that connects the control $u$ with the state $x$, i.e., the equation $x'(t) = f(t, x(t), u(t))$, becomes $x'(t) = u(t)$, so that $u(t)$ is simply a name given to $x'(t)$, we have again the problem of minimizing

$$\int_a^b L(t, x(t), x'(t)) \, dt.$$
In this formulation, however, there appears the new condition that \( x'(t) \) is in this set “of controls” \( U \). Hence, two are the novelties of the problem: the appearance of a dynamic (in general, non-linear) linking the variable in the Lagrangean with the state \( x \) and that of a constraint on the controls \( u \), or, in the case of the Calculus of Variations, on the set of \( x'(t) \) that are allowed.

According to the account given by Boltianski, Pontriagin proposed the Maximum Principle around as a sufficient condition (that it is not!); the proof of the validity of the Maximum Principle as a necessary condition is due to Boltianski. The name of Maximum Principle derives from the basic condition proposed by Pontriagin: there exists a non-trivial vector function \( (p_0, p) \), a solution to

\[
p_0(t) = 0, \quad p'(t) = p_0 \nabla_x L(t, \dot{x}(t), \dot{u}(t)) - p(t) D_x f(t, \dot{x}(t), \dot{u}(t)),
\]

such that, for a.e. \( t \) in \( (a, b) \),

\[
p_0 L(t, \dot{x}(t), \dot{u}(t)) - \langle p(t), f(t, \dot{x}(t), \dot{u}(t)) \rangle = \max_{w \in U} \{p_0 L(t, \dot{x}(t), w) - \langle p(t), f(t, \dot{x}(t), w) \rangle\}.
\]

In the case of the Calculus of Variations, this condition reduces to: there exists \( (p_0, p) \), a solution to

\[
p_0'(t) = 0, \quad p'(t) = \nabla_x L(t, \dot{x}(t), \dot{u}(t))\]

such that, a.e. in \( (a, b) \),

\[
-p_0 L(t, \dot{x}(t), \dot{u}(t)) + p(t) \dot{u}(t) = \max_{w \in U} \{-p_0 L(t, \dot{x}(t), w) + p(t) w\}.
\]

The conditions proposed by Pontriagin differ from the conditions one usually meets even when the control differential equations reduces to \( f(t, x, u) = u \), i.e., in the Calculus of Variations case. The main feature of these conditions are the lack of any differentiability condition w.r.t. \( x' \) and the role of the control set \( U \). About this set, one would expect conditions of regularity: what is surprising is that, for the validity of the Maximum Principle, there are no conditions on \( U \): \( U \) is any set. This fact has two important consequences: first, since \( U \) is not assumed to be an open set, the classical approach, consisting in “exploring” a neighborhood of the solution through variations, is not applicable. Second, since \( U \) can be a closed set, constrained problems with constraints \( x' \in U \) are included.

A sketch of the proof. We wish to present a sketch of the proof of the Maximum Principle in its simplest form, for the problem with free right conditions, for two reasons: first, to point out the conceptual difference in taking variations; second, because it is overall a very beautiful proof. Consider the control system

\[
y'(t) = F(t, y(t), u(t)); \quad y(0) = y^0, \quad \text{and} \quad u(t) \in U
\]

and, having fixed the final time \( T \) (but not the final state), assume that we wish to maximize \( \psi(y(T)) \), a given function of the final state. This special form of the
optimization problem brings into evidence the geometric side of the proof. In the usual case, where we want to minimize
\[ \int_a^b L(t, x(t), u(t)) dt \]
subject to
\[ x'(t) = f(t, x(t), u(t)) \]
and \( u(t) \in U \), it is enough to set:
\[ y = (x^0, x); \quad F(t, (x^0, x), u) = (-L(t, x, u), f(t, x, u)); \quad \psi((x^0, x)) = x^0 \]
to obtain a special case of the problem in this new formulation.

We will take a variation to the optimal control in the following way. Fix an arbitrary variation, taken at time \( t \), a Lebesgue point for the maps \( t \to F(t, y(t), u(t)) \) and \( t \to F(t, y(t), w) \).

For every \( \varepsilon \), define a new control \( u_\varepsilon \) as
\[ u_\varepsilon = \hat{u}(t) + (w - \hat{u}(t))\chi_{[\tau - \varepsilon, \tau]}(t) \]
so that we substitute \( w \) to \( \hat{u}(t) \) on the interval \( [\tau - \varepsilon, \tau] \). As opposite to the classical variations, there is no attempt to let \( w \) “tend to” \( \hat{u}(t) \). Our purpose is to estimate the effect of this variation, taken at time \( \tau \), at the final time \( T \), so as to compare the result, \( y_\varepsilon(T) \), with the solution \( \hat{y}(T) \). Let us first compute the effect of the variation at time \( \tau \), i.e. let us estimate the difference \( v_\varepsilon(\tau) = y_\varepsilon(\tau) - \hat{y}(\tau) \). We have
\[ y_\varepsilon(\tau) - \hat{y}(\tau) = \int_{\tau - \varepsilon}^{\tau} [F(t, y_\varepsilon(t), w) - F(t, \hat{y}(t), \hat{u}(t))] dt = \]
\[ = \int_{\tau - \varepsilon}^{\tau} [F(t, y_\varepsilon(t), w) - F(t, \hat{y}(t), w)] dt + \int_{\tau - \varepsilon}^{\tau} [F(t, \hat{y}(t), w) - F(t, \hat{y}(t), \hat{u}(t))] dt. \]
Assuming some boundedness (local, near the point \( (\tau, \hat{y}(\tau)) \)) of the map \( (t, y) \to F(t, y, w) \), the difference \( y_\varepsilon(t) - \hat{y}(t) \) is bounded on \( [\tau - \varepsilon, \tau] \) by \( K\varepsilon \). Assuming that the map \( y \to F(t, y, w) \) is Lipschitzian, we obtain that the first integral at the r.h.s. satisfies
\[ \left\| \int_{\tau - \varepsilon}^{\tau} [F(t, y_\varepsilon(t), w) - F(t, \hat{y}(t), w)] dt \right\| \leq K_1\varepsilon^2. \]
Here, by the choice of \( \tau \), we have
\[ \int_{\tau - \varepsilon}^{\tau} [F(t, \hat{y}(t), w) - F(t, \hat{y}(t), \hat{u}(t))] dt = \varepsilon((F(\tau, \hat{y}(\tau), w) - F(\tau, \hat{y}(\tau), \hat{u}(\tau)) + O(\varepsilon). \]
Hence we have obtained that
\[ v_\varepsilon(\tau) = \varepsilon((F(t, \hat{y}(t), w) - F(t, \hat{y}(t), \hat{u}(t)) + O(\varepsilon)) = \varepsilon(v_\tau + O(\varepsilon)). \]
On the interval \( [\tau, T] \), the differential equations for \( y_\varepsilon \) and for \( \hat{y} \) are the same, i.e.
\[ y'(t) = F(t, y(t), \hat{u}(t)) \]
and the two solutions differ by the initial (at time $\tau$) condition. By a basic result on the differentiability of a solution with respect to initial conditions, one obtains that the difference $y_\varepsilon(T) - \hat{y}(T)$ can be written as

$$y_\varepsilon(T) - \hat{y}(T) = \varepsilon(v(T) + O(\varepsilon))$$

where $v(T)$ is the solution at time $T$ to the Cauchy problem

$$v'(t) = D_y F(t, \hat{y}(t), \hat{u}(t))v(t); \quad v(\tau) = v_\tau.$$ By $D_y F(t, \hat{y}(t), \hat{u}(t))$ we mean the matrix of partial derivatives of $F$ w.r.t. $y$, computed along the solution $(\hat{y}, \hat{u})$. Since $\hat{y}$ is a maximum, we must have that

$$\langle \nabla \psi(\hat{y}(T)), v(T) \rangle \leq 0.$$ It is this geometric condition at time $T$ that we wish to transfer at time $\tau$: let $p$ be a solution to the Cauchy problem:

$$p'(t) = -p(t)D_y F(t, \hat{y}(t), \hat{u}(t)); \quad y(T) = \nabla \psi(\hat{y}(T)).$$ By the product rule for derivatives, we obtain that

$$\frac{d}{dt} \langle p(t), v(t) \rangle = 0,$$

and, since the maximum condition gives $\langle p(T), v(T) \rangle \leq 0$, it follows that, for every $t$,

$$\langle p(t), v(t) \rangle \leq 0,$$

in particular at $t = \tau$. We have already obtained that, at almost every $\tau$,

$$v(\tau) = F(t, \hat{y}(t), \hat{w}) - F(t, \hat{y}(t), \hat{u}(t)),$$

so that we have obtained that, at almost every $\tau$,

$$\langle p(\tau), F(t, \hat{y}(t), \hat{w}) \rangle \leq \langle p(\tau), F(t, \hat{y}(t), \hat{u}(\tau)) \rangle.$$ Since $\hat{w}$ was arbitrary in $U$, we have proved the Maximum Principle.
Necessary conditions
Chapter 2

Reparametrizations

In this chapter we deal with a tool useful when working with problems involving integrals defined on an interval: the reparametrizations.

Consider a functional of the kind
\[ \int_a^b L(t, x(t), x'(t))dt, \]
(2.1)
where \( x : [a, b] \rightarrow \mathbb{R}^N, x(a) = A \) and \( x(b) = B \).

We are interested to see what happens to the functional (2.1) when we reparametrize the time \( t \).

Let \( s : [a, b] \rightarrow [a, b] \) be an invertible map such that \( s(a) = a \) and \( s(b) = b \). Given an absolutely continuous function \( x \) defined in \([a, b]\), set \( \tilde{x}(s) = x(t(s)) \), where \( t(s) \) is the inverse map of \( s \). \( \tilde{x} \) has been obtained by \( x \) reparametrizing \( t \). We have that \( \{\tilde{x}(s) : s \in [a, b]\} = \{x(t) : t \in [a, b]\} \), \( \tilde{x}(a) = x(a) \) and \( \tilde{x}(b) = x(b) \). Moreover, as we will see in the next section, the vectors \( \tilde{x}' \) and \( x' \) have the same directions and different norms.

2.1 Chain rule for derivatives and change of variables in Lebesgue integrals

In this section we present some results obtained by Serrin and Varberg in [40]. They prove the validity of the chain rule for derivatives for functions possessing not everywhere finite derivative, as the absolutely continuous functions, and a change of variables theorem for the Lebesgue integral.

**Theorem 2.1.** Let \( t : [a, b] \rightarrow [a, b] \) a monotone map and \( x : [a, b] \rightarrow \mathbb{R}^N \) an absolutely continuous function. Consider the composite function \( \tilde{x}(s) = x(t(s)) \). We have that
\[ \frac{d}{ds} \tilde{x}(s) = x'(t(s))t'(s). \]

It follows that the vectors \( \tilde{x}' \) and \( x' \) have the same directions and different norms. Reparametrizing the time in the domain of \( x \), we have a control on the derivatives.
of the function $\tilde{x}$. For instance, in Chapter 5 we obtain a Lipschitzian function $x_\epsilon$ from an absolutely continuous function $x$, defining a reparametrization $t$ such that $t'$ is small where $x'$ is great. In Chapter 6 we seek a solution to a differential inclusion. Starting by a solution of the convexified problem and reparametrizing the time, we are able to obtain a solution for the original problem.

The next theorem, on changing of variables in Lebesgue integrals, gives us a tool to see what happens to the functional (2.1) when we reparametrize the time $t$.

**Theorem 2.2.** Suppose that $t : [a, b] \to [a, b]$ is monotone and absolutely continuous and that $f : [a, b] \to \mathbb{R}^N$ is integrable. Then $(f \circ t)'$ is integrable, and the change of variables formula

$$\int_{t(c)}^{t(d)} f(\tau) d\tau = \int_c^d f(t(s))t'(s) ds$$

holds.

Let us consider the functional

$$\int_a^b L(t, x(t), x'(t)) dt,$$

where $x : [a, b] \to \mathbb{R}^N$, $x(a) = A$ and $x(b) = B$. Let $s : [a, b] \to [a, b]$ be an invertible map such that $s(a) = a$ and $s(b) = b$ and let $t$ be its inverse. Setting $\tilde{x}(s) = x(t(s))$, for every $s \in [a, b]$ and $f(s) = L(s, \tilde{x}(s), \tilde{x}'(s))$, for every $t \in [a, b]$, applying Theorem 2.2, we obtain that

$$\int_a^b L(s(t), \tilde{x}(s), \tilde{x}'(s)) ds = \int_a^b L(s(t), \tilde{x}(s(t)), \tilde{x}'(s(t))) s'(t) dt =$$

$$\int_a^b L \left( s(t), x(t), \frac{x'(t)}{s'(t)} \right) s'(t) dt.$$

Let us consider a Lagrangian $L$ autonomous and homogeneous. In this case, the functional is invariant under reparametrizations, indeed

$$\int_a^b L(\tilde{x}(s), \tilde{x}'(s)) ds = \int_a^b L \left( x(t), \frac{x'(t)}{s'(t)} \right) s'(t) dt = \int_a^b L(x(t), x'(t)) dt.$$ 

In general, for non homogeneous Lagrangian this is not true. However, in some cases, we are able to control the difference between the value of the functional evaluated along a function $x$ and along a reparametrization of $x$. For instance, in Chapter 5 of this thesis, given an absolutely continuous function $x$ and $\epsilon > 0$, we are able to find a reparametrization $t_\epsilon$ such that, setting $\tilde{x}(s) = x(t_\epsilon(s))$, $\tilde{x}$ is Lipschitzian and

$$\left| \int_a^b L(s, \tilde{x}(s), \tilde{x}'(s)) ds - \int_a^b L(t, x(t), x'(t)) dt \right| \leq \epsilon.$$ 

Reparametrizations are a powerful tool when dealing with problems involving integrals defined on an interval. A future challenge could be to try to extend this method in multi-dimensional setting, where, unfortunately, reparametrizing is much more complicated.
Chapter 3

The Strong Maximum Principle

Consider the problem of solving a Euler-Lagrange equation of the kind

\[ F(u) = \sum_{i=1}^{N} g_i(u_{x_i}^2)u_{x_i}x_i = 0, \]

(3.1)

where \( u \in C^2(\Omega) \cap C(\overline{\Omega}) \) and \( \Omega \subset \mathbb{R}^N \) is a connected, open and bounded set. If (3.1) is an elliptic equation, then the Strong Maximum Principle, a well known result, holds:

let \( u \) be such that \( F(u) \leq 0 \) and let \( u \) attain its minimum in \( \Omega \), then \( u \) is a constant.

In this chapter we give a proof of this classical result in the simple case of the Laplace equation, \( \Delta u = 0 \), i.e. where \( g_i \equiv 1 \) for every \( i = 1, \ldots, N \). To find the proof in the general case of elliptic equations, see e.g. [19, 23].

3.1 Hopf’s Lemma

To prove the main result of this chapter we need a comparison lemma.

**Lemma 3.1.** Let \( \omega \subset \Omega \) be a connected, open and bounded set. Let \( u, v \in C^2(\omega) \cap C(\overline{\omega}) \) such that \( \Delta v \geq \Delta u \) in \( \omega \). If \( v_{|\partial\omega} \leq u_{|\partial\omega} \), then \( v \leq u \) in \( \omega \).

**Proof.** See [23].

The following is Hopf’s Lemma in the simple case of the Laplace equation.

**Lemma 3.2 (Hopf’s Lemma).** Let \( \Omega \subset \mathbb{R}^N \) be a connected, open and bounded set. Let \( u \in C^2(\Omega) \cap C(\overline{\Omega}) \) be a solution to

\[ \Delta u \leq 0, \]

such that \( u_{|\partial\Omega} = u_0 \). Suppose that there exists \( z \in \partial\Omega \) such that

\[ u(z) < u(x), \quad \text{for all } x \in \Omega. \]
and that $\Omega$ satisfies the interior ball condition at $z$; that is, there exists an open ball $B \subset \Omega$ with $z \in \partial B$. Then

$$\frac{\partial u}{\partial \nu}(z) < 0,$$

where $\nu$ is the outer unit normal to $B$ at $z$.

**Proof.** a) Assume that $u(z) = 0$ and that $B = B(O, 2r)$. We prove the claim by contradiction. Suppose that

$$\frac{\partial u}{\partial \nu}(z) \geq 0,$$

where $\nu$ is the outer unit normal to $B$ at $z$. Let $\epsilon = \min\{u(x) : x \in \overline{B(O, r)}\}$; we have that $\epsilon > 0$. Set

$$\omega = B(O, 2r) \setminus \overline{B(O, r)}.$$

b) We seek a radial function $v \in C^2(\omega) \cap C(\overline{\omega})$ satisfying the problem

$$\begin{cases}
\Delta v \geq 0 & \text{in } \omega \\
v = 0 & \text{in } \partial B(O, 2r) \\
v \leq \epsilon & \text{in } \partial B(O, r) \\
v_{\rho}(z) < 0
\end{cases} \quad (3.2)$$

If $v$ is a radial function, then

$$\Delta v = v_{\rho\rho} + (N - 1)\frac{v_{\rho}}{\rho}.$$ 

Taking

$$v(\rho) = \epsilon \ln \frac{2r}{\rho}, \quad \text{if } N = 2$$

and

$$v(\rho) = \epsilon \left( \left( \frac{r}{\rho} \right)^{N-2} - \left( \frac{1}{2} \right)^{N-2} \right), \quad \text{if } N > 2$$

we have that $\Delta v = 0$ in $\omega$, $v(2r) = 0$, $v(r) < \epsilon$ and $v_{\rho}(z) < 0$, so that $v$ solves (3.2).

c) Since $u, v \in C^2(\omega) \cap C(\overline{\omega})$, $\Delta v \geq \Delta u$ and $v_{\partial \omega} \leq u_{\partial \omega}$, by Lemma 3.1 we obtain that $u \geq v$ in $\omega$. But, since

$$v_{\rho}(z) = \frac{\partial v}{\partial \nu}(z) < \frac{\partial u}{\partial \nu}(z),$$

it follows that there exists $x^0 \in \omega$ such that $v(x^0) > u(x^0)$, a contradiction. \qed
3.2 The Strong Maximum Principle

By applying Hopf’s Lemma proved in the previous section, we are able to prove the Strong Maximum Principle.

**Theorem 3.3.** Let $\Omega \subset \mathbb{R}^2$ be a connected, open and bounded set. Let $u \in C^2(\Omega) \cap C(\overline{\Omega})$ be such that $\Delta u \leq 0$. If $u$ attains its minimum in $\Omega$, then $u$ is constant.

**Proof.**

a) Suppose that $u$ attains its minimum in $\Omega$. Assume $\min_{\Omega} u = 0$ and set $C = \{x \in \Omega : u(x) = 0\}$. By contradiction, suppose that the open set $\Omega \setminus C \neq \emptyset$.

b) Since $\Omega$ is a connected set, there exist $S \in C$ and $R > 0$ such that $B(S, R) \subset \Omega$ and $B(S, R) \cap (\Omega \setminus C) \neq \emptyset$. Let $P \in B(S, R) \cap (\Omega \setminus C)$. Consider the line $PS$. Moving $P$ along this line, we can assume that $B(P, d(P, C)) \subset (\Omega \setminus C)$ and that there exists one point $z \in C$ such that $d(P, C) = d(P, z)$. Set $r = d(P, C)$. W.l.o.g. suppose that $P = O$.

c) The set $\Omega \setminus C$ satisfies the interior ball condition at $z$, hence Hopf’s Lemma implies

$$\frac{\partial u}{\partial n}(z) < 0.$$ 

But this is a contradiction: since $u$ attains minimum in $z \in \Omega$, we have that $Du(z) = 0$. 

$\square$
Part II

New results on the validity of the Euler-Lagrange equations
Is it true that the validity of the Euler-Lagrange equations (E-L) depends only on suitable assumptions of regularity of the integrand, hence on conditions that can be checked a priori, i.e., without the knowledge of the minimizer $\hat{x}$ and of its properties, in addition, possibly, to the natural assumption that the integral $\int_a^b L(t, \hat{x}(t), \hat{x}'(t))dt$ be finite?

A large number of papers has been devoted to this classical problem, see e.g. [13, 14, 29, 32, 33, 45]. In [31] Ball and Mizel, modifying an earlier example of Maniá [31], built a variational problem possessing a minimum $\hat{x}$, such that $\int_a^b L(t, \hat{x}(t), \hat{x}'(t))dt$ is finite, but not satisfying the Euler-Lagrange equations. What happens in the case of this example is that the integrability of $t \mapsto r_x L(t; \hat{x}(t), \hat{x}'(t))$ does not hold. Hence, some condition on the term $t \mapsto \nabla_x L(t, \hat{x}(t), \hat{x}'(t))$ has to be imposed in order to ensure the validity of (E-L). So far, theorems on the validity of (E-L) require a stronger condition. When the Lagrangean $L$, as well as its gradients w.r.t. $x$ and $x'$, satisfy Carathéodory’s conditions, i.e. they are measurable in $t$ for fixed $(x, x')$ and continuous in $(x, x')$ for a.e. $t$, a suitable condition to assume is the existence of an integral bound for $\|\nabla_x L(t, y, \hat{x}'(t))\|$ for $y$ in a neighborhood of the solution $\hat{x}(t)$. More precisely, one assumes that

there exist a scalar $\delta$ and a function $S(t)$ integrable on $I$ such that, for a.e. $t$ in $I$, if $\|y - \hat{x}(t)\| \leq \delta$ then

$$\|\nabla_x L(t, y, \hat{x}'(t))\| \leq S(t)$$

This condition, in particular, implies that the map $x \mapsto L(t, x, \hat{x}'(t))$ is locally Lipschitzian, in a neighborhood of $\hat{x}(t)$, with Lipschitz constant $S(t)$.

However, there are simple and meaningful examples of variational problems where this Lipschitzianity condition is not verified.

Consider the Lagrangian defined by $L(x, \xi) = (\xi \sqrt{|x|} - \frac{2}{3})^2$, and the problem $(P)$ of minimizing

$$\int_0^1 L(x(t), x'(t))dt$$

over the absolutely continuous functions $x$ with $x(0) = 0, x(1) = 1$. One can easily
verify that $\hat{x}(t) = t^{2/3}$ is a minimizer for $(P)$ (indeed, $L(\hat{x}(t), \hat{x}'(t)) = 0$ on $[0, 1]$, and $L$ is non negative everywhere). In this case, although $L$ is not differentiable everywhere, $L_{\xi}(\hat{x}(t), \hat{x}'(t))$ exists a.e. (it is a.e. zero) and it is integrable.

In this chapter we prove the validity of (E-L) under a weaker condition, that does not imply this Lipschitzianity, [21]. Our result is satisfied by Lagrangeans that are Lipschitzian in $x$, but that applies as well to the non-Lipschitzian cases as the example before.

The method of proof consists in demonstrating first that the fact that $\hat{x}$ is a solution implies the integrability of $t \rightarrow \nabla_\xi L(t, \hat{x}(t), \hat{x}'(t))$, then in deriving further regularity and, using this result, in establishing the validity of (E-L) under Carathéodory’s condition.

## 4.1 Integrability of $\nabla_\xi L$ along the solution

Consider the problem of minimizing the functional:

$$J(x) = \int_I L(t, x(t), x'(t))dt$$

on the set of those absolutely continuous functions $x : I \rightarrow \mathbb{R}^N$ satisfying the boundary conditions $x(a) = A, x(b) = B$. Let $\hat{x}$ be a (weak local) minimizer yielding a finite value for the functional $J$, and set $\mu = \sup_{t \in [a,b]} \|\hat{x}(t)\|$.

Our results will depend on the following assumption:

Assumption A. i) $L$ is differentiable in $x$ along $\hat{x}$, for a.e. $t$, and the map $\nabla_x L(\cdot, \hat{x}(\cdot), \hat{x}'(\cdot))$ is integrable on $I$;

ii) there exists a function $S(t)$ integrable on $I$ such that, for any $y \in B(0, \mu + 1)$,

$$L(t, y, \hat{x}'(t)) \leq L(t, \hat{x}(t), \hat{x}'(t)) + S(t)\|y - \hat{x}(t)\|.$$

Consider problem $(P)$ as presented in the Introduction. $L$ and $\hat{x}$ satisfy the assumption A: $L_{\xi}(\hat{x}(t), \hat{x}'(t))$ exists a.e. (identically zero, hence integrable), $S(t) = t^{-2/3}$ verifies the inequality

$$L(y, \hat{x}'(t)) \leq S(t)\|y - \hat{x}(t)\|,$$

since

$$\left(\frac{2}{3}t^{-1/3} \sqrt{|y|} - \frac{2}{3}\right)^2 = \frac{4(\sqrt{|y|} - t^{1/3})}{9t^{2/3}(\sqrt{|y|} + t^{1/3})}(|y| - t^{2/3}) \leq \frac{4}{9t^{2/3}}|y - t^{2/3}|.$$

This is our first result on the term $\nabla_\xi L(\cdot, \hat{x}(\cdot), \hat{x}'(\cdot))$. In what follows, $\overline{\mathbb{R}}$ denotes $\mathbb{R} \cup \{+\infty\}$.

**Theorem 4.1.** Suppose that $L : I \times \mathbb{R}^N \times \mathbb{R}^N \rightarrow \overline{\mathbb{R}}$ is an extended valued function, finite on its effective domain of the form $\text{dom} L = I \times \mathbb{R}^N \times G$, where $G \subset \mathbb{R}^N$ is an open set, and that it satisfies Carathéodory’s conditions, i.e., $L(\cdot, x, \xi)$ is measurable.
for fixed \((x, \xi)\) and \(L(t, \cdot, \cdot)\) is continuous for almost every \(t\). Moreover assume that \(L\) is differentiable in \(\xi\) on \(\text{dom} L\) and that \(\nabla_\xi L\) satisfies Carathéodory’s conditions on \(\text{dom} L\).

Suppose that assumption \(A\) holds. Then,
\[
\int_I \|\nabla_\xi L(t, \dot{x}(t), \dot{x}'(t))\| dt < +\infty.
\]

**Proof.** 1) By assumption, \(L(\cdot, \dot{x}(\cdot), \dot{x}'(\cdot)) \in L^1(I)\), hence setting \(S_0 = \{t \in I : \dot{x}'(t) \notin G\}\), we have \(m(S_0) = 0\). Given \(\epsilon > 0\), we can cover \(S_0\) by an open set \(O_1\) of measure \(m(O_1) < \epsilon/2\). We have also that \(\nabla_\xi L\) is a Carathéodory’s function and that \(\dot{x}'\) is measurable in \(I\). Hence, by the theorems of Scorza Dragoni and of Lusin, for the given \(\epsilon > 0\) there exists an open set \(O_2\) such that \(m(O_2) < \epsilon/2\) and at once \(\dot{x}'\) is continuous in \(I \setminus O_2\), and \(\nabla_\xi L\) is continuous in \((I \setminus O_2) \times \mathbb{R}^N \times G\). By taking \(K_\epsilon = I \setminus (O_1 \cup O_2)\), we have that \(K_\epsilon\) is a closed set such that on it \(\dot{x}'\) is continuous with values in \(G\), \(\nabla_\xi L\) is continuous on \(K_\epsilon \times \mathbb{R}^N \times G\) and \(m(I \setminus K_\epsilon) < \epsilon\). For \(n \geq 1\) set \(\epsilon_n = (b - a)/2^{n+1}\) and \(K_n = K_{\epsilon_n}\); set also \(C_n = \cup_{j=1}^n K_j\). Then \(C_n\) are closed sets, \(C_n \subset C_{n+1}\), \(\dot{x}'\) is continuous on \(C_n\), with values in \(G\), \(\nabla_\xi L\) is continuous in \(C_n \times \mathbb{R}^N \times G\) and \(\lim_{n \to +\infty} m(I \setminus C_n) = 0\). From these properties it follows that there exists \(k_n > 0\) such that, for all \(t \in C_n\),
\[
\|\nabla_\xi L(t, \dot{x}(t), \dot{x}'(t))\| < k_n.
\]

There is no loss of generality in assuming \(k_n \geq k_{n-1}\). Moreover, we have that \(m(C_1) \geq (b - a)/2\) and \(\sum_{n=2}^{\infty} m(C_n \setminus C_{n-1}) \leq (b - a)/2\).

For all \(n \geq 1\), we set \(A_n = C_n \setminus C_{n-1}\). Hence we obtain that \(C_m = C_1 \cup_{n=2}^m A_n\) and that \(I = E \cup C_1 \cup_{n>1} A_n\), where \(m(E) = 0\).

2) Consider the function
\[
\theta(t) = \begin{cases} 
0 & \text{if } \nabla_\xi L(t, \dot{x}(t), \dot{x}'(t)) = 0 \\
\frac{\nabla_\xi L(t, \dot{x}(t), \dot{x}'(t))}{\|\nabla_\xi L(t, \dot{x}(t), \dot{x}'(t))\|} & \text{otherwise},
\end{cases}
\]

and
\[
v_n = \int_{A_n} \theta(t) dt,
\]
so that \(\|v_n\| \leq m(A_n)\). There exists a closed set \(B_n \subset C_1\) such that \(m(B_n) = \|v_n\|\). Set
\[
\theta_n'(t) = -\theta(t) \chi_{A_n}(t) + \frac{v_n}{\|v_n\|} \chi_{B_n}(t).
\]
We have that
\[
\int_I \theta_n'(t) dt = -\int_{A_n} \theta(t) dt + v_n = 0.
\]
Hence, setting \(\theta_n(t) = \int_a^t \theta_n'(\tau) d\tau\), we see that the functions \(\theta_n(t)\) are admissible variations. Moreover we obtain
\[
\|\theta_n\|_\infty \leq \sup_{t \in I} \int_a^t |\theta_n'(\tau)| d\tau \leq \int_I |\theta_n'(\tau)| d\tau \leq 2m(A_n).
\]
3) We have that, for \( t \in A_n \), \( \| \nabla_{\xi} L(t, \dot{x}(t), \dot{x}'(t)) \| < k_n \), and that, for \( t \in B_n \), \( \| \nabla_{\xi} L(t, \dot{x}(t), \dot{x}'(t)) \| < k_1 \leq k_n \). Recalling that \( \overline{A_n} \subset C_n \), we infer

\[
\| \nabla_{\xi} L(t, \dot{x}(t), \dot{x}'(t)) \| \leq k_n,
\]

for all \( t \in \overline{A_n} \cup B_n \). We wish to obtain an uniform bound for \( \| \nabla_{\xi} L \| \) computed in a suitable neighborhood of the solution \((\dot{x}(-), \dot{x}('))\). Consider the set \((\overline{A_n} \cup B_n) \times \mathbb{R}^N \times G\) as a metric space \( M_n \) with distance \( d((t, x, \xi), (t', x', \xi')) = \sup(|t - t'|, |x - x'|, |\xi - \xi'|) \). On \( M_n \), \( \nabla_{\xi} L \) is continuous. Moreover, its subset

\[
G_n = \{(t, \dot{x}(t), \dot{x}'(t)) : t \in \overline{A_n} \cup B_n\}
\]
is compact and, on \( G_n \), \( \| \nabla_{\xi} L \| \) is bounded by \( k_n \). Hence there exists \( \delta_n > 0 \) such that, for \((t, x, \xi) \in M_n \) with \( d((t, x, \xi), (t, \dot{x}(t), \dot{x}'(t))) < \delta_n \), we have \( \| \nabla_{\xi} L(t, x, \xi) \| < k_n + 1 \).

4) For \( |\lambda| < \min\{1/2m(A_n), \delta_n/2m(A_n), \delta_n\} \), consider the integrals

\[
\int_{I} \frac{1}{\lambda} [L(t, \dot{x}(t) + \lambda \theta_n(t), \dot{x}'(t) + \lambda \theta_n'(t)) - L(t, \dot{x}(t), \dot{x}'(t))] dt =
\]

\[
\int_{I} \frac{1}{\lambda} [L(t, \dot{x}(t) + \lambda \theta_n(t), \dot{x}'(t) + \lambda \theta_n'(t)) - L(t, \dot{x}(t) + \lambda \theta_n(t), \dot{x}'(t))] dt +
\]

\[
\int_{I} \frac{1}{\lambda} [L(t, \dot{x}(t) + \lambda \theta_n(t), \dot{x}'(t)) - L(t, \dot{x}(t), \dot{x}'(t))] dt =
\]

\[
\int_{I} \frac{1}{\lambda} [L(t, \dot{x}(t) + \lambda \theta_n(t), \dot{x}'(t) + \lambda \theta_n'(t)) - L(t, \dot{x}(t) + \lambda \theta_n(t), \dot{x}'(t))] dt +
\]

\[
\int_{I} \frac{1}{\lambda} [L(t, \dot{x}(t) + \lambda \theta_n(t), \dot{x}'(t)) - L(t, \dot{x}(t), \dot{x}'(t))] dt.
\]

For every \( t \in A_n \cup B_n \) there exists \( \zeta_\lambda(t) \in (0, \lambda) \) such that

\[
\frac{1}{\lambda} [L(t, \dot{x}(t) + \lambda \theta_n(t), \dot{x}'(t) + \lambda \theta_n'(t)) - L(t, \dot{x}(t) + \lambda \theta_n(t), \dot{x}'(t))] =
\]

\[
\langle \nabla_{\xi} L(t, \dot{x}(t) + \lambda \theta_n(t), \dot{x}'(t) + \zeta_\lambda(t) \theta_n'(t)), \theta_n'(t) \rangle \leq
\]

\[
\| \nabla_{\xi} L(t, \dot{x}(t) + \lambda \theta_n(t), \dot{x}'(t) + \zeta_\lambda(t) \theta_n'(t)) \|
\]

and from the choice of \( \lambda, \| \nabla_{\xi} L(t, \dot{x}(t) + \lambda \theta_n(t), \dot{x}'(t) + \zeta_\lambda(t) \theta_n'(t)) \| < k_n + 1 \). Hence, we can apply the Dominated Convergence Theorem to obtain that

\[
\lim_{\lambda \to 0} \int_{A_n \cup B_n} \frac{1}{\lambda} [L(t, \dot{x}(t) + \lambda \theta_n(t), \dot{x}'(t) + \lambda \theta_n'(t)) - L(t, \dot{x}(t) + \lambda \theta_n(t), \dot{x}'(t))] dt =
\]

\[
\int_{A_n \cup B_n} \langle \nabla_{\xi} L(t, \dot{x}(t), \dot{x}'(t)), \theta_n'(t) \rangle dt.
\]
5) Set $f^+(s) = \max\{0, f(s)\}$, $f^-(s) = \max\{0, -f(s)\}$. Since

$$0 \leq \frac{1}{\lambda} [L(t, \hat{x}(t) + \lambda \theta_n(t), \hat{x}'(t)) - L(t, \hat{x}(t), \hat{x}'(t))]^+ \leq S(t)\|\theta_n(t)\|,$$

by the Dominated Convergence Theorem,

$$\lim_{\lambda \to 0} \int I \frac{1}{\lambda} [L(t, \hat{x}(t) + \lambda \theta_n(t), \hat{x}'(t)) - L(t, \hat{x}(t), \hat{x}'(t))]^+ dt =$$

$$\int I \lim_{\lambda \to 0} \frac{1}{\lambda} [L(t, \hat{x}(t) + \lambda \theta_n(t), \hat{x}'(t)) - L(t, \hat{x}(t), \hat{x}'(t))]^+ dt.$$

By the Fatou’s Lemma,

$$\liminf_{\lambda \to 0} \int I \frac{1}{\lambda} [L(t, \hat{x}(t) + \lambda \theta_n(t), \hat{x}'(t)) - L(t, \hat{x}(t), \hat{x}'(t))]^- dt \geq$$

$$\int I \liminf_{\lambda \to 0} \frac{1}{\lambda} [L(t, \hat{x}(t) + \lambda \theta_n(t), \hat{x}'(t)) - L(t, \hat{x}(t), \hat{x}'(t))]^- dt.$$

We have obtained that

$$\limsup_{\lambda \to 0} \int I \frac{1}{\lambda} [L(t, \hat{x}(t) + \lambda \theta_n(t), \hat{x}'(t)) - L(t, \hat{x}(t), \hat{x}'(t))] dt \leq$$

$$\lim_{\lambda \to 0} \int I \frac{1}{\lambda} [L(t, \hat{x}(t) + \lambda \theta_n(t), \hat{x}'(t)) - L(t, \hat{x}(t), \hat{x}'(t))]^+ -$$

$$\liminf_{\lambda \to 0} \int I \frac{1}{\lambda} [L(t, \hat{x}(t) + \lambda \theta_n(t), \hat{x}'(t)) - L(t, \hat{x}(t), \hat{x}'(t))]^- dt \leq$$

$$\int I \lim_{\lambda \to 0} \frac{1}{\lambda} [L(t, \hat{x}(t) + \lambda \theta_n(t), \hat{x}'(t)) - L(t, \hat{x}(t), \hat{x}'(t))]^+ dt -$$

$$\int I \liminf_{\lambda \to 0} \frac{1}{\lambda} [L(t, \hat{x}(t) + \lambda \theta_n(t), \hat{x}'(t)) - L(t, \hat{x}(t), \hat{x}'(t))]^- dt =$$

$$\int I \liminf_{\lambda \to 0} \frac{1}{\lambda} [L(t, \hat{x}(t) + \lambda \theta_n(t), \hat{x}'(t)) - L(t, \hat{x}(t), \hat{x}'(t))] dt =$$

$$\int I \langle \nabla_x L(t, \hat{x}(t), \hat{x}'(t)), \theta_n(t) \rangle dt.$$

6) Since $\hat{x}$ is a minimizer, we have

$$0 \leq \int_{A_n \cup B_n} \langle \nabla_x L(t, \hat{x}(t), \hat{x}'(t)), \theta'_n(t) \rangle dt +$$

$$\limsup_{\lambda \to 0} \int I \frac{1}{\lambda} [L(t, \hat{x}(t) + \lambda \theta_n(t), \hat{x}'(t)) - L(t, \hat{x}(t), \hat{x}'(t))] dt \leq$$

$$\int_{A_n \cup B_n} \langle \nabla_x L(t, \hat{x}(t), \hat{x}'(t)), \theta'_n(t) \rangle dt + \int I \langle \nabla_x L(t, \hat{x}(t), \hat{x}'(t)), \theta_n(t) \rangle dt.$$
Since, for any $t$ in $A_n$,

$$\theta_n'(t) = -\frac{\nabla_\xi L(t, \hat{x}(t), \hat{x}'(t))}{\|\nabla_\xi L(t, \hat{x}(t), \hat{x}'(t))\|}$$

it follows that $-(\nabla_\xi L(t, \hat{x}(t), \hat{x}'(t)), \theta_n'(t)) \chi_{A_n}(t) = \|\nabla_\xi L(t, \hat{x}(t), \hat{x}'(t))\| \chi_{A_n}(t)$. Hence, we obtain that

$$\int_{A_n} \|\nabla_\xi L(t, \hat{x}(t), \hat{x}'(t))\| dt = -\int_{A_n} \langle \nabla_\xi L(t, \hat{x}(t), \hat{x}'(t)), \theta_n'(t) \rangle dt \leq \int_{B_n} \langle \nabla_\xi L(t, \hat{x}(t), \hat{x}'(t)), \theta_n'(t) \rangle dt + \int_{I} \langle \nabla_x L(t, \hat{x}(t), \hat{x}'(t)), \theta_n(t) \rangle dt.$$

On $B_n$, $\|\nabla_\xi L(t, \hat{x}(t), \hat{x}'(t))\|$ is bounded by $k_1$; from Hölder’s inequality and the estimate on $\|\theta_n\|_\infty$ obtained in 2) we have that there exists a constant $C$ (independent of $n$) such that

$$\int_{A_n} \|\nabla_\xi L(t, \hat{x}(t), \hat{x}'(t))\| dt \leq Cm(A_n).$$

7) As $m \to +\infty$, the sequence of functions $\left(\|\nabla_\xi L(t, \hat{x}(t), \hat{x}'(t))\| \chi_{\cup_{n=2}^{\infty} A_n}(t)\right)_m$ converges monotonically to the function $\|\nabla_\xi L(t, \hat{x}(t), \hat{x}'(t))\| \chi_{\cup_{n=1}^{\infty} A_n}(t)$. From the estimate above and monotone convergence, we obtain

$$\int_{I_{C_1}} \|\nabla_\xi L(t, \hat{x}(t), \hat{x}'(t))\| dt = \int_{I} \|\nabla_\xi L(t, \hat{x}(t), \hat{x}'(t))\| \chi_{\cup_{n=1}^{\infty} A_n} dt \leq Cm(\cup_{n=1}^{\infty} A_n).$$

On $C_1$, $\|\nabla_\xi L(t, \hat{x}(t), \hat{x}'(t))\| < k_1$. Hence

$$\int_{I} \|\nabla_\xi L(t, \hat{x}(t), \hat{x}'(t))\| dt < +\infty.$$

In order for the Euler-Lagrange equations to make sense, we must have that both $\nabla_\xi L$ and $\nabla_x L$ be integrable along the solution. The integrability of $\nabla_\xi L$ and of $\nabla_x L$ along a given function does not follow from the fact that the integral of $L$ exists finite when computed along that function. For example, when $L(\xi) = e^{\xi^2}$, the integrability of $t \to e^{\|x'(t)\|^2}$ does not imply the integrability of $t \to 2\|x'(t)\|e^{\|x'(t)\|^2}$. The meaning of the reasoning in this chapter is as follows:

to establish the validity of (E-L) we must impose some conditions in order to have the integrability of the terms $\nabla_\xi L$ and $\nabla_x L$ along the solution. By imposing a condition on $\nabla_x L$ and building a suitable variation, we have obtained an integrable bound for $\nabla_\xi L$ along the solution, and this has been accomplished without assuming that the solution and its derivatives are bounded, an assumption that would make the result inapplicable, and without imposing growth assumptions on the dependence of $L$ on $x'$, an assumption that would greatly limit the interest of the result.
### 4.2 Additional regularity and the validity of the Euler-Lagrange equation

**Corollary 4.2.** Under the same assumptions as in Theorem 4.1, for every variation $\eta$, $\eta(a) = 0$, $\eta(b) = 0$ and $\eta' \in L^\infty(I)$, we have

$$
\int_I [(\nabla_x L(t, \hat{x}(t), \hat{x}'(t)), \eta'(t)) + (\nabla_x L(t, \hat{x}(t), \hat{x}'(t)), \eta(t))]dt = 0.
$$

**Proof.** We shall prove that, for every $\eta$ in $AC(I)$ with bounded derivative, such that $\eta(a) = \eta(b) = 0$, we have

$$
\int_I [(\nabla_x L(t, \hat{x}(t), \hat{x}'(t)), \eta'(t)) + (\nabla_x L(t, \hat{x}(t), \hat{x}'(t)), \eta(t))]dt \geq 0.
$$

Fix $\eta$, let $\|\eta'(t)\| \leq K$ for almost every $t$ in $I$.

1) Define $C_n$ and $k_n$ as in point 1) of the proof of Theorem 4.1. Set

$$
v_n = \int_{I \cap C_n} \eta'(t)dt.
$$

We have that $\lim_{n \to +\infty} \|v_n\| = 0$. In particular, for $n \geq \nu$, there exists $B_n \subseteq C_1$ such that $m(B_n) = \|v_n\|$. Set

$$
(\eta_n)'(t) = \begin{cases} 
\eta'(t) & \text{for } t \in I \setminus C_n \\
\frac{v_n}{\|v_n\|} + \eta'(t) & \text{for } t \in C_n \setminus B_n \\
\eta'(t) & \text{for } t \in B_n.
\end{cases}
$$

We obtain

$$
\int_I (\eta_n)'(t)dt = \int_{C_n \setminus B_n} \eta'(t)dt + \int_{B_n} \frac{v_n}{\|v_n\|} + \eta'(t)]dt = \int_{C_n} \eta'(t)dt + v_n \int_I \eta'(t)dt = 0.
$$

Hence, setting $\eta_n(t) = \int_a^t \eta_n'(\tau)d\tau$, we have that the functions $\eta_n(t)$ are variations and that, for almost every $t$ in $I$, $\|\eta_n'(t)\| \leq (1 + K)$, so that $\|\eta_n\|_\infty \leq (1 + K)(b - a)$.

2) As in point 3) of the proof of Theorem 4.1, there exists $\delta_n > 0$ such that for $(t, x, \xi) \in C_n \times \mathbb{R}^N \times G$, with $d\left(\left((t, x, \xi), (t, \hat{x}(t), \hat{x}'(t))\right)\right) < \delta_n$, we have $\|\nabla_x L(t, x, \xi)\| < k_n + 1$.

3) For $|\lambda| < \min\{1/(1 + K)(b - a), \delta_n/(1 + K)(b - a), \delta_n/(1 + K)\}$, consider the integrals

$$
\int_I \frac{1}{\lambda}[L(t, \hat{x}(t) + \lambda \eta_n(t), \hat{x}'(t) + \lambda \eta_n'(t)) - L(t, \hat{x}(t), \hat{x}'(t))]dt =
$$

$$
\int_I \frac{1}{\lambda}[L(t, \hat{x}(t) + \lambda \eta_n(t), \hat{x}'(t) + \lambda \eta_n'(t)) - L(t, \hat{x}(t) + \lambda \eta_n(t), \hat{x}'(t))]dt +
$$
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For almost every \( t \in C_n \), there exists \( \zeta_\lambda(t) \in (0, \lambda) \) such that

\[
\int \frac{1}{\lambda} [L(t, \dot{x}(t) + \lambda \eta_n(t), \dot{x}'(t)) - L(t, \dot{x}(t), \dot{x}'(t))] dt = \int_{C_n} \frac{1}{\lambda} [L(t, \dot{x}(t) + \lambda \eta_n(t), \dot{x}'(t) + \lambda \eta_n'(t)) - L(t, \dot{x}(t) + \lambda \eta_n(t), \dot{x}'(t))] dt +
\]

\[
\int \frac{1}{\lambda} [L(t, \dot{x}(t) + \lambda \eta_n(t), \dot{x}'(t) + \lambda \zeta_\lambda(t) \eta_n'(t)) - L(t, \dot{x}(t), \dot{x}'(t))] dt.
\]

Hence, we can apply the Dominated Convergence Theorem to obtain that

\[
\lim_{\lambda \to 0} \int \frac{1}{\lambda} [L(t, \dot{x}(t) + \lambda \eta_n(t), \dot{x}'(t) + \lambda \eta_n'(t)) - L(t, \dot{x}(t) + \lambda \eta_n(t), \dot{x}'(t))] dt = \int \frac{1}{\lambda} [\nabla_x L(t, \dot{x}(t), \dot{x}'(t)) \cdot \eta_n'(t)] dt.
\]

4) Following the point 5) of Theorem 4.1, we obtain that

\[
\limsup_{\lambda \to 0} \int \frac{1}{\lambda} [L(t, \dot{x}(t) + \lambda \eta_n(t), \dot{x}'(t)) - L(t, \dot{x}(t), \dot{x}'(t))] dt \leq \int [\nabla_x L(t, \dot{x}(t), \dot{x}'(t)) \cdot \eta_n(t)] dt.
\]

5) Since \( \dot{x} \) is a minimizer, we have

\[
0 \leq \int [\nabla_x L(t, \dot{x}(t), \dot{x}'(t)) \cdot \eta_n'(t)] dt + \int [\nabla_x L(t, \dot{x}(t), \dot{x}'(t)) \cdot \eta_n'(t) + \nabla_x L(t, \dot{x}(t), \dot{x}'(t)) \cdot \eta_n(t)] dt.
\]

6) Since

\[
\lim_{n \to +\infty} \eta_n'(t) = \eta'(t),
\]

\[
\|\nabla_x L(t, \dot{x}(t), \dot{x}'(t)) \| \|\eta_n'(t)\| \leq \|\nabla_x L(t, \dot{x}(t), \dot{x}'(t))\| (1 + K)
\]

and

\[
\lim_{n \to +\infty} \eta_n(t) = \eta(t),
\]

\[
\|\nabla_x L(t, \dot{x}(t), \dot{x}'(t)) \| \|\eta_n(t)\| \leq \|\nabla_x L(t, \dot{x}(t), \dot{x}'(t))\| (1 + K)(b - a),
\]
by the dominated convergence we obtain
\[ \lim_{n \to +\infty} \int_I \langle \nabla \xi L(t, \hat{x}(t), \hat{x}'(t)), \eta_n(t) \rangle \, dt = \int_I \langle \nabla \xi L(t, \hat{x}(t), \hat{x}'(t)), \eta(t) \rangle \, dt. \]

and
\[ \lim_{n \to +\infty} \int_I \langle \nabla_x L(t, \hat{x}(t), \hat{x}'(t)), \eta_n(t) \rangle \, dt = \int_I \langle \nabla_x L(t, \hat{x}(t), \hat{x}'(t)), \eta(t) \rangle \, dt. \]

It follows that
\[ \int_I [\langle \nabla \xi L(t, \hat{x}(t), \hat{x}'(t)), \eta(t) \rangle + \langle \nabla_x L(t, \hat{x}(t), \hat{x}'(t)), \eta(t) \rangle] \, dt \geq 0. \]

\[ \square \]

Even if the validity of the Euler-Lagrange equations already follows by the previous Corollary 4.2 and the DuBois-Reymond’s lemma ([8]), we give an alternative proof in Corollary 4.4.

In the following theorem we prove an additional regularity result for the Lagrangian evaluated along the minimizer.

**Theorem 4.3.** Under the same assumptions as in Theorem 4.1, the map \( t \to \nabla \xi L(t, \hat{x}(t), \hat{x}'(t)) \) is in \( L^\infty(I) \).

**Proof.** Using an iteration process, we shall prove that for every \( p \in \mathbb{N}, \nabla \xi L(\cdot, \hat{x}(\cdot), \hat{x}'(\cdot)) \) is in \( L^p(I) \). Since the \( L^p \) are nested, this proves that \( \nabla \xi L(\cdot, \hat{x}(\cdot), \hat{x}'(\cdot)) \in \cap_{p \geq 1} L^p(I) \).

At the same time, we shall prove that there exists a constant \( K > 0 \) such that, for every \( 1 \leq p < +\infty \), \( \|\nabla \xi L\|_p \leq K \), thus proving that \( \nabla \xi L(\cdot, \hat{x}(\cdot), \hat{x}'(\cdot)) \) is in \( L^\infty(I) \).

1) From Theorem 4.1, we know that \( \nabla \xi L(\cdot, \hat{x}(\cdot), \hat{x}'(\cdot)) \) is in \( L^1(I) \). Starting the iteration process, fix \( p \in \mathbb{N} \) and suppose that \( \nabla \xi L(\cdot, \hat{x}(\cdot), \hat{x}'(\cdot)) \in L^p(I) \), to prove that \( \nabla \xi L(\cdot, \hat{x}(\cdot), \hat{x}'(\cdot)) \in L^{p+1}(I) \).

2) We can assume \( \|\nabla \xi L\|_p \neq 0 \). Define \( C_n, A_n, k_n \) as in point 1) of the proof of Theorem 4.1. For all \( n > 1 \), set
\[ v_n^p = \frac{(b-a)}{2\|\nabla \xi L\|_p^p} \int_{A_n} \nabla \xi L(t, \hat{x}(t), \hat{x}'(t)) \|\nabla \xi L(t, \hat{x}(t), \hat{x}'(t))\|^{p-1} \, dt. \]

Since
\[ m(C_1) \geq (b-a)/2 \text{ and } \|v_n^p\| \leq \frac{(b-a)}{2\|\nabla \xi L\|_p^p} \int_{A_n} \|\nabla \xi L(t, \hat{x}(t), \hat{x}'(t))\|^p \leq (b-a)/2, \]

there exists a set \( B_n^p \subset C_1 \) such that \( m(B_n^p) = \|v_n^p\| \), so that
\[ \frac{2\|\nabla \xi L\|_p^p}{(b-a)} \int_{B_n^p} \frac{v_n^p}{\|v_n^p\|} \, dt = \int_{A_n} \nabla \xi L(t, \hat{x}(t), \hat{x}'(t)) \|\nabla \xi L(t, \hat{x}(t), \hat{x}'(t))\|^{p-1}. \]
Set
\[ (\theta_n^p)'(t) = -\nabla_\xi L(t, \hat{x}(t), \hat{x}'(t)) \|
abla_\xi L(t, \hat{x}(t), \hat{x}'(t))\|^{p-1} \chi_{A_n}(t) + \frac{2\|
abla_\xi L\|_p}{(b-a) \|v_n^p\|} \chi_{B_n}(t) \]
and \( \theta_n^p(t) = \int_0^t (\theta_n^p)'(\tau) d\tau \). We obtain that \( \|\theta_n^p\|_\infty \leq 2 \int_{A_n} \|
abla_\xi L(t, \hat{x}(t), \hat{x}'(t))\|^p dt \). The variations \( \theta_n^p \) have bounded derivatives so we can apply Corollary 4.2 to obtain that
\[ \int_I [\langle \nabla_\xi L(t, \hat{x}(t), \hat{x}'(t)), (\theta_n^p)'(t) \rangle + \langle \nabla_x L(t, \hat{x}(t), \hat{x}'(t)), \theta_n^p(t) \rangle] dt = 0. \]
It follows that
\[ \int_{A_n} \|
abla_\xi L(t, \hat{x}(t), \hat{x}'(t))\|^{p+1} dt = \]
\[ \frac{2\|
abla_\xi L\|_p}{(b-a)} \int_{B_n^p} \langle \nabla_\xi L(t, \hat{x}(t), \hat{x}'(t)), v_n^p / \|v_n^p\| \rangle dt + \int_I \langle \nabla_x L(t, \hat{x}(t), \hat{x}'(t)), \theta_n^p(t) \rangle dt \leq \]
\[ k_1 \int_{A_n} \|
abla_\xi L(t, \hat{x}(t), \hat{x}'(t))\|^{p+1} dt + \]
\[ 2 \int_{A_n} \|
abla_\xi L(t, \hat{x}(t), \hat{x}'(t))\|^p dt \int_{A_n} \|
abla_x L(t, \hat{x}(t), \hat{x}'(t))\| dt \leq \]
\[ \tilde{C} \int_{A_n} \|
abla_\xi L(t, \hat{x}(t), \hat{x}'(t))\|^{p+1} dt \]
where \( \tilde{C} \) is independent of \( n \) and \( p \) (suppose \( \tilde{C} \geq 1 \)). The sequence of maps
\[ (\|
abla_\xi L(t, \hat{x}(t), \hat{x}'(t))\|^{p+1} \chi_{\cup_{n=2} A_n}(t))_m \]
converges monotonically to \( \|
abla_\xi L(t, \hat{x}(t), \hat{x}'(t))\|^{p+1} \chi_{\cup_{n>1} A_n}(t) \), and each integral \( \int_I \|
abla_\xi L(t, \hat{x}(t), \hat{x}'(t))\|^{p+1} \chi_{\cup_{n=2} A_n}(t) dt \) is bounded by the same constant
\[ \tilde{C} \sum_{n=2}^\infty \int_{A_n} \|
abla_\xi L(t, \hat{x}(t), \hat{x}'(t))\|^{p+1} dt = \tilde{C} \int_{I \setminus C_1} \|
abla_\xi L(t, \hat{x}(t), \hat{x}'(t))\|^{p+1} dt. \]
Hence, by the Monotone Convergence Theorem,
\[ \int_{I \setminus C_1} \|
abla_\xi L(t, \hat{x}(t), \hat{x}'(t))\|^{p+1} dt \leq \tilde{C} \int_{I \setminus C_1} \|
abla_\xi L(t, \hat{x}(t), \hat{x}'(t))\|^{p+1} dt < +\infty. \]
On \( C_1 \), \( \|
abla_\xi L(t, \hat{x}(t), \hat{x}'(t))\| < k_1 \), proving that \( \nabla_\xi L(\cdot, \hat{x}(\cdot), \hat{x}'(\cdot)) \in L^{p+1}(I) \). Moreover, we have also obtained that
\[ \int_{I \setminus C_1} \|
abla_\xi L(t, \hat{x}(t), \hat{x}'(t))\|^{p+1} dt \leq \tilde{C}^p \int_{I \setminus C_1} \|
abla_\xi L(t, \hat{x}(t), \hat{x}'(t))\| dt, \]
so that
\[ \left( \int_{I \setminus C_1} \|
abla_\xi L(t, \hat{x}(t), \hat{x}'(t))\|^{p+1} dt \right)^{1/(p+1)} \leq \tilde{C} S, \]
where \( S = \max\{1, \int_{I \setminus C_1} \| \nabla_\xi L(t, \hat{x}(t), \hat{x}'(t)) \| dt \} \). Setting \( T = \max\{1, m(C_1)\} \) we have that, for all \( p \in \mathbb{N} \),
\[
\| \nabla_\xi L \|_{(p+1)} \leq \left( k_1^{(p+1)} m(C_1) + \int_{I \setminus C_1} \| \nabla_\xi L(t, \hat{x}(t), \hat{x}'(t)) \|^{p+1} dt \right)^{1/(p+1)} \leq k_1 m(C_1)^{1/(p+1)} + \left( \int_{I \setminus C_1} \| \nabla_\xi L(t, \hat{x}(t), \hat{x}'(t)) \|^{p+1} dt \right)^{1/(p+1)} \leq k_1 T + \tilde{C} S = K.
\]

**Corollary 4.4.** Under the same conditions as in Theorem 4.1, for every variation \( \eta, \eta(a) = 0, \eta(b) = 0 \) and \( \eta' \in L^1(I) \), we have
\[
\int_I [(\nabla_\xi L(t, \hat{x}(t), \hat{x}'(t)), \eta'(t)) + (\nabla_\eta L(t, \hat{x}(t), \hat{x}'(t)), \eta(t))] dt = 0.
\]

As a consequence, \( t \rightarrow \nabla_\xi L(t, \hat{x}(t), \hat{x}'(t)) \) is absolutely continuous.

**Proof.** We shall prove that, for every \( \eta \in AC(I) \), such that \( \eta(a) = \eta(b) = 0 \), we have
\[
\int_I [(\nabla_\xi L(t, \hat{x}(t), \hat{x}'(t)), \eta'(t)) + (\nabla_\eta L(t, \hat{x}(t), \hat{x}'(t)), \eta(t))] dt \geq 0.
\]

1) Fix \( \eta \). Through the same steps as in point 1) of the proof of Theorem 4.1, for every \( n \in \mathbb{N} \) we can define a closed set \( C_n \) such that on it \( \eta' \) is continuous, \( \hat{x}' \) is continuous with values in \( G \), \( \nabla_\xi L \) is continuous in \( C_n \times \mathbb{R}^N \times G \) and \( \lim_{n \to +\infty} m(I \setminus C_n) = 0 \). In particular, it follows that there are constants \( k_n \) and \( c_n > 0 \) such that, for all \( t \in C_n \),
\[
\| \nabla_\xi L(t, \hat{x}(t), \hat{x}'(t)) \| < k_n \quad \text{and} \quad \| \eta'(t) \| < c_n.
\]

Define \( \nu_n, B_n, \eta_n', \) and \( \eta_n \) as in the proof of Corollary 4.2. Since, for all \( t \in I \),
\[
\| \eta_n'(t) \| \leq 1 + \| \eta'(t) \|,\]

it follows that \( \| \eta_n' \|_1 \leq (b - a) + \| \eta' \|_1 \). Moreover, \( \| \eta_n \|_{\infty} \leq \| \eta_n' \|_1 \leq (b - a) + \| \eta' \|_1 \).

2) As in point 3) of the proof of Theorem 4.1, there exists \( \delta_n > 0 \) such that for \( (t, x, \xi) \in C_n \times \mathbb{R}^N \times G \), with \( d((t, x, \xi), (t, \hat{x}(t), \hat{x}'(t))) < \delta_n \), we have \( \| \nabla_\xi L(t, x, \xi) \| < k_n + 1 \).

3) For \( |\lambda| < \min\{1/(b - a) + \| \eta' \|_1, \delta_n/(b - a) + \| \eta' \|_1, \delta_n/(c_n + 1)\} \), consider the integrals
\[
\int_I \frac{1}{\lambda} [L(t, \hat{x}(t) + \lambda \eta_n(t), \hat{x}'(t) + \lambda \eta_n'(t)) - L(t, \hat{x}(t), \hat{x}'(t))] dt =
\]
\[
\int_I \frac{1}{\lambda} [L(t, \hat{x}(t) + \lambda \eta_n(t), \hat{x}'(t) + \lambda \eta_n'(t)) - L(t, \hat{x}(t) + \lambda \eta_n(t), \hat{x}'(t))] dt +
\]
\[
\int_I \frac{1}{\lambda} [L(t, \hat{x}(t) + \lambda \eta_n(t), \hat{x}'(t)) - L(t, \hat{x}(t), \hat{x}'(t))] dt =
\]
For every $t \in C_n$, there exists $\zeta_\lambda(t) \in (0, \lambda)$ such that

$$
\frac{1}{\lambda}[L(t, \dot{x}(t) + \lambda \eta_n(t), \dot{x}'(t) + \lambda \eta'_n(t)) - L(t, \dot{x}(t) + \lambda \eta_n(t), \dot{x}'(t))] = 
$$

$$
\langle \nabla \xi L(t, \dot{x}(t) + \lambda \eta_n(t), \dot{x}'(t) + \zeta_\lambda(t) \eta'_n(t)), \eta'_n(t) \rangle 
$$

$$
\| \nabla \xi L(t, \dot{x}(t) + \lambda \eta_n(t), \dot{x}'(t) + \zeta_\lambda(t) \eta'_n(t)) \| C_n < (k_n + 1)c_n.
$$

Hence, we can apply the Dominated Convergence Theorem to obtain that

$$
\lim_{\lambda \to 0} \int \frac{1}{\lambda}[L(t, \dot{x}(t) + \lambda \eta_n(t), \dot{x}'(t) + \lambda \eta'_n(t)) - L(t, \dot{x}(t) + \lambda \eta_n(t), \dot{x}'(t))]dt = 
$$

$$
\int_{C_n} \langle \nabla \xi L(t, \dot{x}(t), \dot{x}'(t)), \eta'_n(t) \rangle dt = \int \langle \nabla \xi L(t, \dot{x}(t), \dot{x}'(t)), \eta'_n(t) \rangle dt.
$$

4) Following the point 5) of Theorem 4.1, we obtain that

$$
\limsup_{\lambda \to 0} \int \frac{1}{\lambda}[L(t, \dot{x}(t) + \lambda \eta_n(t), \dot{x}'(t) - L(t, \dot{x}(t), \dot{x}'(t))]dt \leq 
$$

$$
\int \langle \nabla \xi L(t, \dot{x}(t), \dot{x}'(t)), \eta_n(t) \rangle dt
$$

and, since $\dot{x}$ is a minimizer, we have

$$
0 \leq \int \langle \nabla \xi L(t, \dot{x}(t), \dot{x}'(t)), \eta'_n(t) \rangle +
$$

$$
\limsup_{\lambda \to 0} \int \frac{1}{\lambda}[L(t, \dot{x}(t) + \lambda \eta_n(t), \dot{x}'(t) - L(t, \dot{x}(t), \dot{x}'(t))]dt \leq 
$$

$$
\int \langle \nabla \xi L(t, \dot{x}(t), \dot{x}'(t)), \eta'_n(t) \rangle + \langle \nabla \xi L(t, \dot{x}(t), \dot{x}'(t)), \eta_n(t) \rangle dt.
$$

5) Finally we have:

$$
\lim_{n \to +\infty} \eta'_n(t) = \eta'(t),
$$

$$
\| \nabla \xi L(t, \dot{x}(t), \dot{x}'(t)) \| \| \eta'_n(t) \| \leq \| \nabla \xi L(\cdot, \dot{x}(\cdot), \dot{x}'(\cdot)) \|_\infty (1 + \| \eta'(t) \|)
$$

and

$$
\lim_{n \to +\infty} \eta_n(t) = \eta(t)
$$

$$
\| \nabla \xi L(t, \dot{x}(t), \dot{x}'(t)) \| \| \eta_n(t) \| \leq \| \nabla \xi L(t, \dot{x}(t), \dot{x}'(t)) \| \| (b - a) + \| \eta' \|_1),
$$

so that, by dominated convergence, we obtain

$$
\lim_{n \to +\infty} \int \langle \nabla \xi L(t, \dot{x}(t), \dot{x}'(t)), \eta'_n(t) \rangle dt = \int \langle \nabla \xi L(t, \dot{x}(t), \dot{x}'(t)), \eta'(t) \rangle dt.
$$
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and

\[ \lim_{n \to +\infty} \int_I \langle \nabla_x L(t, \dot{x}(t), \ddot{x}(t)), \eta_n(t) \rangle dt = \int_I \langle \nabla_x L(t, \dot{x}(t), \ddot{x}(t)), \eta(t) \rangle dt. \]

Hence, it follows that

\[ \int_I [\langle \nabla_x L(t, \dot{x}(t), \ddot{x}(t)), \eta'(t) \rangle + \langle \nabla_x L(t, \dot{x}(t), \ddot{x}(t)), \eta(t) \rangle] dt \geq 0. \]

The absolute continuity of \( t \to \nabla_x L(t, \dot{x}(t), \ddot{x}(t)) \) is classical (e.g. [6]).

\[ \square \]
On the validity of the Euler-Lagrange equations
Part III

Techniques on the domain
Chapter 5

Reparametrizations and approximate values of integrals of the Calculus of Variations

The purpose of this chapter is to prove a general theorem on reparametrizations of an interval onto itself which states that, given an absolutely continuous function $x$ on an interval $[a, b]$ and $\epsilon > 0$, under appropriate conditions on $L$ and $\psi$, there exists a reparametrization $t = t_\epsilon(s)$ of $[a, b]$ such that the composition $x_\epsilon = x \circ t_\epsilon$ is at once Lipschitzian and is such that

$$\int_a^b L(x_\epsilon(s), x'_\epsilon(s))\psi(t, x_\epsilon(s))ds \leq \int_a^b L(x(t), x'(t))\psi(t, x(t))dt + \epsilon,$$

see [11].

An application of our Theorem 5.1 is the non-occurrence of the Lavrentiev Phenomenon for a class of functionals of the Calculus of Variations. We recall that in 1926 M. Lavrentiev [30] published an example of a functional of the kind

$$\int_a^b L(t, x(t), x'(t))dt, \quad x(a) = A, x(b) = B$$

whose infimum taken over the space of absolutely continuous functions was strictly lower than the infimum taken over the space of Lipschitzian functions. The occurrence of this phenomenon in a minimum problem is not a minor nuisance, since de facto it prevents the possibility of computing the true absolute minimum of a variational problem by numerical methods.

In the autonomous case, sufficient conditions to prevent the occurrence of this phenomenon have been given by several authors, by imposing enough growth conditions on the Lagrangean $L$ to insure that solutions themselves (exist and) are Lipschitzian, as in [15] or in [2]: in this case, the question has been finally settled in a paper by Alberti and Serra Cassano [1].

Our result applies to autonomous and non-autonomous problems; it applies to problems with obstacles or with other constraints and it applies to multidimensional rotationally invariant problems, where the measure is $r^D dr$. 
In sections 5.2 and 5.3 of this chapter we prove our main result. Section 5.4 is devoted to applications to avoid the Lavrentiev Phenomenon.

5.1 The main result

The following is our main result, a reparametrization Theorem.

**Theorem 5.1.** Let $x : [a, b] \to \mathbb{R}^N$ be absolutely continuous and set $C = \{x(t) : t \in [a, b]\}$. Let $L : C \times \mathbb{R}^N \to \mathbb{R}$ be continuous and such that $L(x, \cdot)$ is convex, and let $\psi : [a, b] \times C \to [c, +\infty)$ be continuous, with $c > 0$. Then:

i) $\int_a^b L(x(t), x'(t))\psi(t, x(t))dt$

exists, finite or $+\infty$;

ii) given any $\epsilon > 0$, there exists a Lipschitzian function $x_\epsilon$, a reparametrization of $x$, such that $x(a) = x_\epsilon(a), x(b) = x_\epsilon(b)$ and

$$\int_a^b L(x_\epsilon(s), x'_\epsilon(s))\psi(s, x_\epsilon(s))ds \leq \int_a^b L(x(t), x'(t))\psi(t, x(t))dt + \epsilon.$$

**Remark 5.2.** The only technical assumption of theorem 5.1 is the hypothesis that $\psi$ is bounded below by a positive constant. However, in the proof of Theorem 5.1, this assumption is used only to infer that $\int_a^b L(x(t), x'(t))dt$ is finite. Hence the theorem holds under the following more general assumption: $\psi(t, x) \geq 0$ and $\int_a^b L(x(t), x'(t))dt < +\infty$.

To verify how sharp our assumptions are, consider the following example of Manià ([31], [13] and [18]). Consider the problem of minimizing the functional

$$\int_0^1 [t - x(t)]^2[x'(t)]^6dt, \quad x(0) = 0, x(1) = 1.$$ 

Then the infimum taken over the space of absolutely continuous functions (assumed in $x(t) = \sqrt[3]{7}$) is strictly lower than the infimum taken over the space of Lipschitzian functions.

As a consequence, the result of Theorem 5.1 cannot hold for the functional of Manià evaluated along $x(t) = \sqrt[3]{7}$.

Setting $\psi(t, x) = [t - x^3]^2$ and $L(x, \xi) = \xi^6$, we see that $\psi \geq 0$ (but not $\psi \geq c > 0$) and that

$$\int_0^1 [x'(t)]^6dt = \int_0^1 1/(3^6 t^4)dt = +\infty.$$

Hence the assumption $\psi(t, x) \geq 0$ and $\int_a^b L(x(s), x'(s))ds < +\infty$ cannot possibly be dropped.
5.2 Preliminary results

The Proof of Theorem 5.1 is based on some simple properties of the (set-valued) function \((x, \xi) \rightarrow \{L^*(x, p) : p \in \partial \xi L(x, \xi)\}\), where \(L^*(x, p)\) is the polar of \(L\) with respect to its second variable, i.e.

\[
L^*(x, p) = \sup_{\xi \in \mathbb{R}^n} \langle p, \xi \rangle - L(x, \xi).
\]

To establish these properies we shall need some preliminary Propositions. In what follows, by \(B[\xi, r]\) we shall mean the closed ball centered at \(\xi\) and radius \(r\).

**Proposition 5.3.** Let \(f\) be a convex function and \(p \in \partial f(\xi)\). Then \(f^*\) is finite at \(p\) and \(f^*(p) = \langle \xi, p \rangle - f(\xi)\).

**Proof.** See [37], page 218.

**Proposition 5.4.**i) Let \(f, f_n : \mathbb{R}^N \rightarrow \mathbb{R}\) be convex and let \(f_n\) converge pointwise to \(f\); let \(p_n \in \partial f_n(\xi)\). Then the sequence \(\{p_n\}\) admits a subsequence converging to some \(p \in \partial f(\xi)\).

\(\text{ii) Let } L : C \times \mathbb{R}^N \rightarrow \mathbb{R} \text{ be continuous and such that } L(x, \cdot) \text{ is convex; let } x_n \rightarrow x \in C \text{ and set } f(\xi) = L(x, \xi), f_n(\xi) = L(x_n, \xi). \text{ Then the same conclusion as in i) holds for } p_n \in \partial f_n(\xi_n) = \partial \xi L(x_n, \xi_n).\)

**Proof.** We prove i) and ii) at once setting, in case i), \(\xi_n = \xi\) and noticing that, in both cases, we have that, for every \(z\), \(f_n(\xi_n + z) \rightarrow f(\xi + z).\)

The sequence \(\{p_n\}\) cannot be unbounded; if it were, along a subsequence we would have \(|p_n| \rightarrow \infty\); choose a further subsequence so that \(p_n/|p_n| \rightarrow p_0\), where \(|p_0| = 1\). We have

\[
f(\xi + p_0) = \lim_{n \rightarrow \infty} f_n(\xi_n + p_0) \geq \limsup_{n \rightarrow \infty} f_n(\xi_n) + \langle p_n, p_0 \rangle = +\infty
\]

a contradiction, since \(f\) is finite at \(\xi + p_0\). Hence the sequence \(\{p_n\}\) is bounded and we can select a subsequence converging to \(p_0\). If it were \(p_0 \notin \partial f(\xi)\) there would exist \(\xi'\) such that \(f(\xi') < f(\xi) + \langle p_0, \xi' - \xi \rangle\). Since

\[
f(\xi') = f(\xi + (\xi' - \xi)) = \lim_{n \rightarrow \infty} f_n(\xi_n + (\xi' - \xi)) \geq \limsup_{n \rightarrow \infty} f_n(\xi_n) + \langle p_n, \xi' - \xi \rangle \text{ for } p_n \rightarrow p_0,
\]

we would have a contradiction.

**Proposition 5.5.** Let \(f : \mathbb{R}^N \rightarrow \mathbb{R}\) be convex. The map \(t \rightarrow \{\langle \xi t, p \rangle - f(\xi t) : p \in \partial f(\xi t)\}\), from \([0, +\infty)\) to the closed convex subsets of \(\mathbb{R}\), is monotonically increasing.
Proof. a) Assume, in addition, that $f$ is smooth; then we have $\nabla (\langle \xi, \nabla f(\xi) \rangle - f(\xi)) = \xi^T H$, where $H$ is the Hessian matrix of $f$. Hence

$$\frac{d}{dt} (\langle \xi t, \nabla f(\xi t) \rangle - f(\xi t)) = t \xi^T H \xi \geq 0,$$

so that $t_2 \geq t_1$ implies

$$\langle \xi t_2, \nabla f(\xi t_2) \rangle - f(\xi t_2) \geq \langle \xi t_1, \nabla f(\xi t_1) \rangle - f(\xi t_1).$$

b) In general, the map $\phi(t) = f(\xi t)$, being convex, is differentiable for a.e. $t$. Let $t_1^+ > t_1$ and $t_2^- < t_2$ be points where $\phi$ is differentiable. Approximate $f$ by a sequence $\{f_n\}$ of convex smooth maps, converging pointwise to $f$. Set $\phi_n(t) = f_n(\xi t)$: in particular, applying the previous Proposition 5.4, we have that $\phi_n'(t)$ converges to $\phi'(t)$ both at $t_1^+$ and at $t_2^-$. Applying point a) to $f_n$ we obtain that

$$t_2^- \phi_n(t_2^-) - \phi_n(t_2^-) \geq t_1^+ \phi_n(t_1^+) - \phi_n(t_1^+)$$

so that, passing to the limit as $n \to \infty$,

$$t_2^- \phi(t_2^-) - \phi(t_2^-) \geq t_1^+ \phi(t_1^+) - \phi(t_1^+).$$

By the monotonicity of the subdifferential of $\phi$, for every $a_1 \in \partial \phi(t_1)$ and $a_2 \in \partial \phi(t_2)$, we have

$$t_2 a_2 - \phi(t_2) \geq t_2 \phi'(t_2) - \phi(t_2) \geq t_1^+ \phi'(t_1^+) - \phi(t_1^+) \geq t_2 a_1 - \phi(t_1)$$

and passing to the limit as $t_2^+ \to t_2$ and $t_1^- \to t_1$, by the continuity of $\phi$, one has

$$t_2 a_2 - \phi(t_2) \geq t_1 a_1 - \phi(t_1).$$

Since ([8], page 257), $\partial \phi(t) = \{\langle \xi, p \rangle : p \in \partial f(\xi t)\}$, the claim is proved.

Proposition 5.6. Let $f : \mathbb{R}^N \to \mathbb{R}$ be convex. Then the function $f(\xi/(1+\cdot))(1+\cdot)$ is convex in $(-1, \infty)$. Moreover, given $\delta$ there are $\theta, 0 \leq \theta \leq 1$ and $p_0 \in \partial f(\xi/(1+\theta \delta))$, such that

$$f \left( \frac{\xi}{1+\delta} \right) (1+\delta) - f(\xi) = -\delta f^*(p_0).$$

Proof. a) Assume, in addition, that $f$ is $C^2$. Then, computing the derivatives, one obtains

$$\frac{d}{d\alpha} \left( f \left( \frac{\xi}{1+\alpha} \right) (1+\alpha) \right) = \langle \nabla f \left( \frac{\xi}{1+\alpha} \right) , \frac{-\xi}{1+\alpha} \rangle + f \left( \frac{\xi}{1+\alpha} \right) = -f^* \left( \nabla f \left( \frac{\xi}{1+\alpha} \right) \right)$$

and

$$\frac{d^2}{d\alpha^2} \left( f \left( \frac{\xi}{1+\alpha} \right) (1+\alpha) \right) = \frac{1}{(1+\alpha)^3} \xi^T H \xi$$
where $H$ is the Hessian matrix of $f$ computed at $\xi/(1 + \alpha)$, so that the second derivative is non-negative, and the map $f(\xi/(1 + \cdot))(1 + \cdot)$ is convex.

b) In the general case, approximate the convex map $f$ by a sequence of convex differentiable maps $f_n$ converging pointwise to $f$ to obtain the required convexity and to have:

$$f\left(\frac{\xi}{1 + \delta}\right) (1 + \delta) - f(\xi) = \lim_{n \to \infty} f_n\left(\frac{\xi}{1 + \delta}\right) (1 + \delta) - f_n(\xi) =$$

$$\lim_{n \to \infty} \delta \left[ -\langle \frac{\xi}{1 + \theta_n \delta}, \nabla f_n \left( \frac{\xi}{1 + \theta_n \delta} \right) \rangle + f_n \left( \frac{\xi}{1 + \theta_n \delta} \right) \right].$$

c) Applying Proposition 2, let $p_0 \in \partial f(\xi/(1 + \theta \delta))$ be the limit of a converging subsequence of $\{\nabla f_n(\xi/(1 + \theta_n \delta))\}$. We have

$$f\left(\frac{\xi}{1 + \delta}\right) (1 + \delta) - f(\xi) = \delta \left[ -\langle \frac{\xi}{1 + \theta \delta}, p_0 \rangle + f \left( \frac{\xi}{1 + \theta \delta} \right) \right] = -\delta f^*(p).$$

\[\square\]

5.3 Proof of Theorem 5.1

i) For every $t \in [a, b]$, $L(x(t), x'(t)) \geq L(x(t), 0) + \langle p_0(t), x'(t) \rangle$, where $p_0(t)$ is any selection from $\partial \xi L(x(t), 0)$. Let $E_- = \{t \in [a, b] : (L(x(t), x'(t)))^- > 0\}$, let $\chi_-$ be the characteristic function of $E_-$. Then, in particular,

$$-(L(x(t), x'(t)))^- = L(x(t), x'(t))\chi_-(t) \geq [L(x(t), 0) + \langle p_0(t), x'(t) \rangle] \chi_-(t),$$

hence

$$\int_a^b -(L(x(t), x'(t)))^- \psi(t, x(t))dt = \int_a^b L(x(t), x'(t))\chi_-(t)\psi(t, x(t))dt \geq$$

$$\int_a^b [L(x(t), 0) + \langle p_0(t), x'(t) \rangle] \chi_-(t)\psi(t, x(t))dt.$$ Since $\psi$ is bounded and, by Proposition 2, $p_0(t)$ is bounded, the claim follows by Hölder’s inequality.

ii) In case

$$\int_a^b L(x(t), x'(t))\psi(t, x(t))dt = +\infty,$$

any parametrization $t : [a, b] \to [a, b]$ that would make $x \circ t$ Lipschitzian, is acceptable as $x_t$. Hence from now on we shall assume

$$-\infty < \int_a^b L(x(t), x'(t))\psi(t, x(t))dt < +\infty.$$
We have also
\[ +\infty > \int_a^b |L(x(t), x'(t))|\psi(t, x(t))dt \geq c \int_a^b |L(x(t), x'(t))|dt. \]

a) \( C = \{ x(t) : t \in [a, b] \} \) is a compact subset of \( \mathbb{R}^N \); consider the set \( V = \{ (x, p) : x \in C, p \in \partial_x L(x, \xi), |\xi| \leq 1 \} \).

By b) of Proposition 2, arguing by contradiction, we obtain that \( V \) is compact. Then, \( \min_V L^*(x, p) \) is attained and is finite: let \( (x_n, p_n) \in \partial_x L(x_n, \xi_n), |\xi_n| \leq 1 \), be a minimizing sequence; we can assume that \( x_n \to x, x \in C, \xi_n \to \xi, p_n \to p, p \in \partial_x L(x, \xi) \). By Proposition 1, we have that \( L^*(x_n, p_n) = (\xi_n, p_n) - L(x_n, \xi_n) \to (\xi, p) - L(x, \xi) = L^*(x, p) \).

Set \( m = \min_V L^*(x, p) \).

Applying Proposition 3, we obtain that \( L^*(x, p) \geq m, \) any \( x \in C \) and any \( p \in \partial_x L(x, \xi) \), for any \( \xi \in \mathbb{R}^N \). Hence we have that, for every \( x \in C \) and any \( p, \)
\[ L^*(x, p) - m \geq 0. \]

Consider \( \tilde{L}(x, \xi) = L(x, \xi) + m \). Since \( \partial_x L(x, \xi) = \partial_x \tilde{L}(x, \xi) \), we have that \( \tilde{L}^*(x, p) = L^*(x, p) - m, \) and we infer that \( \tilde{L}^*(x, p) \geq 0. \)

b) Set \( \ell = \int_a^b |\tilde{L}(x(t), x'(t))|dt \) and let \( \Psi \) be such that \( |\psi(t, x)| \leq \Psi, \forall (t, x) \in [a, b] \times C. \) From the uniform continuity of \( \psi \) on \( [a, b] \times C \), we infer that we can fix \( k \in \mathbb{N} \) such that \( \forall t_1, t_2 \in [a, b], \text{ with } |t_1 - t_2| \leq (b-a)/2^k, \text{ we have } |\psi(t_2, x) - \psi(t_1, x)| \leq \epsilon/(2\ell), \) any \( x \in C. \) For \( i = 0, \ldots, 2^k - 1 \) set \( I_i = [(b-a)i/2^k, (b-a)(i+1)/2^k], \)
\[ H_i = \int_{I_i} |x'(t)|dt, \mu = \max\{2^{k+1}H_i/(b-a) : i = 0, \ldots, 2^k - 1 \} \] and
\[ T_{H_i} = \left\{ t \in I_i : |x'(t)| \leq \frac{2^{k+1}H_i}{b-a} \right\}; \]
it follows that \( |T_{H_i}| \geq (b-a)/2^{k+1}. \) Set also \( T = \bigcup_{i=0}^{2^k-1} T_{H_i}. \)

Since \( \{(x(t), x'(t)) : t \in T\} \) belongs to a compact set and \( L \) is continuous, there exists a constant \( M, \) such that
\[ \left| \tilde{L}(x(t), 2x'(t)) - \frac{1}{2} + \tilde{L}(x(t), x'(t)) \right| \leq M, \]
for all \( t \in T. \)

c) For every \( n \in N \) set \( S^i_n = \{ t \in I_i : |x'(t)| > n \}, \) \( \epsilon^i_n = \int_{S^i_n} \left( \frac{|x'(t)|}{n} - 1 \right) dt \) and \( \epsilon_n = \sum_{i=0}^{2^k-1} \epsilon^i_n. \) From the integrability of \( |x'|, \) we have that \( \lim_{n \to \infty} \epsilon_n = 0. \)
d) Having defined $\epsilon_n$, for all $n$ such that $\epsilon_n \leq (b-a)/2^{k+2}$, choose $\Sigma_n \subset T$, such that $|\Sigma_n| = 2\epsilon_n^i$. This is possible from point c).

e) Define the absolutely continuous functions $s_n$ by $s_n(t) = a + \int_a^t s_n'(\tau)d\tau$, where

$$s_n'(t) = \begin{cases} 1 + \left(\frac{|x'(t)|}{n} - 1\right) & t \in S_n = \bigcup_{i=0}^{2^k-1} S_n^i \\ 1 - \frac{1}{2} & t \in \Sigma_n = \bigcup_{i=0}^{2^k-1} \Sigma_n^i \\ 1 & \text{otherwise.} \end{cases}$$

One verifies that, $\forall i = 0, \cdots, 2^k - 1$, the restriction of $s_n$ to $I_i$ is an invertible map from $I_i$ onto itself (in particular, each $s_n$ is an invertible map from $[a, b]$ onto itself).

It follows that $|s_n(t) - t| \leq (b-a)/2^k$.

f) We have

$$\int_a^b \mathcal{L}\left(x(t), \frac{x'(t)}{s_n'(t)}\right) s_n'(t)\psi(s_n(t), x(t))dt - \int_a^b \mathcal{L}(x(t), x'(t))\psi(t, x(t))dt =$$

$$\int_a^b \left[ \mathcal{L}\left(x(t), \frac{x'(t)}{s_n'(t)}\right) s_n'(t) - \mathcal{L}(x(t), x'(t)) \right] \psi(s_n(t), x(t))dt +$$

$$\int_a^b \mathcal{L}(x(t), x'(t))\left[\psi(s_n(t), x(t)) - \psi(t, x(t))\right]dt,$$

and, from the definition of $s_n'$,

$$\int_a^b \left[ \mathcal{L}\left(x(t), \frac{x'(t)}{s_n'(t)}\right) s_n'(t) - \mathcal{L}(x(t), x'(t)) \right] \psi(s_n(t), x(t))dt =$$

$$\int_{S_n} \left[ \mathcal{L}\left(x(t), \frac{x'(t)}{|x'(t)|}\right) \frac{|x'(t)|}{n} - \mathcal{L}(x(t), x'(t)) \right] \psi(s_n(t), x(t))dt +$$

$$\int_{\Sigma_n} \left[ \mathcal{L}(x(t), 2x'(t)) \frac{1}{2} - \mathcal{L}(x(t), x'(t)) \right] \psi(s_n(t), x(t))dt.$$

We wish to estimate the above integrals. Since $\Sigma_n \subset T$, we obtain

$$\int_{\Sigma_n} \left[ \mathcal{L}(x(t), 2x'(t)) \frac{1}{2} - \mathcal{L}(x(t), x'(t)) \right] \psi(s_n(t), x(t))dt \leq 2M\Psi\epsilon_n.$$
hence
\[
\int_{S_n} \left[ \tilde{L} \left( x(t), n \frac{x'(t)}{|x'(t)|} \right) \frac{|x'(t)|}{n} - \tilde{L}(x(t), x'(t)) \right] \psi(s_n(t), x(t))dt \leq 0.
\]

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hence, at almost every \( t \), the norm of the derivative of \( x_{\varepsilon} \) is bounded by \( n \). This completes the proof.

g) The choice of \( k \) implies that
\[
\int_a^b \tilde{L}(x(t), x'(t)) \left[ \psi(s_n(t), x(t)) - \psi(t, x(t)) \right] dt \leq \frac{\varepsilon}{2}.
\]

We have obtained
\[
\int_a^b \tilde{L} \left( x(t), \frac{x'(t)}{s_n(t)} \right) \frac{1}{s'_n(t)} \psi(s_n(t), x(t)) dt - \int_a^b \tilde{L}(x(t), x'(t)) \psi(t, x(t)) dt \leq 2M \Psi \varepsilon_n + \frac{\varepsilon}{2}.
\]

h) Fix \( n \) such that \( 2M \Psi \varepsilon_n \leq \varepsilon/2 \).

Then, the conclusion of f) proves the Theorem; in fact, defining \( x_{\varepsilon} = x \circ t_n \), where \( t_n \) is the inverse of the function \( s_n \), we obtain, by the change of variable formula [40], that
\[
\int_a^b \tilde{L}(x_{\varepsilon}(s), x'_{\varepsilon}(s)) \psi(s, x_{\varepsilon}(s)) ds =
\]
\[
\int_a^b \tilde{L} \left( x_{\varepsilon}(s_n(t)), \frac{dx_{\varepsilon}}{ds}(s_n(t)) \right) \frac{1}{s'_n(t)} \psi(s_n(t), x_{\varepsilon}(s_n(t))) dt =
\]
\[
\int_a^b \tilde{L} \left( x(t), \frac{x'(t)}{s'_n(t)} \right) \frac{1}{s'_n(t)} \psi(s_n(t), x(t)) dt \leq \int_a^b \tilde{L}(x(t), x'(t)) \psi(t, x(t)) dt + \varepsilon.
\]

so that
\[
\int_a^b L(x_{\varepsilon}(s), x'_{\varepsilon}(s)) \psi(s, x_{\varepsilon}(s)) ds - \int_a^b L(x(t), x'(t)) \psi(t, x(t)) dt =
\]
\[
\int_a^b [L(x_{\varepsilon}(s), x'_{\varepsilon}(s)) + m] \psi(s, x_{\varepsilon}(s)) ds - \int_a^b [L(x(t), x'(t)) + m] \psi(t, x(t)) dt =
\]
\[
\int_a^b \tilde{L}(x_{\varepsilon}(s), x'_{\varepsilon}(s)) \psi(s, x_{\varepsilon}(s)) ds - \int_a^b \tilde{L}(x(t), x'(t)) \psi(t, x(t)) dt \leq \varepsilon.
\]

Moreover, \( x_{\varepsilon} \) is Lipschitzian. In fact, consider the equality \( x'_{\varepsilon}(s_n(t)) = x'(t)/s'_n(t) \) and fix \( t \) where \( s'_n(t) \) exists; we obtain
\[
\frac{dx_{\varepsilon}}{ds}(s_n(t)) = \begin{cases} 
  n & \text{if } t \in S_n \\
  \leq \mu & \text{if } t \in \Sigma_n \\
  \leq n & \text{otherwise};
\end{cases}
\]

hence, at almost every \( t \), the norm of the derivative of \( x_{\varepsilon} \) is bounded by \( n \). This completes the proof.
5.4 Applications: the non-occurrence of the Lavrentiev Phenomenon

The theorems below present some applications of Theorem 5.1 to prevent the occurrence of the Lavrentiev phenomenon to different classes of Minimum Problems.

Denote by $\text{Lip}([a, b])$ and by $\text{AC}([a, b])$, respectively, the space of all Lipschitzian and absolutely continuous functions from $[a, b]$ to $\mathbb{R}^N$. Let $E \subset \mathbb{R}^N$ and consider the functional

$$ I(x) = \int_a^b L(x(t), x'(t)) \psi(t, x(t)) \, dt. $$

Call $\inf(P)_\infty$ the infimum of \{ $I(x) : x \in \text{Lip}([a, b]), x(t) \in E, x(a) = A, x(b) = B$ \} and $\inf(P)_1$ the infimum of \{ $I(x) : x \in \text{AC}([a, b]), x(t) \in E, x(a) = A, x(b) = B$ \}.

**Theorem 5.7.** Let $L : E \times \mathbb{R}^N \to \mathbb{R}$ be continuous and such that $L(x, \cdot)$ is convex and let $\psi : [a, b] \times E \to [c, +\infty)$ be continuous, with $c > 0$; then $\inf(P)_\infty = \inf(P)_1$.

In the previous Theorem $E$ can be any subset of $\mathbb{R}^N$ such that the set of absolutely continuous functions with values in $E$ and satisfying the boundary conditions is non-empty. In particular, $x \in E$ can describe a problem with an obstacle.

As an application to a problem with a constraint different from an obstacle, let $E = \mathbb{R}^2 \setminus \{0\}$ and call $\inf(P^i)_\infty$ the infimum of \{ $I(x) : x \in \text{Lip}([a, b]), x(t) \in E, x(a) = x(b)$ \} and having prescribed rotation number $i(x) = k$. Call $\inf(P^i)_1$ the infimum of the same problem but for $x \in \text{AC}$.

**Theorem 5.8.** Let $L : E \times \mathbb{R}^2 \to \mathbb{R}$ be continuous and such that $L(x, \cdot)$ is convex and let $\psi : [a, b] \times E \to [c, +\infty)$ be continuous, with $c > 0$; then $\inf(P^i)_\infty = \inf(P^i)_1$.

**Proof.** As it is well known the rotation number $i$ is independent of the parametrizations of $x$.

Theorem 5.8 applies in particular to the case $L(x, \xi) = |\xi|^2/2 + 1/|x|$, the case of the Newtonian potential generated by a body fixed at the origin. Gordon in [25] proved that Keplerian orbits are minima to this problem with $k = 1$.

As a further application, we consider a vectorial case. Let $L : E \times \mathbb{R}^N \to \mathbb{R}$ be a continuous function such that $L(u, \cdot)$ is convex (we shall assume that the Lagrangian is independent of the integration variable). Suppose that $L(u, \cdot)$ has the symmetry of being rotationally invariant, i.e. assuming that there exists a function $h : E \times [0, \infty) \to \mathbb{R}$ such that $L(u, \xi) = h(u, |\xi|)$.

Consider the functional

$$ I(u) = \int_{S[a, b]} L(u(x), \nabla u(x)) \, dx $$

where $S[a, b] = \{ x \in \mathbb{R}^{D+1} : a \leq |x| \leq b \}$. Denote by $\inf(P)_\infty$ the infimum of \{ $I(u) : u \in \text{Lip}(S[a, b]), u(x) \in E, u$ radial, $u|_{\partial B(0, a)} = A, u|_{\partial B(0, b)} = B$ \} and $\inf(P)_1$ the
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infimum of \( \{ I(u) : u \in W^{1,1}(S[a,b]), u(x) \in E, u \text{ radial, } u|_{\partial B(0,a)} = A, u|_{\partial B(0,b)} = B \} \).

It is our purpose to prove that \( \inf(P)_\infty = \inf(P)_1 \).

Observe that if \( w : [a,b] \to E \) is such that \( u(x) = w(|x|) \) then

\[
I(u) = C_D \int_a^b L(w(r), w'(r)) r^p \, dr, \quad w(a) = A, w(b) = B,
\]

where \( C_D = \frac{\pi^{(D+1)/2}}{\Gamma((D+3)/2)} (b^{D+1} - a^{D+1}) \).

**Theorem 5.9.** Let \( L : E \times \mathbb{R}^N \to \mathbb{R} \) be continuous and such that \( L(u, \cdot) \) is convex; then \( \inf(P)_\infty = \inf(P)_1 \).
Chapter 6

On the existence of solutions to a class of minimum time control problems including the Brachystocrona

Around 1696 Jakob Bernoulli raised the following question: find the path from an initial point $x^0$ to a target point $x^f$ such that a body, subject to gravity only, starting from $x^0$ with initial velocity zero, would reach $x^f$ in minimum time. In 1959 [22] A.F. Filippov proved the first general theorem on the existence of solutions to minimum time control problems of the form

$$x'(t) = f(x(t), u(t)) \quad u(t) \in U(x(t))$$

requiring that the set valued map $x \to U(x)$ be upper semicontinuous (with respect to the inclusion) and that the values $F(x) = f(x, U(x))$ be compact and convex. In Theorem 6.2 of the present chapter, we prove the existence of solutions to minimum time problems for differential inclusions, under assumptions that do not require the convexity of the images $F(x) = f(x, U(x))$, see [10]. In section 3 we present a model for the problem raised by Bernoulli and we show that our model (a non-convex control problem) satisfies the assumptions required by our Theorem 6.2 for the existence of solutions to minimum time problems. The case of Brachystocrona as a minimum time control problem has already been amply treated in [41, 42].

6.1 The existence of solutions to minimum time problems

For a compact subset $A \subset \mathbb{R}^N$, set $coA$ be its convex hull. For basic results relating to solutions to differential inclusions, measurable selections and properties of set-valued maps we refer to any standard text on the subject.
Lemma 6.1. Let \( x : [0, t^*] \rightarrow \mathbb{R}^N \) be absolutely continuous; let \( X = \{ x(t) : t \in [0, t^*] \} \) and let \( F \) be defined on \( X \). Assume that, for almost every \( t \in [0, t^*] \),
\[
x'(t) \in F(x(t))
\]
and that there exists \( E \subset [0, t^*] \) of positive measure such that \( x'(t) = 0 \) on \( E \). Then there exist \( \tau^* < t^* \) and an absolutely continuous function \( \tilde{x} : [0, \tau^*] \rightarrow X \), such that \( \tilde{x}(0) = x(0) = x^0 \), \( \tilde{x}(\tau^*) = x(t^*) \) and
\[
\tilde{x}'(t) \in F(\tilde{x}(t))
\]
for almost every \( \tau \in [0, \tau^*] \).

Proof. a) It follows from the assumptions that there exists a closed subset \( K \) of \( E \) of positive measure. The complement of \( K \) consists of at most countably many open non-overlapping intervals \((a_i, b_i), i \in I\). Since the intervals \((a_i, b_i)\) are disjoint, we must have that \( \tau^* = \sum_{i \in I} (b_i - a_i) < t^* \). For each \( i \in I \), set
\[
\tau(b_i) = \sum_{j \in I : b_j < b_i} (b_j - a_j).
\]
If \( b_i > b_m \), we have that \( \tau(b) \geq \tau(b_m) + (b_m - a_m) \).

Consider the intervals \((\tau(b), \tau(b) + b_i - a_i)\); they are disjoint, since in case we had \((\tau(b), \tau(b) + b_i - a_i) \cap (\tau(b_m), \tau(b_m) + b_m - a_m) \neq \emptyset \) with \( b_i > b_m \), we would obtain \( \tau(b_m) + (b_m - a_m) > \tau(b) \), a contradiction. Set \( T = \bigcup_{i \in I} (\tau(b_i), \tau(b_i) + b_i - a_i) \); since \( \tau \leq \sum_{i \in I} (b_j - a_j) = \tau^* \), \( \forall \tau \in T \), we have that \( T \subset [0, \tau^*] \). Moreover, the measure of \( T \) equals \( \tau^* \), in fact
\[
m(T) = \sum_{i \in I} (b_i - a_i) = \tau^*.
\]

b) Define the absolutely continuous function \( \tilde{x} : [0, \tau^*] \rightarrow \mathbb{R}^N \) by \( \tilde{x}(\tau) = x^0 + \int_0^\tau \tilde{x}'(s)ds \), where
\[
\tilde{x}'(s) = \begin{cases} 
x'(s + a_i - \tau(b_i)) & \text{if } s \in (\tau(b_i), \tau(b_i) + b_i - a_i), i \in I \\
0 & \text{if } s \in [0, \tau^*] \setminus T
\end{cases};
\]
we have \( \tilde{x}(0) = x^0 \).

Fix \( \tau \in T \), there exists \( i \in I \) such that \( \tau \in (\tau(b_i), \tau(b_i) + b_i - a_i) \). Notice that \( \tau \in (\tau(b_i), \tau(b_i) + b_i - a_i) \) if and only if \( \tau + a_i - \tau(b_i) \in (a_i, b_i) \). We have
\[
\tilde{x}(\tau) - x^0 = \int_0^{\tau(b_i)} \tilde{x}'(s)ds + \int_{\tau(b_i)}^{\tau} \tilde{x}'(s)ds =
\]
\[
\sum_{j : \tau(b_j) + b_j - a_j \leq \tau(b_i)} \int_{\tau(b_j)}^{\tau(b_j) + b_j - a_j} \tilde{x}'(s)ds + \int_{\tau(b_i)}^{\tau} \tilde{x}'(s)ds =
\]
The previous equality implies that $\bar{x}$ is a solution to the differential inclusion. In fact, we have that, almost everywhere in $[0, \tau^*]$, 

$$\ddot{x}(\tau) = x'(\tau + a_i - \tau(b_i)) \in F(x(\tau + a_i - \tau(b_i)) = F(\bar{x}(\tau)).$$

c) Set $B = \sup\{b_i\}$. Then either the supremum is attained or it is not. In the first case, for some $j_k$, $B = b_{j_k}$ and $\tau(b_{j_k}) + (b_{j_k} - a_{j_k}) = \tau^*$. From b) we have that $x(t) = \bar{x}(t - a_{j_k} + \tau(b_{j_k})), \forall t \in [a_{j_k}, b_{j_k}]$. Since $x'(t) = 0$ on $[B, \tau^*]$, by continuity we have that $x(t^*) = x(B) = x(b_{j_k}) = \bar{x}(b_{j_k} - a_{j_k} + \tau(b_{j_k})) = \bar{x}(\tau^*).

In the second case, let $\{b_{j_k}\}$ be an increasing sequence, converging to $B$. From

$$|x(t^*) - x(a_{j_k})| = \int_{a_{j_k}}^{B} |x'(s)| ds \leq \int_{a_{j_k}}^{B} |x'(s)| ds \leq \sum_{\{j \in I : b_j > b_{j_k} - 1\}} \int_{(a_{j_{k-1}}, b_{j_k})} |x'(s)| ds$$

it follows that $x(a_{j_k}) \to x(t^*)$, while from

$$\sum_{\{j \in I : b_j > b_{j_k}\}} (b_j - a_j) < B - b_{j_k}$$

and

$$\tau^* = \sum_{j \in I} (b_j - a_j) = \tau(b_{j_k}) + \sum_{j \in I, b_j \geq b_{j_k}} (b_j - a_j)$$

we obtain that $\tau(b_{j_k}) \to \tau^*$. By the previous point b) we have that $\bar{x}(\tau(b_{j_k})) = x(a_{j_k})$ and by continuity we infer that $x(t^*) = \bar{x}(\tau^*)$. \hfill \Box
In what follows we shall consider the following minimum time problem for solutions to a differential inclusion: $X$ and $S$ are closed subset of $\mathbb{R}^N$, $S \subset X$, $x^0 \in X$ and $F$ is a set valued map. Consider the problem of reaching the target set $S$ from $x^0$, satisfying the constraint $x(t) \in X$, and $x(.)$ is a solution to the differential inclusion

$$x'(t) \in F(x(t)).$$

**Theorem 6.2.** Let $X \subset \mathbb{R}^N$ be closed and let $F$ be an upper semicontinuous set-valued map defined on $X$ with compact non empty images, with the additional property that, for every $x \in X$, for every $\xi \in \text{co}F(x)$, with $\xi \neq 0$, there exists $\lambda \geq 1$ such that $\lambda \xi \in F(x)$. Assume that there exists $\hat{t} > 0$ and a solution $x$ to

$$x'(t) \in \text{co}F(x(t)), \quad x(0) = x^0 \quad (6.1)$$

such that $x(t) \in X$ for every $t \in [0, \hat{t}]$ and that $x(\hat{t}) \in S$. Then the minimum time problem for

$$x'(t) \in F(x(t))$$

has a solution.

**Proof.** a) Let $t^* = \inf\{t : A_{t^*}^0(t) \cap S \neq \emptyset\}$, where $A_{t^*}^0(t)$ is the attainable set at $t$ of the convexified problem (6.1). Let $(t_n)$ be decreasing to $t^*$ and let $x_n$ be solutions to the differential inclusion

$$x'(t) \in \text{co}F(x(t))$$

such that $x_n(0) = x^0$, $x_n(t_n) \in Sx_n(t) \in X$ for $t \in [0, t_n]$. A subsequence of $(x_n)$ converges uniformly to a function $x_*$. Clearly, $x_*(0) = x^0$, $x_*(t^*) \in S$ and $x_*(t) \in X$ for $t \in [0, t^*]$. It is known that, under the assumptions of the Theorem, $x_*$ is again a solution to

$$x'(t) \in \text{co}F(x(t)).$$

Hence, $x_*$ is a solution to the convexified problem that reaches $S$ in minimum time, and $t^*$ is the value of the minimum time for the convexified problem.

b) Applying Lemma 1, we infer that the set $Z = \{t \in [0, t^*] : x'_*(t) = 0\}$ has measure zero. In fact, otherwise, we could define a different solution to the convexified differential inclusion, defined on an interval $[0, \tau^*]$ with $\tau^* < t^*$, having the same initial and final point: hence $t^*$ would not be the value of the minimum time for the convexified problem.

c) By the previous point and the assumption on $F(x)$, for almost every $t$ there exists a non empty set $\Lambda(t)$ such that $\lambda \in \Lambda(t)$ implies $\lambda x'_*(t) \in F(x_*(t))$ and $\lambda \geq 1$. Reasoning as in [12], we claim that the set valued map $\Lambda(.)$ is measurable on $[0, t^*]$. Write $[0, t^*] \setminus Z$ as a countable union of the sets $M_n = \{t : \|x'_*(t)\| \geq 1/n\}$. On each $M_n$, $\Lambda$ has an upper bound $H_n$, since $F$ is bounded. Applying Lusin’s theorem to $x'_*$, write $M_n$ as $(\cup K_i) \cup N$, where the measure of $N$ is zero, each $K_i$ is compact and the restriction to $K_i$ of $x'_*$ is continuous on $K_i$. Then, by the continuity of $x'_*$ and of $x_*$ and the upper semicontinuity of $F$, it follows that the map $\Lambda$ has closed graph.
on $K_1 \times \mathbb{R}^N$; since, in addition, its values are closed subsets of $[0, H_n]$, it is upper semicontinuous. It follows then that $\Lambda$ is measurable on $[0, t^*]$.

Hence, by standard arguments, that there exists a measurable selection $\lambda(.)$ from $\Lambda(.)$.

Define the absolutely continuous map $s$ by $s(0) = 0$ and $s'(t) = 1/\lambda(t)$: $s$ is an increasing map and maps $[0, t^*]$ onto $[0, s^*]$, where $s^* \leq t^*$. Let $t = t(s)$ be its inverse and consider the map $\tilde{x}(s) = x_s(t(s))$. We obtain in particular that $\tilde{x}(0) = x_s(0)$ and that $\tilde{x}(s^*) = x_s(t(s^*)) = s(t^*)$. We also have

$$\frac{d}{ds}(\tilde{x}(s)) = x'_s(t(s))t'(s) = x'_s(t(s)) \frac{1}{s'(t(s))} = x'_s(t(s))\lambda_1(t(s)) \in F(x_s(t(s))) = F(\tilde{x}(s)).$$

Hence we have obtained that $\tilde{x}$ is a solution to the original differential inclusion but also a solution to the minimum time problem for (6.1). Since every solution to the original problem is also a solution to the convexified problem, the infimum of the times needed to reach $S$ along the solutions to the original problem cannot be lesser than the minimum time for the convexified problem. Hence $\tilde{x}$ is a solution to the minimum time problem for the original differential inclusion.

The following result is on the existence of solutions to initial value problems for differential inclusions with non-convex right hand side.

**Theorem 6.3.** Let $\Omega$ be open, $x^0 \in \Omega$ and let $F$ be as in Theorem 6.2. Assume that there exists $t^* > 0$ such that, on $[0, t^*]$, the Cauchy Problem

$$x'(t) \in \text{co}F(x(t)) \quad x(0) = x^0$$

admits a solution $x \neq x^0$. Then the Cauchy Problem

$$x'(t) \in F(x(t)) \quad x(0) = x^0$$

admits a solution on some interval $[0, \tau^*]$

**Proof.** Since $x \neq x^0$, there exists $t^1 \in [0, t^*]$ such that $x(t^1) \neq x^0$. Consider the minimum time problem for the convexified inclusion, with target set $S = \{x(t^1)\}$. This problem has a solution with minimum time $\tau^*$, where $0 < \tau^* \leq t^1$. By Theorem 6.2, the original non-convexified problem has a solution on $[0, \tau^*]$. 

### 6.2 The Brachystocrone

The scheme commonly used to attack the minimum time problem raised by Bernoulli is to transform it into the problem of minimizing

$$\int \frac{\sqrt{1 + (\xi'_2)^2}}{\sqrt{-\xi_2}} dt$$

...
then to apply necessary conditions to find candidates for the solution; finally to show that the family of such candidates covers the plane, hence that there is one that passes through the final point. This approach, besides being very indirect, is somewhat unsatisfactory, mainly because the integrand is not defined at the initial condition (0, 0) and the necessary conditions are applied without a previous proof of existence of solutions. Instead, Bernoulli’s problem can be stated as follows: in the plane, an initial condition \((\xi_1^0, \xi_2^0)\) is given; consider all the possible oriented rectifiable curves passing through it. Each such curve is defined by assigning \((u_1(\cdot), u_2(\cdot))\), a unit vector describing the direction of its (oriented) tangent. In this way, the parameter \(t\) is the arc-length parametrization of the curve.

The system of equations

\[
\begin{align*}
\xi'_1 &= u_1 \xi_3 \\
\xi'_2 &= u_2 \xi_3 \\
\xi'_3 &= -gu_2
\end{align*}
\]

(6.2)

where the maps \(u_1(\cdot)\) and \(u_2(\cdot)\) are measurable and \(u_1^2(t) + u_2^2(t) = 1\) a.e., describes the motion of a body in the plane, defined by the coordinates \((\xi_1(t), \xi_2(t))\), with (scalar) velocity \(\xi_3(t) = \sqrt{\xi_1'(t)^2 + \xi_2'(t)^2}\), along a curve identified assigning the direction of its tangent vector \((u_1(\cdot), u_2(\cdot))\), subject to the gravity \(g\).

Hence, the problem raised by Bernoulli can be stated as the following minimum time control problem:

The Brachystocrone Minimum Time Problem: find a solution to the control system

\[
\begin{align*}
\xi'_1 &= u_1 \xi_3 \\
\xi'_2 &= u_2 \xi_3 \\
\xi'_3 &= -gu_2
\end{align*}
\]

\((\xi_1(0), \xi_2(0), \xi_3(0)) = (\xi_1^0, \xi_2^0, 0)\), subject to the constraint \(\xi_3 \geq 0\), with control set

\[U = \{(u_1, u_2) : u_1^2 + u_2^2 = 1\},\]

that would reach the target set \(S = (\xi_1^f, \xi_2^f, \mathbb{R}^+)\), where \(\xi_2^f \leq \xi_2^0\), in minimum time.

**Theorem 6.4.** The Brachystocrone Minimum Time Problem admits a solution.

**Proof.** Let \(x = (\xi_1, \xi_2, \xi_3)\), \(X = \{\xi_3 \geq 0\}\), \(S = (\xi_1^f, \xi_2^f, \mathbb{R}^+)\); let \(f(x, u)\) be the right hand side of (6.2) and set \(F(x) = f(x, U)\). Then, as one can check, \(F(x)\) satisfies the assumptions of Theorem 6.2. Hence it is enough to show that there are solutions \(x(t)\) issuing from \(x^0 = (0, 0, 0)\) with \(\xi_3(t) \geq 0\) and such that, at some finite time \(t^*\), \(\xi_1(t^*) = \xi_1^f, \xi_2(t^*) = \xi_2^f\).

This is so in the case where \(\xi_2^f < 0\). In fact, in this case we have that, by choosing the constant control

\[
\begin{align*}
u_1 &= \frac{\xi_1^f}{\sqrt{(\xi_1^f)^2 + (\xi_2^f)^2}} \\
u_2 &= \frac{\xi_2^f}{\sqrt{(\xi_1^f)^2 + (\xi_2^f)^2}}
\end{align*}
\]

we obtain that

\[
\begin{align*}
\xi_1(t) &= u_1(-gu_2^2) \\
\xi_2(t) &= u_2(-gu_2^2)
\end{align*}
\]
so that \( \xi_1(t^f) = \xi_1^f \) and \( \xi_2(t^f) = \xi_2^f \) for \( t^f = \sqrt{\frac{2}{g}} \sqrt{\frac{(\xi_1^f)^2 + (\xi_1^f)^2}{-\xi_2^f}} \). We also obtain that \( \xi_3(t) > 0 \) on \((0, t^f)\) and that \( \xi_3(t^f) = \sqrt{-2g\xi_2^f} \).

Consider the case \( \xi_2^f = 0 \). We can assume \( \xi_1^f \neq 0 \), otherwise \( t^* = 0 \) is the solution to the minimum time problem. In the case \( \xi_1^f > 0 \), consider the solution with the constant control \( u_1 = \frac{1}{\sqrt{2}}, u_2 = -\frac{1}{\sqrt{2}} \) on \([0, \sqrt{2\xi_1^f/g}]\). At time \( t = \sqrt{2\xi_1^f/g} \), we have that \( \xi_1(\sqrt{2\xi_1^f/g}) = \frac{\xi_1^f}{2}, \xi_2(\sqrt{2\xi_1^f/g}) = -\frac{\xi_1^f}{2} \) and \( \xi_3(\sqrt{2\xi_1^f/g}) = \sqrt{g\xi_1^f} \). The solution with constant control \( u_1 = \frac{1}{\sqrt{2}}, u_2 = -\frac{1}{\sqrt{2}} \) on the interval \((\sqrt{2\xi_1^f/g}, 2\sqrt{2\xi_1^f/g})\), with initial conditions \( \xi_1(\sqrt{2\xi_1^f/g}) = \frac{\xi_1^f}{2}, \xi_2(\sqrt{2\xi_1^f/g}) = -\frac{\xi_1^f}{2} \) and \( \xi_3(\sqrt{2\xi_1^f/g}) = \sqrt{g\xi_1^f} \) is such that at \( t = 2\sqrt{2\xi_1^f/g} \), \( \xi_1(2\sqrt{2\xi_1^f/g}) = \xi_1^f \) and \( \xi_2(2\sqrt{2\xi_1^f/g}) = 0 \). Hence \( t^f = 2\sqrt{2\xi_1^f/g} \) and \( \xi_3(t) > 0 \) on \((0, t^f)\).
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Chapter 7

On Hopf’s Lemma and the Strong Maximum Principle

Let $\Omega \subset \mathbb{R}^N$ be a connected, open and bounded set; we call $\Omega$ regular if for every $z \in \partial \Omega$, there exists a tangent plane, continuously depending on $z$. We say that $\Omega$ satisfies the interior ball condition at $z$ if there exists an open ball $B \subset \Omega$ with $z \in \partial B$. On $\Omega$, consider the operator

$$F(u) = \sum_{i=1}^{N} g_i(u_{x_i}^2)u_{x_i x_i},$$

where $g_i : [0, +\infty) \to [0, +\infty)$ are continuous functions. When $F$ is elliptic, two classical results hold.

**Hopf’s Lemma:**

let $\Omega$ be regular, let $u$ be such that $F(u) \leq 0$ on $\Omega$. Suppose that there exists $z \in \partial \Omega$ such that

$$u(z) < u(x), \quad \text{for all } x \in \Omega.$$

If, in addition, $\Omega$ satisfies the interior ball condition at $z$, we have

$$\frac{\partial u}{\partial \nu}(z) < 0,$$

where $\nu$ is the outer unit normal to $B$ at $z$.

**The Strong Maximum Principle:**

let $u$ be such that $F(u) \leq 0$ on $\Omega$, then if $u$ it attains minimum in $\Omega$, it is a constant.

In 1927 Hopf proved the Strong Maximum Principle in the case of second order elliptic partial differential equations, by applying a comparison technique, see [28]. For the class of quasilinear elliptic problems, many contributions have been given, to extend the validity of the previous results, as in [4, 17, 18, 20, 23, 25, 34, 35, 36, 38, 39, 44]. In the case in equation (7.1) we have $g_i \equiv 1$, for every $i$, then $F(u) = \Delta u$, and we find the classical problem of the Laplacian, see [19, 23]. On the other hand,
when there exists \( i \in \{1, \ldots, N\} \) such that \( g_i \equiv 0 \) on an interval \( I = [0, T] \subset \mathbb{R} \), the Strong Maximum Principle does not hold. Indeed, in this case, it is always possible to define a function \( u \) assuming minimum in \( \Omega \) and such that \( \sum_{i=1}^{N} g_i(u_{x_i}^2)u_{x_i}x_i = 0 \).

For instance, let \( g_N(t) = 0 \), for every \( t \in [0, 2] \). The function

\[
 u(x_1, \ldots, x_N) = \begin{cases} 
 -(x_N^2 - 1)^4 & \text{if } -1 \leq x_N \leq 1 \\
 0 & \text{otherwise}
\end{cases}
\]

satisfies (7.1) in \( \mathbb{R}^N \). We are interested in the case when \( 0 \leq g_i(t) \leq 1 \) and it does not exist \( i \) such that \( g_i \equiv 0 \) on an interval. Since \( g_i \) could assume value zero, the equation (7.1) is non elliptic. The results known so far, for the validity of Hopf’s Lemma and of the Strong Maximum Principle, suggest that, for possibly non elliptic equations, but arising from a functional having rotational symmetry, this validity shall depend only on the behaviour of the functions \( g_i \) near zero, see [9]. In this chapter, we prove, in section 7.2, a sufficient condition for the validity of Hopf’s Lemma and of the Strong Maximum Principle; a remarkable feature of this condition is that it concerns only the behaviour of the function \( g_i(t) \) that goes fastest to zero, as \( t \) goes to zero. Hopf’s lemma and the Strong Maximum Principle are essentially the same result as long as we can build subsolutions whose level lines can have arbitrarily large curvature. This need not be always possible for problems not possessing rotational symmetry. This difficulty will be evident in sections 7.3 and 7.4. In these sections, a more restricted class of equations is considered, namely when all the functions \( g_i \), for \( i = 1, \ldots, N - 1 \), are 1 and only \( g_N \) is allowed to go to zero. In this simpler class of equations we are able to show that the condition

\[
 \lim_{t \to 0^+} \frac{(g_N(t))^{3/2}}{tg_N'(t)} > 0
\]

is at once necessary for the validity of Hopf’s Lemma and sufficient for the validity of the Strong Maximum Principle.

### 7.1 Preliminary results

We impose the following local assumptions.

**Assumptions (L):**

there exists \( \overline{T} > 0 \) such that:

i) on \( [0, \overline{T}] \), for every \( i = 1, \ldots, N - 1 \),

\[
 0 \leq g_N(t) \leq g_i(t) \leq 1;
\]

ii) \( g_N \) is continuous on \( [0, \overline{T}] \); positive and differentiable on \( (0, \overline{T}] \);

iii) on \( (0, \overline{T}] \), the function \( t \to g_N(t) + g_N'(t)t \) is non decreasing.

Notice that, in case ii) above is violated, the Strong Maximum Principle does not hold; and that condition iii) above includes the case of the Laplacian, \( g_i(t) \equiv 1 \);
and, finally, that under these assumptions, $g_i$ could assume value zero at most for $t = 0$. Moreover, we can consider the equation
\[ \sum_{i=1}^{N} g_i(u_{x_i}^{2})u_{x,x_i} = 0 \]
as the Euler-Lagrange equation associated to the functional
\[ J(u) = \int_{\Omega} L(\nabla u) d\Omega = \int_{\Omega} \frac{1}{2} \left( \sum_{i=1}^{N} f_i(u_{x_i}^{2})u_{x_i}^{2} \right) d\Omega, \]
where $L(\nabla u)$ is strictly convex in $\{ (u_{x_1}, \ldots, u_{x_N}) : u_{x_i}^{2} \leq \bar{T}, \text{ for every } i = 1, \ldots, N \}$. Indeed, fix $i$. Let $f_i$ be a solution to the differential equation
\[ g_i(t) = f_i(t) + 5tf_i'(t) + 2t^2f_i''(t), \tag{7.2} \]
for $t \in [0, \bar{T}]$. Since
\[ \frac{\partial^2 L}{\partial u_{x_i}^{2}}(u_{x_i}^{2}) = f_i(u_{x_i}^{2}) + 5u_{x_i}^{2}f_i'(u_{x_i}^{2}) + 2u_{x_i}^{4}f_i''(u_{x_i}^{2}) = g_i(u_{x_i}^{2}), \]
we have that
\[ \text{div} \nabla_{\nabla} L(\nabla u) = \sum_{i=1}^{N} \frac{\partial^2 L}{\partial u_{x_i}^{2}}(u_{x_i}^{2})u_{x,x_i} = \sum_{i=1}^{N} g_i(u_{x_i}^{2})u_{x,x_i}. \]

Moreover, the strict convexity of $L(\nabla u)$ in $\{ (u_{x_1}, \ldots, u_{x_N}) : u_{x_i}^{2} \leq \bar{T}, \text{ for every } i = 1, \ldots, N \}$ follows by the fact that $g_i$ is positive $(0, \bar{T}]$. Since we will need general comparison theorems that depend on the global properties of the solutions, i.e. on their belonging to a Sobolev space, we will need also a growth assumption on $g_i$ (assumption $(G)$) to insure these properties of the solutions.

\textbf{Assumption (G):}
\[ \text{each function } f_i \text{ as defined in (7.2), is bounded and } f_i(u_{x_i}^{2})u_{x_i}^{2} \text{ is strictly convex.} \]

Any function $g_i$ satisfying assumptions $(L)$ on $[0, \bar{T}]$ can be extended so as to satisfy assumption $(G)$ on $[0, +\infty)$. In fact, it is enough to extend $g_i$ to $(\bar{T}, +\infty)$ by setting $g_i(t) = f_i(\bar{T})$, for $t > \bar{T}$.

\textbf{Definition 7.1.} Let $\Omega$ be open, and let $u \in W^{1,2}(\Omega)$. The map $u$ is a weak solution to the equation $F(u) = 0$ if, for every $\eta \in C_{0}^\infty(\Omega),$
\[ \int_{\Omega} \langle \nabla L(\nabla u(x)), \nabla \eta(x) \rangle dx = 0. \]
u is a weak subsolution $(F(u) \geq 0)$ if, for every $\eta \in C_{0}^\infty(\Omega), \eta \geq 0,$
\[ \int_{\Omega} \langle \nabla L(\nabla u(x)), \nabla \eta(x) \rangle dx \leq 0. \]
$u$ is a weak supersolution ($F(u) \leq 0$) if, for every $\eta \in C_0^\infty(\Omega)$, $\eta \geq 0$,

$$\int_\Omega \langle \nabla L(\nabla u(x)), \nabla \eta(x) \rangle dx \geq 0.$$  

We say that a function $w \in W^{1,2}(\Omega)$ is such that $w_{|\partial\Omega} \leq 0$ if $w^+ \in W^{1,2}_0(\Omega)$.

The growth assumption $(G)$ assures that, if $u \in W^{1,2}(\Omega)$, then $\nabla L(\nabla u(x)) \in L^2(\Omega)$. The strict convexity of $L$ implies the following comparison lemma.

**Lemma 7.2.** Let $\Omega$ be a open and bounded set, let $v \in W^{1,2}(\Omega)$ be a subsolution and let $u \in W^{1,2}(\Omega)$ be a supersolution to the equation $F(u) = 0$. If $v_{|\partial\Omega} \leq u_{|\partial\Omega}$, then $v \leq u$ a.e. in $\Omega$.

**Proof.** Let $\eta = (v - u)^+ = \max(0, v - u)$. By assumption, $\eta \in W^{1,2}_0(\omega)$ and $\eta \geq 0$. Suppose that the claim is false, i.e., that the support of $\eta$ has positive measure. By the strict convexity of $L$, we have

$$\int_\omega \langle \nabla L(\nabla v(x) - \nabla u(x)), \nabla \eta(x) \rangle dx =$$

$$\int_{\text{supp} \eta} \langle \nabla L(\nabla v(x) - \nabla u(x)), \nabla v(x) - \nabla u(x) \rangle dx > 0.$$  

However, since $v$ is a subsolution, we have that

$$\int_\omega \langle \nabla L(\nabla v(x)), \nabla \phi(x) \rangle dx \leq 0,$$

for every $\phi \in C_0^\infty(\omega), \phi \geq 0$, and the same is true for $\eta \in W^{1,2}_0(\omega), \eta \geq 0$, so that

$$0 \geq \int_\omega \langle \nabla L(\nabla v(x)), \nabla \eta(x) \rangle dx > \int_\omega \langle \nabla L(\nabla u(x)), \nabla \eta(x) \rangle dx \geq 0,$$

a contradiction.  

We wish to express the operator

$$F(v) = \sum_{i=1}^N g_i(v_{x_i}^2)v_{xix_i}$$

in polar coordinates. Set

$$\begin{cases}
  x_1 = \rho \cos \theta_{N-1} \cdots \cos \theta_2 \cos \theta_1 \\
  x_2 = \rho \cos \theta_{N-1} \cdots \cos \theta_2 \sin \theta_1 \\
  \cdots \\
  x_N = \rho \sin \theta_{N-1}
\end{cases}$$

so that

$$v_{x_i} = v_\rho \frac{x_i}{\rho} \quad \text{and} \quad v_{x_ix_i} = v_\rho \frac{x_i^2}{\rho} + v_\rho \left[ 1 - \left( \frac{x_i}{\rho} \right)^2 \right].$$
When \( v \) is a radial function, \( F \) reduces to
\[
F(v) = \sum_{i=1}^{N} g_i \left( \frac{v^2}{\rho} \left( \frac{x_i}{\rho} \right)^2 \right) \left[ v_{pp} \left( \frac{x_i}{\rho} \right)^2 + \frac{v_{\rho}}{\rho} \left( 1 - \left( \frac{x_i}{\rho} \right)^2 \right) \right] =
\]
\[
v_{pp} \sum_{i=1}^{N} g_i \left( \frac{v^2}{\rho} \left( \frac{x_i}{\rho} \right)^2 \right) \left( \frac{x_i}{\rho} \right)^2 + \frac{v_{\rho}}{\rho} \sum_{i=1}^{N} g_i \left( \frac{v^2}{\rho} \left( \frac{x_i}{\rho} \right)^2 \right) \left( 1 - \left( \frac{x_i}{\rho} \right)^2 \right).
\]

In general, we do not expect that the equation \( F(v) = 0 \) admits radial solutions. However we will use the expression of \( F \) valid for radial functions in order to reach our results. The following technical lemmas will be used later.

**Lemma 7.3.** Let \( n = 2, \ldots, N \) and set
\[
h_n(a) = g_N \left( \frac{t(1-a)}{n-1} \right) (1-a) + g_N(ta)a.
\]
For every \( 0 < t \leq \overline{t} \) (\( \overline{t} \) defined in assumptions (L)), \( h_n(a) \geq h_n(1/n) \), for every \( a \in [0,1] \).

**Proof.** Since, on \((0, \overline{t}]\), the function \( t \rightarrow g_N(t) + g_N'(t)t \) is non decreasing, we have that
\[
h'_n(a) = -g_N \left( \frac{t(1-a)}{n-1} \right) - g_N' \left( \frac{t(1-a)}{n-1} \right) \frac{t(1-a)}{n-1} + g_N(ta) + g_N'(ta)ta \geq 0
\]
if and only if \( a \geq 1/n \), so that \( h_n(a) \geq h_n(1/n) \), for every \( a \in [0,1] \).

**Lemma 7.4.** For every \( 0 < t \leq \overline{t} \) (\( \overline{t} \) defined in assumptions (L)), we have that
\[
\sum_{i=1}^{N} g_N \left( t \left( \frac{x_i}{\rho} \right)^2 \right) \left( \frac{x_i}{\rho} \right)^2 \geq g_N \left( \frac{t}{N} \right).
\]

**Proof.** We prove the claim by induction on \( N \). Let \( N = 2 \). Set \( a = \sin^2 \theta_1 \). Applying Lemma 7.3 we obtain that
\[
g_N \left( t \left( \frac{x_1}{\rho} \right)^2 \right) \left( \frac{x_1}{\rho} \right)^2 + g_N \left( t \left( \frac{x_2}{\rho} \right)^2 \right) \left( \frac{x_2}{\rho} \right)^2 =
\]
\[
g_N(t(1-a))(1-a) + g_N(ta)a \geq g_N \left( \frac{t}{2} \right).
\]
Suppose that the claim is true for \( N - 1 \), i.e.
\[
\sum_{i=1}^{N-1} g_N \left( t \left( \frac{x_i}{\rho} \right)^2 \right) \left( \frac{x_i}{\rho} \right)^2 \geq g_N \left( \frac{t}{N-1} \right).
\]
Let us prove it for $N$. Set
\[
\begin{align*}
y_1 &= \rho \cos \theta_{N-2} \ldots \cos \theta_2 \cos \theta_1 \\
y_2 &= \rho \cos \theta_{N-2} \ldots \cos \theta_2 \sin \theta_1 \\
& \quad \vdots \\
y_{N-1} &= \rho \sin \theta_{N-2}
\end{align*}
\]
and set $a = \sin^2 \theta_{N-1}$. Applying Lemma 7.3 we obtain that
\[
\sum_{i=1}^{N} g_N \left( t \left( \frac{x_i}{\rho} \right)^2 \right) \left( \frac{x_i}{\rho} \right)^2 =
\]
\[
\sum_{i=1}^{N-1} g_N \left( t \left( \frac{y_i}{\rho} \right)^2 \right) (1-a) \left( \frac{y_i}{\rho} \right)^2 (1-a) + g_N(ta) a \geq
\]
\[
g_N \left( t \left( \frac{N}{N-1} \right) \right) (1-a) + g_N(ta) a \geq g_N \left( t \frac{N}{N} \right),
\]
and the claim is proved.

\[\Box\]

### 7.2 A sufficient condition for the validity of Hopf’s Lemma and of the Strong Maximum Principle

Consider the improper Riemann integral
\[
\int_0^\xi \frac{g_N(\zeta^2/N)}{\zeta} d\zeta = \lim_{\xi \to 0} \int_0^\xi \frac{g_N(\zeta^2/N)}{\zeta} d\zeta
\]
as an extended valued function $G$,
\[
G(\xi) = \int_0^\xi \frac{g_N(\zeta^2/N)}{\zeta} d\zeta,
\]
where we mean that $G(\xi) \equiv +\infty$ whenever the integral diverges.

We wish to prove the following lemma.

**Lemma 7.5 (Hopf’s Lemma).** Let $\Omega \subset \mathbb{R}^N$ be a connected, open and bounded set. Let $u \in W^{1,2}(\Omega) \ intersection C^1(\overline{\Omega})$ be a weak solution to
\[
\sum_{i=1}^{N} g_i(u_x^2) u_{x_i, x_i} \leq 0.
\]

In addition to the assumptions (L) and (G) on $g_i$, assume that $G(\xi) \equiv +\infty$. Suppose that there exists $z \in \partial \Omega$ such that
\[
u(z) < u(x), \quad \text{for all } x \ in \ \Omega
\]
and that $\Omega$ satisfies the interior ball condition at $z$. Then

$$\frac{\partial u}{\partial \nu}(z) < 0,$$

where $\nu$ is the outer unit normal to $B$ at $z$.

As an example of an equation satisfying the assumptions of the theorem above, consider the Laplace equation $\Delta u = 0$. The functions $g_i \equiv 1$ satisfy the assumptions $(L)$ and $(G)$, and

$$G(\xi) = \int_0^\xi \frac{1}{\zeta} d\zeta = +\infty.$$  

Another example is obtained setting

$$g_N(t) = \frac{1}{|\ln(t)|}$$

for $0 \leq t \leq 1/e$. The assumptions $(L)$ and $(G)$ are satisfied; moreover, for $0 \leq \xi^2/N \leq 1/e$,

$$G(\xi) = \int_0^\xi \frac{d\zeta}{\zeta |\ln(\xi^2/N)|} = +\infty.$$  

Proof of Lemma 7.5. a) Assume that $u(z) = 0$ and that $B = B(O, r)$. We prove the claim by contradiction. Suppose that

$$\frac{\partial u}{\partial \nu}(z) \geq 0,$$

where $\nu$ is the outer unit normal to $B$ at $z$. Let $\epsilon = \min\{u(x) : x \in B(O, r/2)\}$; we have that $\epsilon > 0$. Set

$$\omega = B(O, r) \setminus B(O, r/2).$$

b) We seek a radial function $v \in W^{1,2}(\omega) \cap C(\overline{\omega})$ satisfying

$$\begin{cases}
  v \text{ is a weak solution to } F(v) \geq 0 \quad &\text{in } \omega \\
  v > 0 \quad &\text{in } \omega \\
  v = 0 \quad &\text{in } \partial B(O, r) \\
  v \leq \epsilon \quad &\text{in } \partial B(O, r/2) \\
  v_{\nu}(z) < 0.
\end{cases}$$

Consider the Cauchy problem

$$\begin{cases}
  \zeta' = -\frac{N - 1}{\rho g_N(\xi^2/N)} \zeta \\
  \zeta(r/2) = -\frac{\epsilon}{r}.
\end{cases}$$
There exists a unique local solution $\zeta$ of (7.4), such that 
\[
\int_{-\frac{1}{2}}^{\zeta(\rho)} \frac{g_N(\zeta^2/N)}{\zeta} d\zeta = \int_{r/2}^{\rho} \frac{N-1}{s} ds = -(N-1) \ln \left( \frac{2\rho}{r} \right).
\]
We claim that $\zeta$ is defined in $[r/2, +\infty)$. Indeed, suppose that $\zeta$ is defined in $[r/2, \tau)$, with $\tau < +\infty$. Since $\zeta' > 0$, $\zeta$ is an increasing function, so that $\tau < +\infty$ if and only if $\lim_{\rho \to \tau} \zeta(\rho) = 0$. But 
\[
-\infty = \lim_{\rho \to \tau} \int_{-\frac{1}{2}}^{\zeta(\rho)} \frac{g_N(\zeta^2/N)}{\zeta} d\zeta = \lim_{\rho \to \tau} -(N-1) \ln \left( \frac{2\rho}{r} \right),
\]
a contradiction. Hence, the solution $\zeta$ of (7.4) is defined in $[r/2, +\infty)$. Setting $v_\rho = \zeta$, since, for every $\rho \in (r/2, 1)$,
\[
-\frac{\epsilon}{r} < v_\rho(\rho) < 0,
\]
we have that the function 
\[
v(\rho) = \int_{r}^{\rho} v_\rho(s) ds
\]
solves the problem
\[
v_{pp} g_N \left( \frac{v_\rho^2}{N} \right) + \frac{v_\rho}{\rho} (N-1) = 0,
\]
in particular, $v(\rho) > 0$ and $v_\rho(\rho) < 0$, for every $\rho \in (r/2, 1)$, $v(1) = 0$ and $v(r/2) \leq \epsilon$. Since $v_{pp} \geq 0$ and $-\sqrt{\pi} \leq v_\rho \leq 0$, for every $\rho \in (r/2, 1)$, by the hypotheses on $g_i$ and by Lemma 7.4, we have that 
\[
F(v) = v_{pp} \sum_{i=1}^{N} g_i \left( v_\rho^2 \left( \frac{x_i}{\rho} \right) \right) \left( \frac{x_i}{\rho} \right)^2 + \frac{v_\rho}{\rho} \sum_{i=1}^{N} g_i \left( v_\rho^2 \left( \frac{x_i}{\rho} \right) \right) \left( 1 - \left( \frac{x_i}{\rho} \right)^2 \right) \geq
\]
\[
v_{pp} \sum_{i=1}^{N} g_N \left( v_\rho^2 \left( \frac{x_i}{\rho} \right) \right) \left( \frac{x_i}{\rho} \right)^2 + \frac{v_\rho}{\rho} \sum_{i=1}^{N} \left( 1 - \left( \frac{x_i}{\rho} \right)^2 \right)\]
\[
v_{pp} \sum_{i=1}^{N} g_N \left( v_\rho^2 \left( \frac{x_i}{\rho} \right) \right) \left( \frac{x_i}{\rho} \right)^2 + \frac{v_\rho}{\rho} (N-1) \geq v_{pp} g_N \left( \frac{v_\rho^2}{N} \right) + \frac{v_\rho}{\rho} (N-1).
\]
The function $v$ solves (7.3), indeed, $v$ is in $C^2(\overline{\omega})$ and it is such that $F(v) \geq 0$ and $v > 0$ in $\omega$, $v(r) = 0$, $v(r/2) \leq \epsilon$ and $v_\rho(z) < 0$.

\noindent c) Since $u, v \in W^{1,2}(\omega) \cap C(\overline{\omega})$, $v$ is a weak subsolution and $u$ is a weak solution to $F(u) = 0$, and $v_{|\partial \omega} \leq u_{|\partial \omega}$, applying Lemma 7.2, we obtain that $v \leq u$ in $\omega$. From 
\[
v_\rho(z) = \frac{\partial v}{\partial v}(z) < \frac{\partial u}{\partial v}(z),
\]
it follows that there exists $x^0 \in \omega$ such that $v(x^0) > u(x^0)$, a contradiction. \hfill \Box
From Hopf’s Lemma we derive:

**Theorem 7.6 (Strong Maximum Principle).** Let $\Omega \subset \mathbb{R}^N$ be a connected, open and bounded set. Let $u \in W^{1,2}(\Omega) \cap C(\overline{\Omega})$ be a weak supersolution to

$$
\sum_{i=1}^{N} g_i(u_{x_i}^2)u_{x_i x_i} = 0.
$$

In addition to the assumptions $(L)$ and $(G)$ on $g_i$, assume that $G(\xi) \equiv +\infty$. Then, if $u$ attains its minimum in $\Omega$, it is a constant.

**Proof.** a) Assume $\min_{\Omega} u = 0$ and set $C = \{x \in \Omega : u(x) = 0\}$. By contradiction, suppose that the open set $\Omega \setminus C \neq \emptyset$.

b) Since $\Omega$ is a connected set, there exist $s \in C$ and $R > 0$ such that $B(s, R) \subset \Omega$ and $B(s, R) \cap (\Omega \setminus C) \neq \emptyset$. Let $p \in B(s, R) \cap (\Omega \setminus C)$. Consider the line $\overline{ps}$. Moving $p$ along this line, we can assume that $B(p, d(p, C)) \subset (\Omega \setminus C)$ and that there exists one point $z \in C$ such that $d(p, C) = d(p, z)$. Set $r = d(p, C)$. W.l.o.g. suppose that $p = O$.

c) The set $\Omega \setminus C$ satisfies the interior ball condition at $z$, hence Hopf’s Lemma implies

$$
\frac{\partial u}{\partial \nu}(z) < 0.
$$

But this is a contradiction: since $u$ attains minimum at $z \in \Omega$, we have that $Du(z) = 0$.

\[\square\]

### 7.3 A necessary condition for the validity of Hopf’s Lemma

In this and the following section we consider the operator

$$
F(u) = \sum_{i=1}^{N-1} u_{x_i x_i} + g(u_{x_N}^2)u_{x_N x_N},
$$

(7.5)

We wish to provide a necessary condition for the validity of Hopf’s Lemma in a class of non elliptic equations. Consider the case

$$
G(\xi) = \int_0^\xi \frac{g(\zeta^2/N)}{\zeta} d\zeta < +\infty.
$$
Theorem 7.7. Consider the operator (7.5), where \( g \) satisfies assumptions \((L)\) and \((G)\), and on \((0, \bar{T}]\) (\(\bar{T}\) defined in assumptions \((L)\)),
\[
g'(t) > 0 \quad \text{and} \quad g(t) + g'(t)t \leq 1.
\]
If
\[
\lim_{t \to 0^+} \frac{(g(t))^{3/2}}{tg'(t)} = 0,
\]
then there exist: an open regular region \( \Omega \subset \mathbb{R}^N \); a radial function \( u \in C^2(\Omega) \) such that \( F(u) \leq 0 \) in \( \Omega \) and a point \( z \in \partial \Omega \) such that \( u(z) = 0, u(z) \leq u(x) \) for every \( x \in \Omega \) and
\[
\frac{\partial u}{\partial \vec{n}}(z) = 0
\]
where \( \vec{n} \) is the outer unit normal to \( \Omega \) at \( z \).
If, in addition, we assume that
\[
\frac{g(t)}{tg'(t)} \quad \text{is bounded in} \ (0, \bar{T}]
\]
then \( \Omega \) satisfies the interior ball condition at \( z \).

Remark 7.8. When \( \lim_{t \to 0^+} g'(t)t \) exists, it follows that
\[
\lim_{t \to 0^+} \frac{(g(t))^{3/2}}{tg'(t)}
\]
exists, and that
\[
g(t) + g'(t)t \leq 1, \ \text{on} \ (0, \bar{T}].
\]
Indeed, we have that
\[
\lim_{t \to 0^+} (g(t) + g'(t)t) = 0.
\]
Otherwise, there exists \( K > 0 \) such that, when \( 0 < t \leq \bar{T}, g'(t)t \geq K \), so that
\[
g(t)t = \int_0^t (g(s)+g'(s)s) \, ds \geq Kt,
\]
and \( g(t) \geq K \). From
\[
\int_0^\epsilon \frac{g(t)}{t} \, dt < +\infty,
\]
it follows that \( \lim_{t \to 0^+} g(t) = 0 \), a contradiction.
The map
\[
g(t) = \frac{1}{|\ln(t)|^k},
\]
with \( k > 2 \), for \( 0 \leq t \leq 1/e \), satisfies the assumption
\[
\lim_{t \to 0^+} \frac{(g(t))^{3/2}}{tg'(t)} = 0.
\]
The following lemma is instrumental to the proofs of the main results.
Lemma 7.9. Let $g$ satisfies assumptions (L) and (G). Suppose that for every $0 < t \leq \overline{t}$,

$$g'(t) \geq 0 \quad \text{and} \quad g(t) + g'(t)t \leq 1.$$  

Set

$$k_1(a) = (1 - a) + ag(ta) \quad \text{and} \quad k_2(a) = -a - (1 - a)g(ta)$$

For every $0 < t \leq \overline{t}$ ($\overline{t}$ defined in assumptions (L)), $k_1$ and $k_2$ are non increasing in $[0, 1]$.

Proof. Since, for every $0 < t \leq \overline{t}$,

$$g'(t) \geq 0 \quad \text{and} \quad g(t) + g'(t)t \leq 1,$$

we have that, for every $0 \leq a \leq 1$,

$$k_1'(a) = -1 + g(ta) + g'(ta)ta \leq 0$$

and

$$k_2'(a) = -1 + g(ta) - (1 - a)g'(ta)t = -1 + g(ta) + g'(ta)ta - g'(ta)t \leq 0.$$  

Proof of Theorem 7.7. a) Let $v$ be a radial function. Setting $a = \sin^2 \theta_{N-1}$, (7.5) reduces to

$$F(v) = v_{\rho \rho} \left(1 - a + ag(v^2_{\rho\rho})\right) + \frac{v_\rho}{\rho} \left(N - 2 + a + (1 - a)g(v^2_{\rho\rho})\right).$$

Let $a = 1$, we seek a solution to

$$v_{\rho \rho}g(v^2_\rho) + (N - 1)\frac{v_\rho}{\rho} = 0$$  

(7.7)

such that $v_\rho(R(1) + 1) = 0$ and $v_\rho(\rho) < 0$, for every $\rho \in [2, R(1) + 1)$. Consider the Cauchy problem

$$\begin{cases}
\zeta' = -\frac{N - 1}{\rho} \frac{\zeta}{g(\zeta^2)} \\
\zeta(2) = -1.
\end{cases}$$  

(7.8)

We are interested in a negative solution $\zeta$. Define $R(1)$ to be the unique positive real solution to

$$G(-1) - (N - 1)\ln \left(\frac{R(1) + 1}{2}\right) = 0,$$

i.e.

$$R(1) = 2^{\frac{G(-1)}{N - 1}} - 1.$$  

The solution $\zeta$ of (7.8), satisfies

$$G(\zeta(\rho)) - G(-1) = \int_{\zeta(2)}^{\zeta(\rho)} \frac{g(t^2)}{t} dt = \int_{2}^{\rho} -\frac{N - 1}{s} ds = -(N - 1)\ln \left(\frac{\rho}{2}\right).$$
Then, for every \( \rho \in (2, R(1) + 1) \), \( G(\zeta(\rho)) > 0 \) and \( \zeta(\rho) < 0 \), while \( \zeta(R(1) + 1) = 0 \). Setting \( v_\rho(\rho) = \zeta(\rho) \) and

\[
v(\rho) = \int_{R(1)+1}^{\rho} v_\rho(s)ds,
\]

we obtain that \( v \) solves (7.7) and, for every \( \rho \in (2, R(1) + 1) \),

\[
v_\rho(\rho) < v_\rho(R(1) + 1) = 0 \quad \text{and} \quad v(\rho) > v(R(1) + 1) = 0.
\]

b) Set, for \( \rho \in (1, R(1)] \), \( u(\rho) = v(\rho + 1) \). Since, for the function \( v \), we have

\[
v_{\rho\rho}(\rho)g(v_\rho^2(\rho)) + (N - 1)\frac{v_\rho(\rho)}{\rho} = 0,
\]

at \( \rho + 1 \) we obtain

\[
u_{\rho\rho}(\rho)g(u_\rho^2(\rho)) + (N - 1)\frac{u_\rho(\rho)}{\rho + 1} = 0. \quad (7.9)
\]

This equality yields, for \( \rho \in (1, R(1)) \),

\[
u_{\rho\rho}(u_\rho^2) + (N - 1)\frac{u_\rho}{\rho} = -(N - 1)u_\rho \left( \frac{1}{\rho + 1} - \frac{1}{\rho} \right) < 0.
\]

c) Let \( 1/2 < a < 1 \). We wish to find \( R(a) \leq R(1) \) such that \( u \) is a solution to

\[
F(u) = u_{\rho\rho} \left( 1 - a + ag(u_\rho^2 a) \right) + \frac{u_\rho}{\rho} \left( N - 2 + a + (1 - a)g(u_\rho^2 a) \right) \leq 0,
\]

for \( \rho \in (1, R(a)) \). Since

\[
F(u) = \frac{-u_\rho}{\rho(\rho + 1)g((u_\rho)^2)} \left[ \rho(N - 1) \left( 1 - a + ag((u_\rho)^2 a) \right) - \right.
\]

\[
\left. (\rho + 1)g((u_\rho)^2) \left( N - 2 + a + (1 - a)g((u_\rho)^2 a) \right) - \right],
\]

setting

\[
k(\rho) = \rho(N - 1) \left( 1 - a + ag((u_\rho)^2 a) \right) - (\rho + 1)g((u_\rho)^2) \left( N - 2 + a + (1 - a)g((u_\rho)^2 a) \right),
\]

we have that \( F(u) \leq 0 \) if and only if \( k(\rho) \leq 0 \). From

\[
\frac{d}{d\rho}g((u_\rho)^2) \leq \frac{d}{d\rho}g((u_\rho)^2 a) \leq 0,
\]

and \( N - 2 + a \geq (N - 1)a \) for \( N \geq 2 \), applying Lemma 7.9, we have that

\[
\begin{align*}
k'(\rho) &= (N - 1)(1 - a + ag((u_\rho)^2 a) - g((u_\rho)^2)(N - 2 + a + (1 - a)g((u_\rho)^2 a) + \\
&\rho(N - 1)a\frac{d}{d\rho}g((u_\rho)^2) - (\rho + 1)\frac{d}{d\rho}g((u_\rho)^2)(N - 2 + a + (1 - a)g((u_\rho)^2) -
\end{align*}
\]
7.3 A necessary condition for the validity of Hopf’s Lemma

\[(\rho + 1)g((u_\rho)^2)(1 - a)\frac{d}{d\rho}g((u_\rho)^2)\geq \]

\[\rho \left((N - 1)a \frac{d}{d\rho}g((u_\rho)^2)a - \frac{d}{d\rho}g((u_\rho)^2)(N - 2 + a + (1 - a)g((u_\rho)^2)a)\right) \geq \]

\[g((u_\rho)^2)(1 - a)\frac{d}{d\rho}g((u_\rho)^2)a \geq \]

\[\rho(N - 1)a \left(\frac{d}{d\rho}g((u_\rho)^2a) - \frac{d}{d\rho}g((u_\rho)^2)\right) \geq 0.\]

Since the function \(k(\rho)\) is non decreasing, it follows that \(F(u) \leq 0\), for every \(\rho \in (1, R(a))\), if and only if \(k(R(a)) \leq 0\).

We have that

\[k(R(a)) = R(a)(N - 1) \left(1 - a + ag((u_\rho(R(a)))^2)a)\right) - \]

\[(R(a) + 1)g((u_\rho(R(a)))^2) \left(N - 2 + a + (1 - a)g((u_\rho(R(a)))^2)a\right) \leq \]

\[(N - 1) \left[R(a)(1 - a) - g((u_\rho(R(a)))^2)a\right] \leq \]

\[(N - 1) \left[R(1)(1 - a) - \frac{g((u_\rho(R(a)))^2)}{2}\right].\]

We define \(R(a)\) to be a solution to

\[R(1)(1 - a) - \frac{g((u_\rho(R(a)))^2)}{2} = 0. \quad (7.10)\]

\[d) \text{ In order to solve (7.10) for the unknown } R(a), \text{ recalling that } 1 - a = \cos^2 \theta_{N-1} = c^2, \text{ let} \]

\[h(r) = \sqrt{\frac{g(u_\rho(r))^2}{2R(1)}}.\]

The function \(h\) is decreasing, differentiable and with inverse differentiable. We have that \(|c| = h(R(1 - c^2))\), so that \(R(1 - c^2) = h^{-1}(|c|)\), \(R(1 - c^2)\) is increasing in \(|c|\) and

\[\lim_{c \to 0} R(1 - c^2) = R(1).\]

Let \(0 < |c| < 1/2\) be such that \(R(1 - c^2) \geq 1\), so that, for \(c^2 \leq \bar{c}^2\), we have \(R(1 - c^2) \geq R(1 - \bar{c}^2) \geq 1\). We have obtained that, for every \(1 \geq a \geq 1 - |c|^2\), there exists \(R(a)\) such that (7.10) holds. It follows that

\[k(R(a)) \leq \left[R(1)(1 - a) - \frac{g((u_\rho(R(a)))^2)}{2}\right] = 0,\]
so that the function $u$ solves $F(u) \leq 0$ for every $\rho \in (1, R(a))$.

e) Set $\Omega = \{(x_1, \ldots, x_N) \in \mathbb{R}^N : \rho \in (1, R(1 - c^2))$ and $|c| < |\tilde{c}|\}$. $\Omega \subset \mathbb{R}^N$ is a connected, open and bounded set and $u \in W^{1,2}(\Omega) \cap C(\overline{\Omega})$ is a weak solution to $F(u) \leq 0$. The point $z = (R, 0, \ldots, 0) \in \partial \Omega$ is such that $u(z) < u(x)$, for all $x \in \Omega$. We wish to show that $\Omega$ is regular in a neighborhood of $z = (R, 0, \ldots, 0)$. Since $\frac{d}{dc}R(1 - c^2)$ exists, in $(0, |\tilde{c}|)$, to prove our claim it is sufficient to show that

$$\lim_{c \to 0} \frac{d}{dc}R(1 - c^2) = 0.$$  

Recalling (7.9), we have that

$$\frac{d}{dc}R(1 - c^2) = h^{-1}(c) = \frac{1}{h'(R(1 - c^2))} = -\frac{\sqrt{2}R(1)}{N - 1} \left( R(1 - c^2) + 1 \right) \frac{(g((u_{\rho}(R(1 - c^2)))^2)^{3/2}}{(u_{\rho}(R(1 - c^2)))^2 g'((u_{\rho}(R(1 - c^2)))^2). \tag{7.11}$$

Since

$$\lim_{c \to 0} u_{\rho}(R(1 - c^2)) = 0$$

and

$$\lim_{t \to 0^+} \frac{(g(t))^{3/2}}{tg'(t)} = 0,$$

it follows that

$$\lim_{c \to 0} \frac{d}{dc}R(1 - c^2) = 0.$$

Since $\frac{d}{dt} \cos \theta_{N-1}|_{\theta_{N-1}=\frac{\pi}{2}} = 1$, this shows that $\frac{d}{dt}R(1 - \cos^2 \theta_{N-1}) = 0$, and $\Omega$ is regular.

f) To prove the validity of the interior ball condition at $z = (R, 0, \ldots, 0)$, it is enough to show that the second derivative of $R(1 - c^2)$ is bounded at $c = 0$, i.e. that

$$\left| \frac{1}{c} \frac{d}{dc}R(1 - c^2) \right|$$

is bounded. Set

$$t(c) = (u_{\rho}(R(1 - c^2)))^2,$$

from (7.11) we obtain

$$\frac{d}{dc}R(1 - c^2) = -\frac{\sqrt{2}R(1)}{N - 1} \left( R(1 - c^2) + 1 \right) \frac{(g(t(c)))^{3/2}}{t(c)g'(t(c))},$$

and from (7.9)

$$\frac{dt(c)}{dc} = 2u_{\rho}(R(1 - c^2))u_{\rho\rho}(R(1 - c^2)) \frac{d}{dc}(R(1 - c^2)) = \ldots$$
7.3 A necessary condition for the validity of Hopf’s Lemma

Figure 7.1: $\Omega$ in the case $N = 2$.

\[
2\sqrt{2}R(1) \frac{(g(t(c)))^{1/2}}{g'(t(c))}
\]

and

\[
\frac{d}{dc} (g(t(c)))^{1/2} = \frac{g'(t(c))}{2(g(t(c)))^{1/2}} \frac{dt(c)}{dc} = \sqrt{2}R(1).
\]

From $g(t(0)) = 0$, we obtain that

\[
(g(t(c)))^{1/2} = \sqrt{2}R(1)c
\]

and

\[
\frac{(g(t(c)))^{3/2}}{ct(c)g'(t(c))} = \sqrt{2}R(1) \frac{g(t(c))}{t(c)g'(t(c))}.
\]

From condition (7.6) we obtain

\[
\left| \frac{1}{c} \frac{d}{dc} R(1 - c^2) \right| \leq M.
\]
7.4 A sufficient condition for the validity of the Strong Maximum Principle

Consider the case

\[ G(\xi) = \int_0^\xi \frac{g(\zeta^2/N)}{\zeta} d\zeta < +\infty. \]

We wish to prove the following theorem.

**Theorem 7.10 (The Strong Maximum Principle).** Let \( \Omega \subset \mathbb{R}^N \) be a connected, open and bounded set. Let \( u \in W^{1,2}(\Omega) \cap C(\overline{\Omega}) \) be a weak supersolution to

\[ \sum_{i=1}^{N-1} u_{x_i} + g(u_{x_N}) u_{x_N} = 0, \]

where \( g \) satisfies assumptions (L) and (G) and, on \((0, T] \) (\( T \) defined in assumptions (L)),

\[ g'(t) \geq 0 \quad \text{and} \quad g(t) + g'(t)t \leq 1. \]

Moreover, suppose that there exists \( K > 0 \) such that, for every \( 0 < \xi^2/N \leq T \), we have

\[ \sqrt{g(\xi^2/N)} \leq K (e^{G(\xi)} - 1). \]

Then, if \( u \) attains its minimum in \( \Omega \), it is a constant.

**Remark 7.11.** When the function \( g \) satisfies the condition

\[ g'(t)t \leq 2K (g(t))^{3/2}, \]

for every \( 0 < t \leq T \), then it satisfies

\[ g(t) + g'(t)t \leq 1 \]

and

\[ \sqrt{g(\xi^2/N)} \leq K (e^{G(\xi)} - 1), \]

for every \( 0 < \xi^2/N \leq T \).

Indeed, since \( G(\xi) < +\infty \), we have that \( \lim_{t \to 0} g(t) = 0 \). Hence, we can suppose that \( g(t) \leq 1/(2K + 1) \), for \( 0 < t \leq T \), so that

\[ g(t) + g'(t)t \leq g(t) + 2K (g(t))^{3/2} \leq 1. \]

Moreover, since \( g(0) = 0 \), \( G(0) = 0 \) and

\[ \left( \sqrt{g(\xi^2/N)} \right)' \leq 2K \frac{g(\xi^2/N)}{\xi} \leq K (e^{G(\xi)} - 1)', \]

we obtain that

\[ \sqrt{g(\xi^2/N)} \leq K (e^{G(\xi)} - 1). \]
Remark 7.12. Among the functions \( g \) such that

\[
\lim_{t \to 0^+} \frac{(g(t))^{3/2}}{tg'(t)}
\]

exists, there exists \( K > 0 \) such that, for every \( 0 < \xi^2/N \leq \gamma \), we have

\[
\sqrt{g(\xi^2/N)} \leq K \left(e^{G(\xi)} - 1\right)
\]

if and only if

\[
\lim_{t \to 0^+} \frac{(g(t))^{3/2}}{tg'(t)} > 0.
\]

An example of a map satisfying the assumptions of the theorem above, is given by

\[
g(t) = \frac{1}{(\ln(t))^2},
\]

for \( 0 \leq t \leq 1/e^4 \). For \( 0 \leq \xi^2/N \leq 1/e^4 \), we have

\[
G(\xi) = \int_0^\xi \frac{1}{\zeta(\ln(\xi^2/N))^2} d\zeta = -\frac{1}{2\ln(\xi^2/N)},
\]

and

\[
\sqrt{g(\xi^2/N)} = \frac{1}{|\ln(\xi^2/N)|} \leq 2 \left(e^{2|\ln(\xi^2/N)|} - 1\right) = e^{G(\xi)} - 1.
\]

Set

\[
\mathcal{R}(\lambda, \lambda_N) = \{(x_1, \ldots, x_N) : |x_i| \leq \lambda, \text{ for } i = 1, \ldots, N - 1, |x_N| \leq \lambda_N\}.
\]

To the opposite of the proof of Lemma 7.5 and Theorem 7.6, we will build a subsolution that is not radially symmetric. This construction is provided by next theorem.

**Theorem 7.13.** Under the same assumptions on \( g \) as on Theorem 7.10, for every \( r > 0 \) and every \( \epsilon \), there exist: \( l, l_N \); an open convex region \( \mathcal{A} \subset \mathcal{R}(l, l_N) \); a function \( v \in W^{1,2}(\omega) \cap C^1(\omega) \cap C(\overline{\omega}) \), where \( \omega = B(\mathcal{A}, r) \setminus \mathcal{A} \), such that i) \( 0 \leq l \leq 2Kr \), and \( 0 \leq l_N \leq r/4 \); ii)

\[
\begin{cases}
  v \text{ is a weak solution to } F(v) \geq 0 & \text{in } \omega \\
  v > 0 & \text{in } \omega \\
  v = 0 & \text{in } \partial B(\mathcal{A}, r) \\
  v \leq \epsilon & \text{in } \partial \mathcal{A}.
\end{cases}
\]  

(7.12)

**Proof.** Fix \( r \); we can assume that \( \epsilon \) is such that \( 0 < \epsilon^2/r^2 \leq \gamma \) and that

\[
2 \sqrt{g \left( \frac{\epsilon^2}{r^2} \right)} + \frac{1}{2} \sqrt{g \left( \frac{\epsilon^2}{r^2} \right)} \left| \ln g \left( \frac{\epsilon^2}{r^2} \right) \right| \leq \frac{1}{4K}.
\]
Fix the origin $O^0 = (0, \ldots, 0)$, and set polar coordinates as
\[
\begin{align*}
    x_1 &= \rho \cos \theta_{N-1} \ldots \cos \theta_2 \cos \theta_1 \\
    x_2 &= \rho \cos \theta_{N-1} \ldots \cos \theta_2 \sin \theta_1 \\
    \vdots \\
    x_N &= \rho \sin \theta_{N-1}.
\end{align*}
\]

1) When $w$ is a radial function, setting $a = \sin^2 \theta_1$, $F$ reduces to
\[
F(w) = w_{\rho \rho} \left( 1 - a + ag(w_{\rho}a) \right) + \frac{w_{\rho}^2}{\rho} \left( N - 2 + a + (1 - a)g(w_{\rho}^2a) \right).
\]

For $a = 1$, we seek a solution to
\[
(N - 1)\frac{w_{\rho}}{\rho} + w_{\rho \rho}g(w_{\rho}^2) = 0. \tag{7.13}
\]
such that $w_{\rho}(R(1)) = -\epsilon/r$ and $w_{\rho}(\rho) < 0$, for every $\rho \in [R(1), R(1) + r)$. Consider the Cauchy problem
\[
\begin{align*}
    \zeta' &= -\frac{N - 1}{\rho} \frac{\zeta}{g(\zeta^2)} \\
    \zeta(R(1)) &= -\frac{\epsilon}{r}.
\end{align*} \tag{7.14}
\]
We are interested in a negative solution $\zeta$. Define $R(1)$ to be the unique positive real solution to
\[
G(-\epsilon/r) - (N - 1) \ln \left( \frac{R(1) + r}{R(1)} \right) = 0,
\]
i.e.
\[
R(1) = \frac{r}{\exp(-\epsilon/r)} - 1.
\]
Consider the unique solution $\zeta$ of (7.14), such that $\zeta(R(1)) = -\epsilon/r$, i.e., such that
\[
G(\zeta(\rho)) - G(-\epsilon/r) = \int_{\zeta(R(1))}^{\zeta(\rho)} \frac{g(t^2)}{t} dt = \int_{R(1)}^{\rho} \frac{N - 1}{s} ds = -(N - 1) \ln \left( \frac{\rho}{R(1)} \right).
\]
Then, for every $\rho \in [R(1), R(1)+r)$, $G(\zeta(\rho)) > 0$ and $\zeta(\rho) < 0$, while $\zeta(R(1)+r) = 0$. Setting $w_{\rho}(\rho) = \zeta(\rho)$ and
\[
w(\rho) = \int_{R(1)+r}^{\rho} w_{\rho}(s) ds,
\]
we obtain that $w$ solves (7.13) and, for every $\rho \in (R(1), R(1) + r),$
\[
-\epsilon/r = w_{\rho}(R(1)) < w_{\rho}(\rho) < w_{\rho}(R(1) + r) = 0
\]
and
\[
0 = w(R(1) + r) < w(\rho) < w(R(1)) \leq \epsilon.
\]
2) Applying Lemma 7.9, we infer that the function \( w \) defined in 1) is actually a solution to
\[
F(w) = w_{pp}(1 - a) + \frac{w}{\rho}(N - 2 + a) + g(w^2)\left( w_{pp}a + \frac{w}{\rho}(1 - a) \right) \geq 0,
\]
for every \( 0 \leq a \leq 1 \) and every \( \rho \in (R(1), R(1) + r) \).

3) Let \( a < 1 \). We wish to find the smallest \( R(\bar{a}) > 0 \) such that, setting
\[
w^{\bar{a}}(\rho) = w(\rho - R(\bar{a}) + R(1)),
\]
the function \( w^{\bar{a}} \) is a solution to
\[
F(w^{\bar{a}}) = w_{pp}(1 - \bar{a}) + \frac{w^{\bar{a}}}{\rho}(N - 2 + \bar{a}) + g((w^{\bar{a}})^2)\left( w_{pp}^{\bar{a}}a + \frac{w^{\bar{a}}}{\rho}(1 - \bar{a}) \right) \geq 0, \tag{7.15}
\]
for every \( \rho \in (R(\bar{a}), R(\bar{a}) + r) \).

Since, for the function \( w \), we have
\[
(N - 1)\frac{w(\rho)}{\rho} + g(w^2)w_{pp}(\rho) = 0,
\]
at \( \rho - R(\bar{a}) + R(1) \) we obtain
\[
(N - 1)\frac{w^{\bar{a}}(\rho)}{\rho} + g((w^{\bar{a}})^2)w_{pp}^{\bar{a}}(\rho) = 0.
\]
This equality yields
\[
(N - 2 + \bar{a})\frac{w^{\bar{a}}}{\rho} + \bar{a}g((w^{\bar{a}})^2)w_{pp}^{\bar{a}} =
\]
\[
\frac{w^{\bar{a}}}{\rho(\rho - R(\bar{a}) + R(1))} \left( (N - 2 + \bar{a})(\rho - R(\bar{a}) + R(1)) - \bar{a}\rho(N - 1)g((w^{\bar{a}})^2) \right)
\]
and
\[
(1 - \bar{a}) \left( w_{pp}^{\bar{a}} + \frac{w^{\bar{a}}}{\rho}g((w^{\bar{a}})^2) \right) =
\]
\[
\frac{(1 - \bar{a})w_{pp}^{\bar{a}}}{\rho(\rho - R(\bar{a}) + R(1))} - \frac{(N - 1)\rho}{g((w^{\bar{a}})^2)}.
\]
Since
\[
F(w^{\bar{a}}) = (N - 2 + \bar{a})\frac{w^{\bar{a}}}{\rho} + \bar{a}g((w^{\bar{a}})^2)w_{pp}^{\bar{a}} + (1 - \bar{a}) \left( w_{pp}^{\bar{a}} + \frac{w^{\bar{a}}}{\rho}g((w^{\bar{a}})^2) \right),
\]
we obtain that \( F(w^{\bar{a}}) \geq 0 \) if and only if
\[
(\rho - R(\bar{a}) + R(1)) (N - 2 + \bar{a} + (1 - \bar{a})g((w^{\bar{a}})^2)) -
\]
\[
\frac{(N - 1)p}{g((w_{\rho})^2)} \left(1 - \bar{a} + \bar{a}g((w_{\rho})^2\bar{a})\right) \leq 0.
\]

Set

\[
k'(\rho) = (\rho - R(\bar{a}) + R(1)) \left(N - 2 + \bar{a} + (1 - \bar{a})g((w_{\rho})^2\bar{a})\right) - \frac{(N - 1)p}{g((w_{\rho})^2)} \left(1 - \bar{a} + \bar{a}g((w_{\rho})^2\bar{a})\right)
\]

Since

\[
\frac{d}{d\rho} g((w_{\rho})^2) \leq \frac{d}{d\rho} g((w_{\rho})^2\bar{a}) \leq 0,
\]

applying Lemma 7.9, we have that

\[
k' = \left(N - 2 + \bar{a} + (1 - \bar{a})g((w_{\rho})^2\bar{a})\right) + (\rho - R(\bar{a}) + R(1))(1 - \bar{a}) \frac{d}{d\rho} g((w_{\rho})^2\bar{a}) - \frac{(N - 1)p}{g((w_{\rho})^2)} \frac{d}{d\rho} g((w_{\rho})^2\bar{a}) \leq \frac{(N - 1)p}{g((w_{\rho})^2)} \frac{d}{d\rho} g((w_{\rho})^2) \left(1 - \bar{a} + \bar{a}g((w_{\rho})^2\bar{a})\right) - \frac{(N - 1)p}{g((w_{\rho})^2)} \frac{d}{d\rho} g((w_{\rho})^2\bar{a}) \bar{a} \leq \frac{(N - 1)p}{g((w_{\rho})^2)} \frac{d}{d\rho} g((w_{\rho})^2) \left(1 - \bar{a} + \bar{a}g((w_{\rho})^2\bar{a})\right) - \frac{(N - 1)p}{g((w_{\rho})^2)} \frac{d}{d\rho} g((w_{\rho})^2) \left(1 - \bar{a} + \bar{a}g((w_{\rho})^2\bar{a})\right) \leq \frac{(N - 1)p}{g((w_{\rho})^2)} \frac{d}{d\rho} g((w_{\rho})^2) \left(1 - \bar{a} + \bar{a}g((w_{\rho})^2\bar{a})\right) \leq 0.
\]

Since the function \(k'(\rho)\) is non-increasing, it follows that \(F(w_{\rho}) \geq 0\), for every \(\rho \in (R(\bar{a}), R(\bar{a}) + r)\), if and only if

\[
R(1) \left(N - 2 + \bar{a} + (1 - \bar{a})g((w_{\rho}(R(\bar{a})))^2\bar{a})\right) - \frac{(N - 1)pR(\bar{a})}{g((w_{\rho}(R(\bar{a})))^2)} \left(1 - \bar{a} + \bar{a}g((w_{\rho}(R(\bar{a})))^2\bar{a})\right) = R(1) \left(N - 2 + \bar{a} + (1 - \bar{a})g \left(\frac{\alpha^2}{r^2\bar{a}}\right)\right) - \frac{(N - 1)pR(\bar{a})}{g \left(\frac{\alpha^2}{r^2\bar{a}}\right)} \left(1 - \bar{a} + \bar{a}g \left(\frac{\alpha^2}{r^2\bar{a}}\right)\right) \leq 0,
\]

i.e. if and only if

\[
R(\bar{a}) \geq g \left(\frac{\alpha^2}{r^2}\right) \frac{R(1)}{N - 1} \frac{N - 2 + \bar{a} + g \left(\frac{\alpha^2}{r^2\bar{a}}\right)(1 - \bar{a})}{1 - \bar{a} + g \left(\frac{\alpha^2}{r^2\bar{a}}\right) \bar{a}}.
\]
Hence, we define
\[
R(\bar{a}) = g \left( \frac{e^2}{r^2} \right) \frac{R(1)}{N-1} \frac{N-2 + \bar{a} + g \left( \frac{e^2}{r^2} \bar{a} \right) (1 - \bar{a})}{1 - \bar{a} + g \left( \frac{e^2}{r^2} \bar{a} \right) \bar{a}}.
\]

4) The function \( w^{\bar{a}} \) defined in point 3) is a solution to
\[
F(w^{\bar{a}}) = w^{\bar{a}}_{pp}(1-a) + \frac{w^{\bar{a}}}{\rho} (N-2+a) + g((w^{\bar{a}}_\rho)^2 a) \left( w^{\bar{a}}_{pp} a + \frac{w^{\bar{a}}}{\rho} (1-a) \right) \geq 0,
\]
for every \( a < \bar{a} \). Indeed, applying Lemma 7.9 we obtain that, for every \( \rho \in (R(\bar{a}), R(\bar{a}) + r) \),
\[
(p - R(\bar{a}) + R(1)) (N - 2 + a + (1 - a) g((w^{\bar{a}}_\rho)^2 a)) - \frac{(N-1)\rho}{g((w^{\bar{a}}_\rho)^2)} (1 - a + a g((w^{\bar{a}}_\rho)^2 a)) \leq 0.
\]

5) Assume we have a partition \( \alpha \) of \([0, \pi/2]\), \( \alpha = \{0 = \alpha_0 < \ldots < \alpha_1 < \alpha_0 = \pi/2\} \). This partition defines two partitions of \([0, 1]\), given by \( c_i = \cos \alpha_i \) and \( s_i = \sin \alpha_i \). Consider the sums
\[
S_1(\alpha) = \sum_{i=1}^{n} (R(1 - c_i^2) - R(1 - c_i^2)) c_i = R(1)c_1 + \sum_{i=1}^{n-1} R(1 - c_i^2) (c_{i+1} - c_i)
\]
and
\[
S_2(\alpha) = \sum_{i=1}^{n} R(c_i^2) (s_{i-1} - s_i) = R(1)(1 - s_1) + \sum_{i=1}^{n-1} R(s_i^2)(s_i - s_{i+1}),
\]
where, in the previous equalities, we have taken into account that \( R(1 - c_i^2) = R(0) = 0 \). Our purpose is to provide a partition \( \alpha \) and corresponding estimates for \( S_1(\alpha) \) and \( S_2(\alpha) \) that are independent of \( \epsilon \). The sums
\[
\sum_{i=1}^{n-1} R(1 - c_i^2) (c_{i+1} - c_i) \quad \text{and} \quad \sum_{i=1}^{n-1} R(s_i^2)(s_i - s_{i+1})
\]
are Riemann sums for the integrals
\[
\int_{c_1}^{1} R(1 - c^2) dc \quad \text{and} \quad \int_{0}^{s_1} R(s^2) ds.
\]
Consider the first integral. From
\[
\frac{N - 1 - c^2 + g \left( \frac{e^2}{r^2} (1 - c^2) \right) c^2}{c^2 + g \left( \frac{e^2}{r^2} (1 - c^2) \right) (1 - c^2)} \leq \frac{N - 1}{c^2}
\]
we obtain that
\[
\int_{c_1}^{1} R(1 - c^2) dc = g \left( \frac{e^2}{r^2} \right) \frac{R(1)}{N - 1} \int_{c_1}^{1} \frac{N - 1 - c^2 + g \left( \frac{e^2}{r^2} (1 - c^2) \right) c^2}{c^2 + g \left( \frac{e^2}{r^2} (1 - c^2) \right) (1 - c^2)} dc \leq
\]
\[
R(1) g \left( \frac{e^2}{r^2} \right) \int_{c_1}^{1} \frac{dc}{c^2}.
\]
Set
\[
S_x^*(c) = R(1)c + R(1)g \left( \frac{e^2}{r^2} \right) \int_{c}^{1} \frac{db}{b^2} = R(1) \left( c + g \left( \frac{e^2}{r^2} \right) \left( \frac{1}{c} - 1 \right) \right) =
\]
\[
R(1) g \left( \frac{e^2}{r^2} \right) \left( \frac{c}{g \left( \frac{e^2}{r^2} \right)} + \frac{1}{c} - 1 \right).
\]
Evaluating the last term at the minimum point \( c = \sqrt{g \left( \frac{e^2}{r^2} \right)} \), we obtain
\[
S_x^* \left( \sqrt{g \left( \frac{e^2}{r^2} \right)} \right) = R(1) g \left( \frac{e^2}{r^2} \right) \left( \frac{2}{\sqrt{g \left( \frac{e^2}{r^2} \right)}} - 1 \right) =
\]
\[
\frac{2r \sqrt{g \left( \frac{e^2}{r^2} \right)}}{e^{G(c/r)} - 1} - R(1) g \left( \frac{e^2}{r^2} \right).
\]
We fix \( c_1 = \sqrt{g \left( \frac{e^2}{r^2} \right)} \), so that \( \alpha_1 = \arccos \sqrt{g \left( \frac{e^2}{r^2} \right)} \). Consider the second integral
\[
\int_{0}^{s_1} R(s^2) ds.
\]
From
\[
\frac{N - 2 + s^2 + g \left( \frac{e^2}{r^2} s^2 \right) (1 - s^2)}{1 - s^2 + g \left( \frac{e^2}{r^2} \right) s^2} \leq \frac{N - 2 + s^2 + g \left( \frac{e^2}{r^2} s^2 \right) (1 - s^2)}{1 - s^2}
\]
we obtain that
\[
\int_{0}^{s_1} R(s^2) ds = g \left( \frac{e^2}{r^2} \right) \frac{R(1)}{N - 1} \int_{0}^{s_1} \frac{N - 2 + s^2 + g \left( \frac{e^2}{r^2} s^2 \right) (1 - s^2)}{1 - s^2 + g \left( \frac{e^2}{r^2} s^2 \right) s^2} ds \leq
\]
7.4 A sufficient condition for the validity of the Strong Maximum Principle

\[ g \left( \frac{\epsilon^2}{r^2} \right) \frac{R(1)}{N-1} \int_0^{s_1} \frac{N-2+s^2+g \left( \frac{\epsilon^2}{r^2} \right) (1-s^2)}{1-s^2} ds. \]

Set

\[ S_g^*(s) = R(1)(1-s) + g \left( \frac{\epsilon^2}{r^2} \right) \frac{R(1)}{N-1} \int_0^s \frac{N-2+b^2+g \left( \frac{\epsilon^2}{r^2} \right) (1-b^2)}{1-b^2} db = R(1)(1-s) + g \left( \frac{\epsilon^2}{r^2} \right) \frac{R(1)}{N-1} \left[ \left( g \left( \frac{\epsilon^2}{r^2} \right) - 1 \right) s + \frac{N-1}{2} \ln \left( \frac{1+s}{1-s} \right) \right]. \]

Since

\[ 1 - \sqrt{1 - g \left( \frac{\epsilon^2}{r^2} \right)} \leq g \left( \frac{\epsilon^2}{r^2} \right), \]

evaluating the last term at the point \( s_1 = \sin \alpha_1 = \sqrt{1 - g \left( \frac{\epsilon^2}{r^2} \right)} \), we obtain

\[ S_x^* \left( \sqrt{1 - g \left( \frac{\epsilon^2}{r^2} \right)} \right) = R(1) \left( 1 - \sqrt{1 - g \left( \frac{\epsilon^2}{r^2} \right)} \right) + g \left( \frac{\epsilon^2}{r^2} \right) \frac{R(1)}{N-1} \left[ \left( g \left( \frac{\epsilon^2}{r^2} \right) - 1 \right) \sqrt{1 - g \left( \frac{\epsilon^2}{r^2} \right)} - \frac{1}{2} \ln \left( \frac{1 - \sqrt{1 - g \left( \frac{\epsilon^2}{r^2} \right)}}{1 + \sqrt{1 - g \left( \frac{\epsilon^2}{r^2} \right)}} \right) \right] \leq R(1) \sqrt{g \left( \frac{\epsilon^2}{r^2} \right)} \left( \sqrt{g \left( \frac{\epsilon^2}{r^2} \right)} + \frac{1}{2} \sqrt{g \left( \frac{\epsilon^2}{r^2} \right)} \ln g \left( \frac{\epsilon^2}{r^2} \right) \right). \]

To define the other points of the required partition \( \alpha \), consider the integrals

\[ \int_{c_1}^1 R(1-c^2) dc \quad \text{and} \quad \int_0^{s_1} R(s^2) ds. \]

Set

\[ \sigma = R(1)g \left( \frac{\epsilon^2}{r^2} \right). \]

By the basic theorem of Riemann integration, taking a partition \( \alpha \) with mesh size small enough, the value of the Riemann sums

\[ \sum_{i=1}^{n-1} R(1-c_i^2)(c_{i+1} - c_i) \quad \text{and} \quad \sum_{i=1}^{n-1} R(s_i^2)(s_i - s_{i+1}) \]

differs from

\[ \int_{c_1}^1 R(1-c^2) dc \quad \text{and} \quad \int_0^{s_1} R(s^2) ds. \]
by less than $\sigma$. In particular we obtain

$$ S_1(\alpha) = R(1)c_1 + \sum_{i=1}^{n-1} R(1 - c_i^2) (c_{i+1} - c_i) \leq R(1)c_1 + \int_{c_1}^1 R(1 - c^2) dc + \sigma \leq$$

$$\frac{2r \sqrt{g(\frac{e^2}{r^2})}}{e^N - 1} \leq 2Kr$$

and

$$ S_2(\alpha) = R(1)(1 - s_1) + \sum_{i=1}^{n-1} R(s_i^2)(s_i - s_{i+1}) \leq$$

$$R(1)(1 - s_1) + \int_{s_1}^1 R(s^2) ds + \sigma \leq$$

$$\frac{r \sqrt{g(\frac{e^2}{r^2})}}{e^N - 1} \left( 2 \sqrt{g(\frac{e^2}{r^2})} + \frac{1}{2} \sqrt{g(\frac{e^2}{r^2})} \ln g(\frac{e^2}{r^2}) \right) \leq \frac{r}{4}$$

6) With respect to the coordinates fixed at the beginning of the proof, consider $x_i \geq 0$. Set

$$ D_0 = \{(x_1, \ldots, x_N) : R(1) < \rho < R(1) + r, \sqrt{a_1} \leq \sin \theta_{N-1} \leq \sqrt{a_0} = 1\} $$

and on $D_0$ define the function

$$ v_0(x_1, \ldots, x_N) = w(\rho(x_1, \ldots, x_N)). $$

By point 1), the function $v_0$ is of class $C^2(int(D_0))$ and satisfies, pointwise, the inequality $F(v_0) \geq 0$. Having defined $v_0$, define $v_1$ as follows. Set:

$$ O_1(\theta_1, \ldots, \theta_{N-2}) = (O_{1z_1}, \ldots, O_{1z_N}) =$$

$$(R(1) - R(a_1)) \left( \sqrt{1 - a_1 \cos \theta_{N-2} \ldots \cos \theta_1}, \sqrt{1 - a_1 \cos \theta_{N-2} \ldots \sin \theta_1, \ldots, \sin a_1} \right),$$

$$ \rho_1(x_1, \ldots, x_N) = \sqrt{(x_1 - O_{1z_1})^2 + \cdots + (x_N - O_{1z_N})^2} $$

and

$$ \sin \theta_{N-1}^1(x_1, \ldots, x_N) = \frac{x_N - O_{1x_N}}{\rho_1(x_1, \ldots, x_N)} $$

Recalling the definition of $w_1$ in 3), consider

$$ D_1 = \{(x_1, \ldots, x_N) : R(a_1) < \rho_1(x_1, \ldots, x_N) < R(a_1) + r, $$

$$ \sqrt{a_2} \leq \sin \theta_{N-1}^1(x_1, \ldots, x_N) \leq \sqrt{a_1} \} $$

and, on $D_1$, set

$$ v_1(x_1, \ldots, x_N) = w^1(\rho_1(x_1, \ldots, x_N)). $$
The function $v_1$ is of class $C^2(\text{int}(D_1))$. We claim that $v_1$ still satisfies $F(v_1) \geq 0$. Remark that the set of the points $O^1$ is equal to

$$O^1 = \{(x_1, \ldots, x_N) : \rho = R(1) - R(a_1), \quad \sin \theta_{N-1} = \sqrt{a_1}\}$$

and that for every point $p \in D_1$, the corresponding point $O^1(p)$ is the projection of $p$ on $O^1$, while $\rho_1(p) = d(p, O^1)$. Then we obtain

$$\frac{\partial \rho_1}{\partial \theta_i} = 0 \quad \text{for every} \quad i = 1, \ldots, N - 2,$$

$$\frac{\partial O^1_{x_i}}{\partial x_i} \geq 0 \quad \text{for every} \quad i = 1, \ldots, N - 1,$$

$$\frac{\partial O^1_{x_N}}{\partial x_N} = 0.$$

and

$$\nabla v_1 = \frac{w_{\rho}^{a_1}(\rho_1)}{\rho_1} (x_1 - O^1_{x_1}, \ldots, x_N - O^1_{x_N}) =$$

$$w_{\rho}^{a_1}(\rho_1) \left( \cos \theta_{N-1}^1 \cdots \cos \theta_1^1, \cos \theta_{N-1}^1 \cdots \sin \theta_1^1, \ldots, \sin \theta_{N-1}^1 \right),$$

$$(v_1)_{x_{x_1}} = w_{\rho \rho}^{a_1}(\rho_1) \left( \frac{x_i - O^1_{x_i}}{\rho_1} \right)^2 + w_{\rho}^{a_1}(\rho_1) \left( 1 - \left( \frac{x_i - O^1_{x_i}}{\rho_1} \right)^2 - \frac{\partial O^1_{x_i}}{\partial x_i} \right) =$$

$$w_{\rho \rho}^{a_1}(\rho_1) \cos^2 \theta_{N-1} \cdots \sin^2 \theta_1 + \frac{w_{\rho}^{a_1}(\rho_1)}{\rho_1} \left( 1 - \cos^2 \theta_{N-1} \cdots \sin^2 \theta_1 - \frac{\partial O^1_{x_i}}{\partial x_i} \right).$$

Then

$$F(v_1) = \sum_{i=1}^{N-1} (v_1)_{x_{x_1}} + (v_1)_{x_N \times x_N} g((v_1)_{x_N}^2) =$$

$$w_{\rho \rho}^{a_1}(\rho_1) \left( \cos^2 \theta_{N-1}^1 + \sin^2 \theta_{N-1}^1 g (w_{\rho}^{a_1}(\rho_1)^2 \sin^2 \theta_{N-1}^1) \right) +$$

$$\frac{w_{\rho}^{a_1}(\rho_1)}{\rho_1} \left( N - 2 + \sin^2 \theta_{N-1}^1 + \cos^2 \theta_{N-1}^1 g (w_{\rho}^{a_1}(\rho_1)^2 \sin^2 \theta_{N-1}^1) \right) -$$

$$\frac{w_{\rho}^{a_1}(\rho_1)}{\rho_1} \sum_{i=1}^{N-1} \frac{\partial O^1_{x_i}}{\partial x_i} \geq 0$$

since $w^{a_1}(\rho_1)$ verifies equation (7.15). The sets $D_0$ and $D_1$ intersect on $\sin \theta_2(x_1, \ldots, x_N) = \sin \theta_2^1(x_1, \ldots, x_N) = \sqrt{a_1}$. For a point $(x_1, \ldots, x_N)$ in this intersection we have

$$\rho_1(x_1, \ldots, x_N) = \rho(x_1, \ldots, x_N) - (R(1) - R(a_1)).$$

Hence, on $D_0 \cap D_1$

$$R(a_1) \leq \rho(x_1, \ldots, x_N) \leq R(a_1) + r$$

if and only if

$$R(1) \leq \rho(x_1, \ldots, x_N) \leq R(1) + r,$$
and the functions $v_0$ and $v_1$ coincide:

$$v_1(x_1, \ldots, x_N) = \varphi^\alpha(\rho_1(x_1, \ldots, x_N)) = w(\rho_1(x_1, \ldots, x_N) + R(1) - R(a_1)) = w(\rho(x_1, \ldots, x_N)) = v_0(x_1, \ldots, x_N).$$

The formula

$$\bar{v}(x_1, \ldots, x_N) = \begin{cases} v_0(x_1, \ldots, x_N) & \text{on } D_0 \\
 v_1(x_1, \ldots, x_N) & \text{on } D_1 \end{cases}$$

defines a function $\bar{v}$ in $C^0(\text{int}(D_0 \cup D_1))$. We claim that it is also in $C^1(\text{int}(D_0 \cup D_1))$. In fact, we have

$$\nabla v_0(x_1, \ldots, x_N) = \frac{\varphi^\alpha(\rho_1(x_1, \ldots, x_N))}{\rho(x_1, \ldots, x_N)}(x_1, \ldots, x_N),$$

$$\nabla v_1(x_1, \ldots, x_N) = \frac{\varphi^\alpha(\rho_1(x_1, \ldots, x_N))}{\rho_1(x_1, \ldots, x_N)}(x_1 - O^1_{x_1}, \ldots, x_N - O^1_{x_N})$$

and, on $D_0 \cap D_1$,

$$\frac{1}{\rho(x_1, \ldots, x_N)}(x_1 - O^1_{x_1}, \ldots, x_N - O^1_{x_N}) = \frac{1}{\rho_1(x_1, \ldots, x_N)}(x_1, \ldots, x_N).$$

On $\text{int}(D_0)$ and $\text{int}(D_1)$, the function $\bar{v}$ is of class $C^2$ and satisfies, pointwise, the inequality $F(\bar{v}) \geq 0$. We claim that $\bar{v}$ is also in $W^{1,2}(\text{int}(D_0 \cup D_1))$ and that it is a weak solution to $F(\bar{v}) \geq 0$ on $\text{int}(D_0 \cup D_1)$. In fact, for every $\eta \in C_0^\infty(\text{int}(D_0 \cup D_1))$, applying the divergence theorem separately to $\text{int}(D_0)$ and to $\text{int}(D_1)$, we obtain

$$\int_{\text{int}(D_0 \cup D_1)} [\text{div}\nabla \bar{v}]L(\nabla \bar{v}(x))\eta(x) + \langle \nabla L(\nabla \bar{v}(x)), \nabla \eta(x) \rangle \, dx = 0,$$

$$\int_{\text{int}(D_0 \cup D_1)} [\text{div}\nabla \bar{v}]L(\nabla \bar{v}(x))\eta(x) + \langle \nabla L(\nabla \bar{v}(x)), \nabla \eta(x) \rangle \, dx = 0,$$

$$\int_{\partial(\text{int}(D_0))} \eta(x)\langle \nabla L(\nabla \bar{v}(x)), \mathbf{n}(x) \rangle \, dl + \int_{\partial(\text{int}(D_1))} \eta(x)\langle \nabla L(\nabla \bar{v}(x)), \mathbf{n}(x) \rangle \, dl =$$

$$\int_{D_0 \cap \{\sin \theta_{N-1} = \sqrt{\alpha}1\}} \eta(x)\langle \nabla L(\nabla \bar{v}(x)), \mathbf{n}(x) \rangle \, dl +$$

$$\int_{D_1 \cap \{\sin \theta_{N-1} = \sqrt{\alpha}1\}} \eta(x)\langle \nabla L(\nabla \bar{v}(x)), \mathbf{n}(x) \rangle \, dl.$$

The last term equals zero, since $\bar{v} \in C^1(\text{int}(D_0 \cup D_1))$. Hence, when if $\eta \geq 0$, we have that

$$\int_{\text{int}(D_0 \cup D_1)} \langle \nabla L(\nabla \bar{v}(x)), \nabla \eta(x) \rangle \, dx \leq 0,$$
as we wanted to show. Assuming defined \( O^{n-2}(\theta_1, \ldots, \theta_{N-2}) \) and a function \( v \in C^1(int(\mathcal{D}_0 \cup \cdots \cup \mathcal{D}_{n-2})) \), consider

\[
O^{n-1}(\theta_1, \ldots, \theta_{N-2}) = (O^{n-1}_{x_1}, \ldots, O^{n-1}_{x_N}) = O^{n-2}(\theta_1, \ldots, \theta_{N-2}) + (R(a_{n-2}) - R(a_{n-1}))
\]

\[
\left( \sqrt{1 - a_{n-1} \cos \theta_{N-2}} \cdots \cos \theta_1, \sqrt{1 - a_{n-1} \cos \theta_{N-2}} \cdots \sin \theta_1, \ldots, \sqrt{a_{n-1}} \right).
\]

Set

\[
\rho_{n-1}(x_1, \ldots, x_N) = \sqrt{(x_1 - O^{n-1}_{x_1})^2 + \cdots + (x_N - O^{n-1}_{x_N})^2},
\]

\[
sin \theta_{n-1}(x_1, \ldots, x_N) = \frac{x_N - O^{n-1}_{x_N}}{\rho_{n-1}(x_1, \ldots, x_N)}
\]

\(\mathcal{D}_{n-1} = \{(x_1, \ldots, x_N) : R(a_n) < \rho_{n-1}(x_1, \ldots, x_N) < R(a_{n-1}) + r, 0 = \sqrt{a_n} \leq \sin \theta_{n-1}(x_1, \ldots, x_N) \leq \sqrt{a_{n-1}}\}\)

and define on \(\mathcal{D}_{n-1}\) the function

\[
v_{n-1}(x_1, \ldots, x_N) = w^{\rho_{n-1}}(\rho_{n-1}(x_1, \ldots, x_N))
\]

Set \(\mathcal{D} = int(\mathcal{D}_0 \cup \cdots \cup \mathcal{D}_{n-1})\), the same considerations as before imply that the function

\[
\tilde{v}(x_1, \ldots, x_N) = \begin{cases} v_0(x_1, \ldots, x_N) & \text{on } \mathcal{D}_0 \\ \cdots & \text{on } \mathcal{D}_{n-1} \\ v_{n-1}(x_1, \ldots, x_N) & \text{on } \mathcal{D}_{n-1} \end{cases}
\]

is such that \(\tilde{v} \in W^{1,2}(\mathcal{D}) \cap C^1(\mathcal{D}) \cap C(\overline{\mathcal{D}})\) and it is a weak solution to \( F(\tilde{v}) \geq 0 \) on \(\mathcal{D}\). This completes the construction of \(\tilde{v}\) as a weak solution to \( F(\tilde{v}) \geq 0 \) on \(\mathcal{D}_0 \cup \cdots \cup \mathcal{D}_{n-1}\). Set \(O^* = (O^*_{x_1}, \ldots, O^*_{x_N}) = (0, \ldots, 0, R(1) - l_N)\). We have that

\[
\mathcal{D}_0 \cup \cdots \cup \mathcal{D}_{n-1} \subset \{(x_1, \ldots, x_N) : 0 \leq x_i \leq l + r \text{ for } i = 1, \ldots, N - 1, O^*_{x_N} \leq x_N \leq O^*_{x_N} + l_N + r\}
\]

Define the full domain \(\omega\) and the solution by symmetry with respect to the point \(O^*\). Figure 7.2 shows this construction in dimension \(N = 2\) and for \(n - 1 = 2\). Hence the solution will be in \(W^{1,2}(\omega) \cap C^1(\omega) \cap C(\overline{\omega})\) and a weak solution of \( F(v) \geq 0 \) on \(\omega\).

7) The previous construction yields a region \(\mathcal{A}\) centered in \(O^*\), a corresponding region \(\omega\) and a function \(v\) that solves (7.12). The change of coordinates \(\hat{x}_1 = x_1, \ldots, \hat{x}_{N-1} = x_{N-1}, \hat{x}_N = x_N - O^*_{x_N}\), centers \(\mathcal{A}\) at the origin and proves the theorem.

In order to prove Theorem 7.10, we need this further lemma.

**Lemma 7.14.** Consider the sets \(\mathcal{A}\) and \(\mathcal{R}(O^*, l, l_N)\), where \(\mathcal{A}, O^*, l, l_N\) have been defined in Theorem 7.10. Then, for every \(p \in \partial\mathcal{R}\),

\[
d(p, \mathcal{A}) < l_N.
\]
Proof. Set \( q = O^* + (l, \ldots, l, l_N) \). We prove that
\[
d(q, A) < l_N.
\]
Set \( p_i = O^* + (0, \ldots, 0, l, 0, \ldots, 0) \) and let \( \Pi^{N-1} \) the hyperplane passing through \( p_1, \ldots, p_N \). Since \( A \) is convex and \( p_i \in A \), we obtain that
\[
d(q, A) < d(q, \Pi^{N-1}) < l_N.
\]
See Figure 7.3. \( \square \)

Proof of Theorem 7.10. a) Suppose that \( u \) attains its minimum in \( \Omega \), and assume \( \min_{\Omega} u = 0 \) and set \( C = \{ x \in \Omega : u(x) = 0 \} \). By contradiction, suppose that the open set \( \Omega \setminus C \neq \emptyset \).

b) Since \( \Omega \) is a connected set, there exist \( s \in C \) and \( R > 0 \) such that \( B(s, R) \subset \Omega \) and \( B(s, R) \cap (\Omega \setminus C) \neq \emptyset \). Let \( p \in B(s, R) \cap (\Omega \setminus C) \). Consider the line \( \overline{ps} \). Moving \( p \) along this line, we can assume that \( B(p, d(p, C)) \subset (\Omega \setminus C) \), and that there exists one point \( z \in C \) such that \( d(p, C) = d(p, z) \).

c) Fix \( r \):
\[
0 < r < \frac{d(p, C)}{32(N - 1)K^2 + \frac{7}{8}}.
\]
7.4 A sufficient condition for the validity of the Strong Maximum Principle

Figure 7.3: \( \mathcal{A} \) and \( R(O^*, l, l_N) \) in the case \( N = 2 \).

Set
\[
\epsilon(r) = \min \left\{ u(z) : z \in B \left( p, d(p, \mathcal{C}) - \frac{r}{4} \right) \right\},
\]
we have that \( \epsilon(r) > 0 \), and we set \( \epsilon = \min\{\epsilon(r), r\xi\} \).

d) For \( r \) and \( \epsilon \) as defined in c), consider: \( l, l_N, \mathcal{A} \) and \( v \) as defined in Theorem 7.13. Without loss of generality, since the set \( \mathcal{A} \) is symmetric with respect to both coordinate axis, we can suppose that \( p_\mathcal{C} \) belongs to the first quadrant, i.e. that, for every \( i = 1, \ldots, N, \ z_i \geq p_i \), where \( z = (z_1, \ldots, z_N) \) and \( p = (p_1, \ldots, p_N) \). Define the point \( q \) on the segment \( p_\mathcal{C} \) such that \( d(q, p) = d(p, z) - \frac{r}{2} \). Set \( q^* = q - (l, \ldots, l, l_N) \), \( R(q^*, l, l_N) = q^* + R(l, l_N) \), \( \mathcal{A}^* = q^* + \mathcal{A} \) and \( v^*(x + q^*) = v(x) \). We first claim that
\[
R(q^*, l, l_N) \subset B \left( p, d(p, \mathcal{C}) - \frac{r}{4} \right).
\]

Let \( t \in R(q^*, l, l_N) \), then \( t \) can be written as \( (q_1 - 2\alpha_1l, \ldots, q_{N-1} - 2\alpha_{N-1}l, q_N - 2\alpha_Nl_N) \), with \( 0 \leq \alpha_i \leq 1 \), for \( i = 1, \ldots, N \). Since \( r < \frac{d(p, \mathcal{C})}{32(N-1)K^2 + \epsilon} \), we have that
\[
d(t, p)^2 = \sum_{i=1}^{N} (q_i - 2\alpha_i l_i - p_i)^2 = d(q, p)^2 + \sum_{i=1}^{N} 4\alpha_i^2 l_i^2 - \sum_{i=1}^{N} 4\alpha_i l_i (q_i - p_i) \leq
\]
\[
\left( d(p, \mathcal{C}) - \frac{r}{2} \right)^2 + \sum_{i=1}^{N} l_i^2 \leq \left( d(p, \mathcal{C}) - \frac{r}{2} \right)^2 + 16(N-1)K^2 r^2 + \frac{r^2}{4} < \left( d(p, \mathcal{C}) - \frac{r}{4} \right)^2.
\]

See Figure 7.4.
Since $\mathcal{A}^* \subset \mathcal{R}(q^*, l, l_N)$, we have obtained that
\[ \mathcal{A}^* \subset B \left( p, d(p, C) - \frac{r}{4} \right), \]
so that $u \geq \epsilon$ in $\partial \mathcal{A}^*$. By Lemma 7.14,
\[ d(q, \mathcal{A}^*) < l_N \leq \frac{r}{4}, \]
we have that
\[ d(z, \mathcal{A}^*) \leq d(z, q) + d(q, \mathcal{A}^*) < \frac{3}{4} r, \]
so that
\[ z \in \omega^* = B(\mathcal{A}^*, r) \setminus \overline{\mathcal{A}^*}. \]

e) The function $v^*$ satisfies
\[ \begin{cases} 
  v^* \text{ is a weak solution to } F(v) \geq 0 & \text{ in } \omega^* \\
  v^* > 0 & \text{ in } \omega^* \\
  v^* = 0 & \text{ in } \partial B(\mathcal{A}^*, r) \\
  v^* \leq \epsilon & \text{ in } \partial \mathcal{A}^*. 
\end{cases} \]
Since $u, v^* \in W^{1,2}(\omega^*) \cap C(\overline{\omega^*})$, $v^*$ is a weak subsolution and $u$ is a weak solution to $F(u) = 0$, and $u_{|\partial \omega^*} \leq v^*_{|\partial \omega^*}$, applying Lemma 7.2, we obtain that $u \geq v^*$ in $\omega^*$. But $u(z) = 0 < v^*(z)$, a contradiction.
Bibliography


