Average decay of the Fourier transform

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1 General results for characteristic functions of convex bodies

1.1 L^1 spherical means for polygons

Consider a convex planar body B and the Fourier transform of its characteristic function

$$\widehat{\chi}_B(\xi) = \int_B e^{-2\pi i\xi\cdot x}\,dx\ .$$

A familiar problem concerning $\widehat{\chi}_B(\xi)$ is the study of its decay as $|\xi| \to +\infty$. As a first example, let us consider the unit disc D. In this case

$$\widehat{\chi}_D(\xi) = |\xi|^{-1} J_1\left(2\pi\left|\xi\right|\right) \sim \pi^{-1}\left|\xi\right|^{-3/2}\cos\left(2\pi\left|\xi\right| - 3\pi/4\right) \tag{1}$$

where J_1 is the Bessel function of order 1. If B is not a disc, the decay of $\widehat{\chi}_B(\xi)$ depends, mildly or strongly, on the direction of ξ. Choose e.g. $\xi = (\xi_1, 0)$, then

$$\begin{aligned}\widehat{\chi}_B(\xi_1, 0) &= \iint_B e^{-2\pi i\xi_1 x_1}\,dx_1 dx_2 \qquad (2)\\ &= \int_{\mathbb{R}} e^{-2\pi i\xi_1 x_1}\left(\int_{\mathbb{R}}\chi_B(x_1,x_2)\,dx_2\right)dx_1\\ &= \widehat{f}(\xi_1)\end{aligned}$$

where $f(s)$ is the length of the chord in Figure 1.

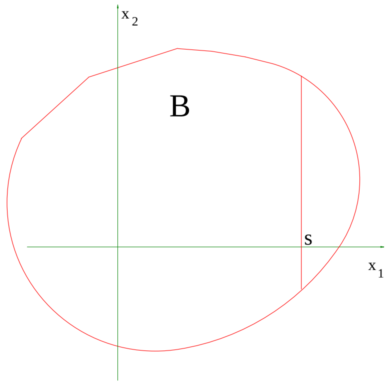

Figure 1

Sometimes (2) is described by saying that the 2-dimensional Fourier transform is a 1-dimensional Fourier transform of a Radon transform.

Let us now replace B by a square Q. If we consider (Figure 2) the Fourier transform in a direction orthogonal to a side of Q, then the graph of $f(s)$ is represented in Figure 3.

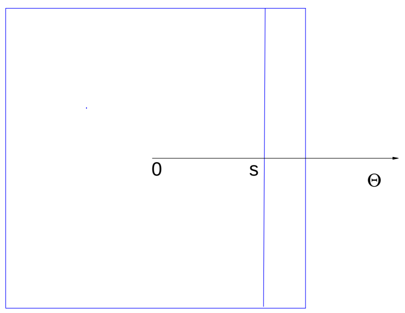

Figure 2

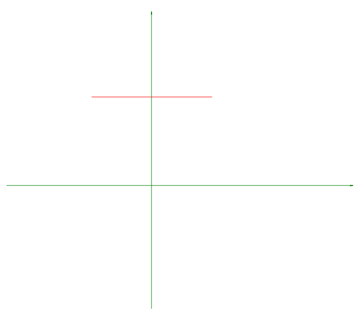

Figure 3

and therefore $\widehat{f}$ decays of order 1. On the other hand, for most directions (Figure 4) $f(s)$ is piecewise smooth and continuous (Figure 5) and $\widehat{f}$ decays of order 2.

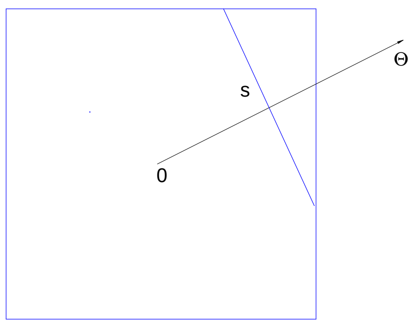

Figure 4

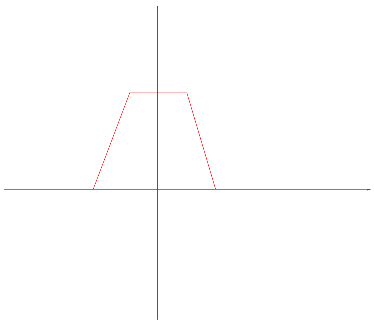

Figure 5

As one might expect, the study of $\widehat{\chi}_B(\xi)$ in a given direction is essentially intractable for the case of an arbitrary convex body B. On the other hand, several problems do not require precise estimates in all directions, but rather a global information, since usually one just needs to integrate the Fourier transform in a suitable sense. We start considering the following L^1 spherical average:

$$\int_0^{2\pi} |\widehat{\chi}_B(\rho\Theta)| \, d\theta$$

where, from now on, $\Theta = (\cos\theta . \sin\theta)$. To see what is going on, we look at the case of a polygon P. By the divergence theorem,

$$\int_P e^{-2\pi i t\cdot\xi} \, dt = \int_P \operatorname{div}\left\{ \frac{e^{-2\pi i t\cdot\xi}}{-2\pi i \, |\xi|^2} \xi \right\} dt = \frac{1}{-2\pi i \, |\xi|^2} \int_{\partial P} e^{-2\pi i t\cdot\xi} \, \xi \cdot \nu(t) \, dS_t$$

where ν is the outward normal vector and dS_t is the boundary measure. We split the integral according to the sides of P and we consider one of them, which we may assume to have extremes $(-1, 0)$ and $(0, 1)$. Then $|\widehat{\chi}_P(\rho\Theta)|$ is controlled by a finite sum of terms of the form

$$\rho^{-1} \left| \int_{-1}^{1} e^{-2\pi i t \rho \cos\theta} \, dt \right| = \left| \frac{\sin(2\pi\rho\cos\theta)}{\pi\rho^2 \cos\theta} \right|$$

and it easy to see that

$$\int_0^{2\pi} |\widehat{\chi}_P(\rho\Theta)| \, d\theta \le c\rho^{-2} \int_0^{2\pi} \left| \frac{\sin(2\pi\rho\cos\theta)}{\cos\theta} \right| d\theta \le c\rho^{-2} \log\rho \, . \qquad (3)$$

There is a more geometrical way to get the same result and it may interesting to describe it. Consider a convex planar body B (Figure 6)

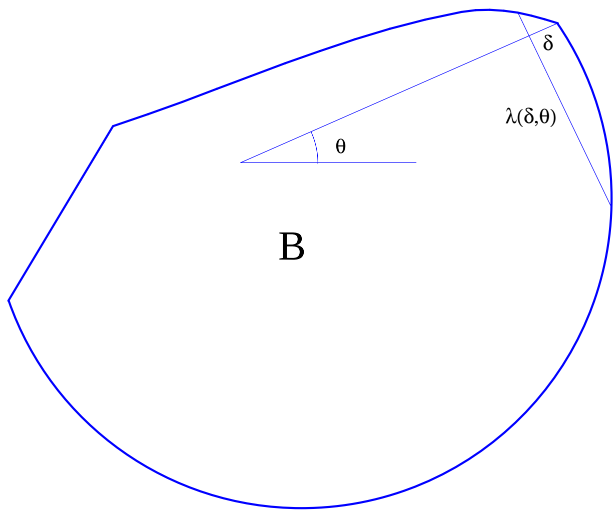

Figure 6

(4)

and let $\lambda(\delta,\theta)$ be the chord at distance δ from the boundary of B, along the direction Θ. The following geometric bound for the Fourier transform has been pointed out by several authors.

Theorem 1 *Let B be a convex planar body. Then*

$$\left|\widehat{\chi}_B(\rho\Theta)\right| \leq c\rho^{-1}\left\{\left|\lambda(\rho^{-1},\theta)\right| + \left|\lambda(\rho^{-1},\theta+\pi)\right|\right\} \tag{5}$$

with c depending only on B.

In order to prove (5) it is useful to observe that the function $f(s)$ in (2) is concave, say with support $[-1,1]$. Then (5) is equivalent to

$$\left|\widehat{f}(t)\right| \leq c\left|t\right|^{-1}\left\{f(-1+\left|t\right|^{-1}) + f(1-\left|t\right|^{-1})\right\}.$$

Assume for simplicity f continuous. Integrating by parts, one essentially reduces to bound a term of the form

$$\left|t^{-1}\int_0^1 -f'(s)\sin\left(2\pi st\right)\,ds\right| \tag{6}$$

where $-f'(s)$ is increasing in $[0,1]$. The graphs of the functions $-f'(s)$ and $-f'(s)\sin(2\pi st)$ are as in Figure 7

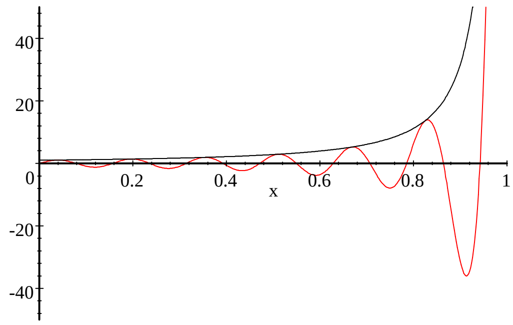

Figure 7

and the integral in (6) is essentially a Leibniz sum, thereby bounded by its largest term

$$\int_{1-|t|^{-1}}^{1} -f'(s)\,ds = f\left(1-|t|^{-1}\right).$$

Going back to the case of a polygon, it is an exercise to estimate the lengths of the chords $\lambda(\rho^{-1},\theta)$ defined in (4) and then deduce (3).

As another application of (5), let us recall the bound

$$|\widehat{\chi}_B(\xi)| \le c\,|\xi|^{-3/2} \tag{7}$$

which holds true for a planar body B, the boundary of which is smooth with everywhere positive curvature. We point out that (7) follows from the obvious geometric inequality $|\lambda(\rho^{-1},\theta)| \le c\rho^{-1/2}$.

It can be shown that (3) is sharp and indeed, for any planar convex body B,

$$\limsup_{\rho\to+\infty} \rho^2 \log^{-1}\rho \int_0^{2\pi} |\widehat{\chi}_B(\rho\Theta)|\,d\theta > 0\,. \tag{8}$$

The proof of (8) depends on a modification of an argument used by Yudin in the study of Lebesgue constants.

The bound in (3) is essentially sharp also in a perhaps more subtle sense. If we assume $P = P_N$ to be a polygon with N sides, say contained in the unit disc, then the above argument (divergence theorem + splitting the boundary integral according to the sides) shows that

$$\int_0^{2\pi} |\widehat{\chi}_{P_N}(\rho\Theta)|\, d\theta \leq cN\rho^{-2}\log\rho\,, \tag{9}$$

where c is independent of N. We are going to show that this bound is essentially sharp since, for any $\varepsilon > 0$ we cannot replace N in the above R.H.S. by $N^{1-\varepsilon}$. Indeed, assume P_N is a regular N-gon inscribed in the unit disc D and consider $\widehat{\chi}_{D\setminus P_N}(\rho\Theta)$. Here $f(s)$ is the sum of the lengths of the two small segments enhanced in Figure 8.

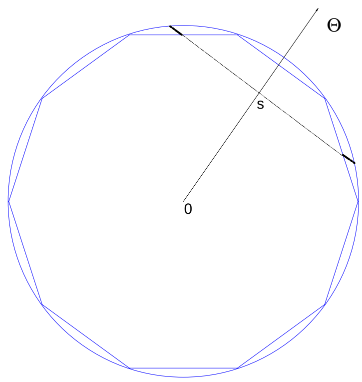

Figure 8

The set $D\setminus P_N$ is the union of N "lunes". Let us number them counterclockwise starting from Θ and let us consider only the first $[N/2]$ lunes for simplicity. Figure 9 shows that the contribution of the k^{th} lune $(1 \leq k \leq [N/2])$ to the total variation of $f(s)$ is

$$\approx N^{-2}\sin^{-1}\frac{k}{N} \approx \frac{1}{Nk}\,.$$

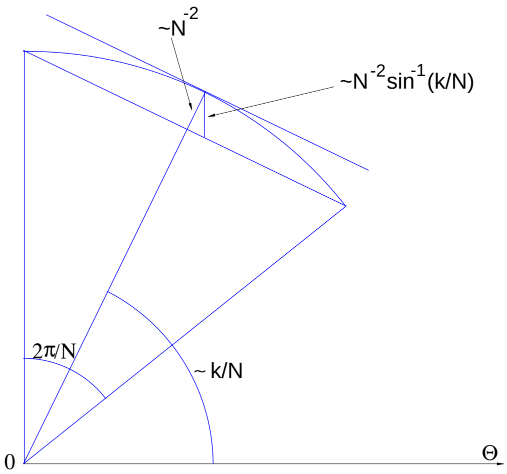

Figure 9

Adding on k and using the theorem on the Fourier transform of a function of bounded variation we get, uniformly in θ,

$$\left|\widehat{\chi}_{D\setminus P_N}(\rho\Theta)\right| = \left|\widehat{f}(\rho)\right| \leq c\rho^{-1}\frac{\log N}{N} \ .$$

On the other hand (1) implies

$$\int_0^{2\pi} \left|\widehat{\chi}_D(\rho\Theta)\right| \, d\theta \leq c_1\rho^{-3/2}$$
$$\int_0^{2\pi} \left|\widehat{\chi}_D(\rho_j\Theta)\right| \, d\theta \geq c_2\rho_j^{-3/2}$$

(for a suitable sequence ρ_j). Now choose j so that $\rho_j \approx N^{2-\varepsilon}$ (ε small). Then

$$\begin{aligned}\int_0^{2\pi} \left|\widehat{\chi}_{P_N}(\rho_j\Theta)\right| \, d\theta &\geq \left|\int_0^{2\pi} \left|\widehat{\chi}_D(\rho_j\Theta)\right| \, d\theta - \int_0^{2\pi} \left|\widehat{\chi}_{D-P_N}(\rho_j\Theta)\right| \, d\theta\right| \\ &\geq \left|c_1N^{-3+3\varepsilon/2} - c_2\frac{\log N}{N}N^{-2+\varepsilon}\right| \\ &\geq cN^{-3+3\varepsilon/2}.\end{aligned}$$

In order to end the proof let us assume that, for a suitable constant $K = K(N)$, we have

$$\int_0^{2\pi} |\widehat{\chi}_{P_N}(\rho\Theta)|\ d\theta \leq K(N)\rho^{-2}\log\rho\ .$$

Then choosing again $\rho = \rho_j \approx N^{2-\varepsilon}$ we get

$$cN^{-3+3\varepsilon/2} \leq K(N)\rho_j^{-2}\log\rho_j \approx K(N)N^{-4+2\varepsilon}\log N\ ,$$

hence

$$K(N) \geq cN^{1-\varepsilon/2}\log^{-1}N \geq cN^{1-\varepsilon}. \tag{10}$$

1.2 L^p means for convex bodies having piecewise smooth boundaries

Let us go back to the case of a planar convex body B and let us consider more general L^p means,

$$\left\{\int_0^{2\pi} |\widehat{\chi}_B(\rho\Theta)|^p\ d\theta\right\}^{1/p}\ .$$

Arguing as in the L^1 case we can prove the following sharp bound for a polygon P when $1 < p < \infty$.

$$\left\{\int_0^{2\pi} |\widehat{\chi}_P(\rho\Theta)|^p\ d\theta\right\}^{1/p} \leq c\rho^{-1-1/p}\ .$$

Note that for a disc D we obviously have

$$\left\{\int_0^{2\pi} |\widehat{\chi}_D(\rho\Theta)|^p\ d\theta\right\}^{1/p} \leq c\rho^{-3/2}\ .$$

The above estimates agree when $p = 2$ and this is a general fact. Indeed Podkorytov proved the following theorem.

Theorem 2 *Let B be a planar convex body. Then*

$$\left\{\int_0^{2\pi} |\widehat{\chi}_B(\rho\Theta)|^2\ d\theta\right\}^{1/2} \leq c\rho^{-3/2}\ . \tag{11}$$

We stress that Podkorytov's theorem requires no regularity assumptions on B. The proof of (11) uses (5) and it is rather difficult. A perhaps simpler argument comes out studying the essentially equivalent inequality

$$\left\{ \int_0^{2\pi} \left| \widehat{\mu}(\rho\Theta) \right|^2 \, d\theta \right\}^{1/2} \leq c\rho^{-1/2} \, , \tag{12}$$

where μ denotes the restriction of the Lebesgue measure to a "convex" finite arc. To prove (12) we write

$$\begin{aligned} \int_0^{2\pi} \left| \widehat{\mu}(\rho\Theta) \right|^2 \, d\theta &= \int_0^{2\pi} \int_{\mathbb{R}^2} e^{2\pi i x \cdot \rho\Theta} \, d\mu(x) \int_{\mathbb{R}^2} e^{-2\pi i y \cdot \rho\Theta} \, d\mu(y) \, d\theta \\ &= \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \int_0^{2\pi} e^{2\pi i (x-y) \cdot \rho\Theta} \, d\theta \, d\mu(x) \, d\mu(y) \\ &= 2\pi \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} J_0 \left(2\pi\rho \left| x - y \right| \right) \, d\mu(x) \, d\mu(y) \end{aligned}$$

where J_0 is the Bessel function. By convexity $|x-y|$ is "close" to the arc length and one can reduce the two integrals on $\mathbb{R}^2$ to one (which indeed is a line integral),

$$\int_0^1 J_0 \left(\rho s \right) \, ds = \rho^{-1} \int_0^{\rho} J_0(u) \, du \leq c\rho^{-1} \, .$$

If we deal with the case $p \neq 2$, a rather interesting phenomenon appears. Let us consider the simplest class which contains polygons and discs, i.e. the family of convex planar bodies with piecewise smooth boundary. Let us say that such a body B has p-order ($p > 1$) of decay equal to a if

$$\left\{ \int_0^{2\pi} \left| \widehat{\chi}_B(\rho\Theta) \right|^p \, d\theta \right\}^{1/p} \leq c\rho^{-a}$$

and

$$\limsup_{\rho \to +\infty} \rho^a \left\{ \int_0^{2\pi} \left| \widehat{\chi}_B(\rho\Theta) \right|^p \, d\theta \right\}^{1/p} > 0 \, .$$

Then we have the following result.

Theorem 3 *Let*

$$S = \left\{ \left(\frac{1}{p}, a\right) : 1 < p \leq 2 \quad , \quad a = \frac{3}{2} \ or \ a = 1 + \frac{1}{p} \right\}$$
$$T = \left\{ \left(\frac{1}{p}, a\right) : 2 \leq p \leq +\infty \quad , \quad 1 + \frac{1}{p} \leq a \leq \frac{3}{2} \right\} .$$

Then the pair $\left(\frac{1}{p}, a\right)$ *belongs to the set* $S \cup T$ *if and only if there exists a convex body* B *(having piecewise smooth boundary) with* p*-order of decay equal to* a*.*

The set $S \cup T$ is represented in Figure 10.

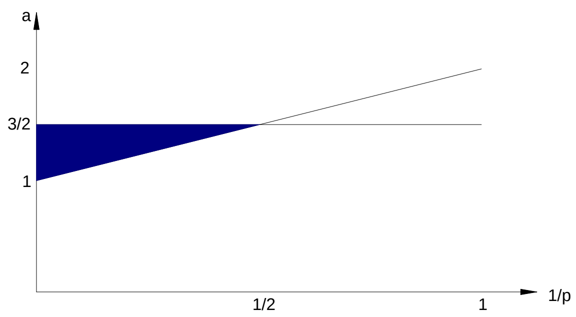

Figure 10

The difference between the cases $p < 2$ and $p > 2$ can be explained in the following way.

When $p < 2$ the proof of the above result essentially says that polygons have p-order of decay equal to $1 + 1/p$, while all other bodies have $3/2$, i.e. they behave like discs. The point is that if B is not a polygon, then its boundary must contain an arc with positive curvature. Using asymptotic of Fourier transforms this is enough to get uniform order of decay $3/2$ on a positive interval of θ's, and this (together with (11) implies the L^p average $3/2$ over $[0, 2\pi)$ when $p \leq 2$.

The situation is different when $p > 2$, where (consider for a moment the case $p = \infty$) the flat points in the boundary are the relevant ones. Here one can produce a scaling between the polygon and the disc by constructing suitable bodies containing a piece of the graph of x^γ ($\gamma > 2$) in their boundaries.

The above dichotomy is no longer valid for arbitrary convex bodies (let us focus on L^1, L^2 and L^∞, the other cases being esseantailly obtained through interpolation): the L^∞ average decay reflects the presence of flat points in the boundary of B, the L^2 average decay is generic, and, finally, the L^1 average decay describes global geometric properties of B, as we shall see in the two coming sections.

1.3 Inscribed polygons

Let B be a convex planar body. Choose any chord at distance δ from the boundary (briefly a δ-chord) and name it ℓ_1. Move counterclockwise constructing a finite sequence of consecutive δ-chords until you reach ℓ_1. Then, if necessary, replace the last chord by one consecutive to ℓ_1. In this way we get a polygon inscribed in B and we denote it by P_δ^B. Let M_δ^B be the number of sides of P_δ^B. It is known that $M_\delta^B \lesssim \delta^{-1/2}$.

Theorem 4 *Let B be a convex planar body and assume $M_{\rho^{-1}}^B \lesssim \rho^\alpha$ (where $0 \le \alpha < 1/2$, the cases $\alpha = 1/2$ being covered by (11). Then*

$$\|\widehat{\chi}_B(\rho\cdot)\|_{L^1(\Sigma_1)} \lesssim \rho^{\alpha-2}\log\rho. \tag{13}$$

Moreover, for any $0 < \alpha < 1/2$, there exists a convex planar body B such that $M_{\rho^{-1}}^B \lesssim \rho^\alpha$ and, for any $\varepsilon > 0$,

$$\limsup_{\rho\to+\infty} \rho^{-\alpha+2+\varepsilon} \|\widehat{\chi}_B(\rho\cdot)\|_{L^1(\Sigma_1)} > 0.$$

See Figure 11.

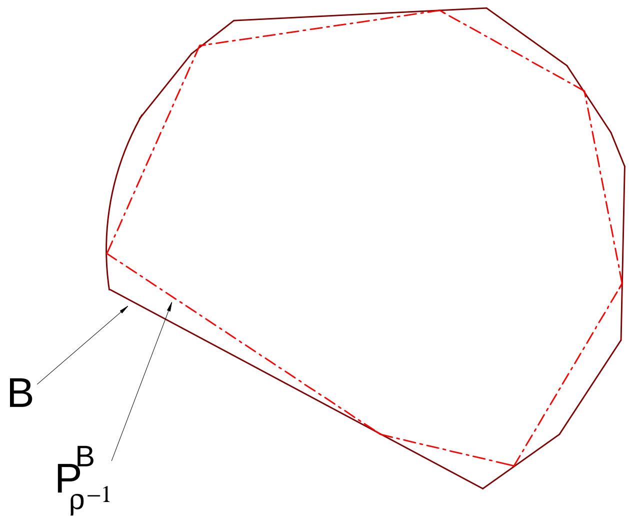

Figure 11

The first step in the proof of Theorem 4 is to prove the inequality

$$\int_0^{2\pi} |\widehat{\chi}_B(\rho\Theta)|\ d\theta \lesssim \int_0^{2\pi} \left|\widehat{\chi}_{P^B_{\rho^{-1}}}(\rho\Theta)\right|\ d\theta.$$

We are therefore reduced to estimating the average decay for a polygon with $\lesssim \rho^\alpha$ sides. We then apply the "trivial" estimate (9) with ρ^α in place of N.

At this point one should expect to have gotten a *poor* result using the trivial estimate (**??**). However, it is not so. The counterexample follows the idea which has been used to prove (10).

1.4 Measuring the image of the Gauss map

We can also take the following "dual" point of view. If B is close to a polygon, then its boundary ∂B has relatively few normals. By convexity, at every point of ∂B there is a left and a right tangent, therefore a left and a right outward normal. Let $\Delta^B \subset [0, 2\pi)$ be the set of the directions appearing either as a left or as right outward normal. In other words, Δ^B is the image of the (generalized) Gauss map. It is reasonable to measure Δ^B in fractal way and in order to do so we define its ε-neighborhood

$$\Delta_\varepsilon^B = \left\{\theta \in [0, 2\pi) : \text{dist}\left(\theta, \Delta^B\right) < \varepsilon\right\}$$

and we assume an inequality of the form

$$\left|\Delta_\varepsilon^B\right| \leq c\, \varepsilon^{1-d} .$$

If B is a disc, we can only take $d = 1$. On the other hand, we can choose $d = 0$ if an only if B is a polygon with finitely many sides. As an exercise, let us consider a polygon with infinitely many sides, such that the set of the normal directions to its sides is

$$\Delta^B = \left\{n^{-\gamma}\right\}_{n=1}^{+\infty}$$

($\gamma > 0$). Let us measure Δ_ε^B. The distance between two consecutive points in the sequence Δ^B is $\approx n^{-\gamma-1}$. Let us cover $[0,1]$ with ε^{-1} essentially disjoint intervals of length ε (we call them (ε)-intervals). Note that they may cover more than one point in Δ^B when $n^{-\gamma-1} \lesssim \varepsilon$, i.e. $n \gtrsim \varepsilon^{-1/(\gamma+1)}$. We set $n_0 \approx \varepsilon^{-1/(\gamma+1)}$ and we keep all the $\approx n_0^{-\gamma}\varepsilon^{-1}$ (ε)-intervals between 0 and $n_0^{-\gamma}$. We observe that on the right of $n_0^{-\gamma}$ only n_0 (ε)-intervals are necessary. Then

$$\left|\Delta_\varepsilon^B\right| \approx n_0^{-\gamma} + \varepsilon n_0 \approx \varepsilon^{\gamma/(\gamma+1)} .$$

We have the following essentially sharp result.

Theorem 5 *Let $0 \leq d \leq 1$ and assume that*

$$\left|\Delta_\varepsilon^B\right| \leq c\varepsilon^{1-d}. \tag{14}$$

Then

$$\int_0^{2\pi} \left|\widehat{\chi}_B(\rho\Theta)\right| \, d\theta \leq \begin{cases} c_d \rho^{-1-1/(d+1)} & \textit{if } 0 < d \leq 1 \\ \\ c\rho^{-2}\log\rho & \textit{if } d = 0 \end{cases} .$$

A slightly weaker form of this theorem can be proved using Theorem 4 and (15) below. We sketch the first part of a direct proof. We can assume $d > 0$ and write

$$\int_0^{2\pi} \left|\widehat{\chi}_B(\rho\Theta)\right| \, d\theta = \int_{\Delta^B_{\rho^{-1/(d+1)}}} \left|\widehat{\chi}_B(\rho\Theta)\right| \, d\theta + \int_{[0,2\pi)\setminus\Delta^B_{\rho^{-1/(d+1)}}} \left|\widehat{\chi}_B(\rho\Theta)\right| \, d\theta$$
$$= I_1 + I_2 .$$

By Schwartz inequality, (14) and Podkorytov's L^2 result (11) we have

$$\begin{aligned} I_1 &\le \left| \Delta^B_{\rho^{-1/(d+1)}} \right|^{1/2} \left\{ \int_0^{2\pi} \left|\widehat{\chi}_B(\rho\Theta)\right|^2 \, d\theta \right\}^{1/2} \\ &\le c\rho^{(d-1)/(2d+2)} \rho^{-3/2} \\ &= c\rho^{-1-1/(d+1)}. \end{aligned}$$

The bound for I_2 depends on (5) and it is more technical.

Introducing the infima α^* and d^* (note that d^* is the upper Minkowski dimension of Δ^B) we can prove that

$$\alpha^* \le \frac{d^*}{d^* + 1}. \tag{15}$$

2 Lattice points and irregularities of distribution

2.1 Lattice points

Let ρB be the dilated copy of a convex planar body B. Some elementary geometric considerations show that

$$\operatorname{card}\left(\rho B \cap \mathbb{Z}^2\right) \sim \rho^2 \, |B|$$

and

$$D_B(\rho) \stackrel{\text{def}}{=} \operatorname{card}\left(\rho B \cap \mathbb{Z}^2\right) - \rho^2 \, |B| = \mathcal{O}(\rho) \, . \tag{16}$$

as $\rho \to +\infty$. $D_B(\rho)$ is called *discrepancy*.

Let us replace B by a disc D. Then the problem of improving (16) is known as the circle problem, and it has counted several milestones during the last century. Among them we point out Sierpinski's early estimate (1906)

$$D_D(\rho) = \mathcal{O}(\rho^{2/3}) \tag{17}$$

and Hardy's Ω-result

$$D_D(\rho) = \Omega\left(\rho^{1/2} \log^{1/4} \rho\right)$$

which, in particular, says that the bound $D_D(\rho) = \mathcal{O}(\rho^{1/2})$ is false. The conjectured result is $\mathcal{O}\left(\rho^{\frac{1}{2}+\varepsilon}\right)$ and the best estimate so far is $\mathcal{O}\left(\rho^{\frac{46}{73}+\varepsilon}\right)$ (Huxley).

Sierpinski's estimate can be proved through a by now classical Fourier analysis argument. It may be worthwhile to recall it.

Proof of (17). This proof works for every convex planar body B having smooth boundary with everywhere positive curvature. We can write

$$D_B(\rho) = \left(\sum_{m \in \mathbb{Z}^2} \chi_{\rho B}(m)\right) - \rho^2 \, |B|$$

and we want to use the Poisson summation formula. This can be done after smoothing the discontinuous function $\chi_{\rho B}$. We consider the convolution $\chi_{\rho B} * \varphi_{\rho^{-1/3}}$ (here φ is a smooth positive function supported in the unit disc and such that $\varphi(0)=1$, $\int\varphi=1$, while $\varphi_{\rho^{-1/3}}(t)=\rho^{2/3}\varphi\left(\rho^{1/3}t\right)$. Then, by the Poisson summation formula and (7),

$$\begin{aligned}\sum_{m\in\mathbb{Z}^2}\chi_{\rho B}(m) &\leq \sum_{m\in\mathbb{Z}^2}\left(\chi_{\left(\rho+\rho^{-1/3}\right)B}*\varphi_{\rho^{-1/3}}\right)(m)\\ &= \sum_{m\in\mathbb{Z}^2}\widehat{\chi}_{\left(\rho+\rho^{-1/3}\right)B}(m)\,\widehat{\varphi}_{\rho^{-1/3}}(m)\\ &= \left(\rho+\rho^{-1/3}\right)^2\sum_{m\in\mathbb{Z}^2}\widehat{\chi}_B\left(\left(\rho+\rho^{-1/3}\right)m\right)\,\widehat{\varphi}\left(\rho^{-1/3}m\right)\\ &= \left(\rho+\rho^{-1/3}\right)^2|B|+\mathcal{O}\left(\rho^2\sum_{m\neq 0}\left(\rho\,|m|\right)^{-3/2}\frac{1}{1+|\rho^{-1/3}m|}\right).\end{aligned}$$

By splitting the last sum one gets

$$\sum_{m\in\mathbb{Z}^2}\chi_{\rho B}(m)\leq\rho^2\,|B|+\mathcal{O}(\rho^{2/3})$$

and since a similar bounds holds from below, the proof is complete.

Here the positive curvature has been crucial. If one replaces B by a unit square Q having sides parallel to the axes, then $D_Q(\rho)$ can be nothing better than $\mathcal{O}(\rho)$. The same happens when Q has a rational slope, but when the slope of Q is irrational, the problem becomes far less trivial. Hardy and Littlewood have proved that in this case the discrepancy $D_Q(\rho)$ can be $\mathcal{O}(\log\rho)$. Actually an even better bound may occur. Davenport has essentially proved that when the slope of Q is algebraic of order 2, then

$$\left\{\int_{\mathbb{T}^2}|D_{Q-t}(\rho)|^2\,dt\right\}^{1/2}=\mathcal{O}\left(\sqrt{\log\rho}\right). \tag{18}$$

Davenport's argument is not difficult, but dealing with more general irrational slopes seems to be extremely hard. On the other hand, mixing the argument in the proof of Sierpinski's results and (3) one can prove the following average result (which extends to several variables).

Theorem 6 *Let P be a polygon in $\mathbb{R}^2$. Then*

$$\int_{SO(2)}\left|D_{\sigma^{-1}(P)}(\rho)\right|\,d\sigma=\mathcal{O}\left(\log^2\rho\right).$$

Averaging over rotations *and* translations (i.e. considering a convex body thrown at random in plane or in the space) leads to sharper results. We describe two of them. Note that it is enough to translate inside a torus.

Theorem 7 *Let T be a triangle and $1 < p < \infty$. Then*

$$c_1(p)\,\rho^{1-1/p} \le \left\{ \int_{\mathbb{T}^2} \int_{SO(2)} \left| D_{\sigma^{-1}(Q)-t}(\rho) \right|^p \, d\sigma dt \right\}^{1/p} \le c_2(p)\,\rho^{1-1/p} . \tag{19}$$

Theorem 8 *Let B be a convex planar body different from a polygon and having piecewise smooth boundary. Let $1 \le p \le 2$. Then*

$$c_1\,\rho^{1/2} \le \left\{ \int_{\mathbb{T}^2} \int_{SO(2)} \left| D_{\sigma^{-1}(B)-t}(\rho) \right|^p \, d\sigma dt \right\}^{1/p} \le c_2\,\rho^{1/2} . \tag{20}$$

Let us look at the proof of the last theorem which, as one may expect, depends on a suitable double Fourier series. Indeed, consider the function

$$\begin{aligned} t \longrightarrow D_{\sigma^{-1}(B)-t}(\rho) &= \operatorname{card}\left(\rho\sigma^{-1}(B) - t \cap \mathbb{Z}^2\right) - \rho^2\,|B| \\ &= \left(\sum_{k\in\mathbb{Z}^2} \chi_{\rho\sigma^{-1}(B)}(k+t) \right) - \rho^2\,|B| \ . \end{aligned}$$

Then, if $m \neq 0$,

$$\begin{aligned} \left(D_{\sigma^{-1}(B)-(\cdot)}(\rho)\right)\hat{\ }(m) &= \sum_{k\in\mathbb{Z}^2} \int_{\mathbb{T}^2} \chi_{\rho\sigma^{-1}(B)}(k+t)\, e^{-2\pi i m\cdot t}\, dt \\ &= \int_{\mathbb{R}^2} \chi_{\rho\sigma^{-1}(B)}(t)\, e^{-2\pi i m\cdot t}\, dt \\ &= \rho^2 \widehat{\chi}_B\left(\rho\sigma(m)\right) , \end{aligned} \tag{21}$$

while $\left(D_{\sigma^{-1}(B)-(\cdot)}(\rho)\right)\hat{\ }(0) = 0$. Then, by (11) and Theorem 3,

$$\begin{aligned} \int_{\mathbb{T}^2} \int_{SO(2)} \left| D_{\sigma^{-1}(B)-t}(\rho) \right|^2 d\sigma dt &= \rho^4 \int_{SO(2)} \sum_{m\neq 0} \left|\widehat{\chi}_B\left(\rho\sigma(m)\right)\right|^2 d\sigma \\ &= \rho^4 \sum_{m\neq 0} \int_{SO(2)} \left|\widehat{\chi}_B\left(\rho\sigma(m)\right)\right|^2 d\sigma \\ &\le c\rho^4 \sum_{m\neq 0} \left(\rho\,|m|\right)^{-3} \\ &\le c\rho\,. \end{aligned}$$

As for the estimate from below, one observes that, for any $\tilde{m} \neq 0$,

$$\int_{\mathbb{T}^2} \int_{SO(2)} \left| D_{\sigma^{-1}(B)-t}(\rho) \right| \, d\sigma dt \geq \rho^2 \int_{SO(2)} \left| \widehat{\chi}_B \left(\rho\sigma(\tilde{m}) \right) \right| \, d\sigma \, .$$

Playing with different $\tilde{m}$'s one gets the desired lower bound.

2.2 Irregularities of distribution

Suppose $\mathcal{P}_N = \{u(j)\}_{j=1}^N$ is a distribution of N points in the unit n-dimensional cube $U = [0,1]^n$, treated as a torus. "Irregularities of distribution" usually means that the above N points cannot, in a certain sense, be too evenly distributed. A well known result in this field is the following theorem, due to Roth.

Theorem 9 *For any distribution $\mathcal{P}_N$ there exists a cube $Q \subset \mathbb{T}^n$, with sides parallel to the axes, such that*

$$\left| \left(\sum_{j=1}^{N} \chi_Q(u(j)) \right) - N \left| Q \right| \right| \geq c \log^{(n-1)/2} N \, .$$

This is a consequence of the following stronger result. Assume $Q = \left[-\frac{1}{2}, \frac{1}{2}\right]^n$, so that the set

$$\{sQ - t\}_{\substack{s\in[0,1] \\ t\in\mathbb{T}^n}}$$

consists of all cubes in $\mathbb{T}^n$, having sides parallel to the axes. Then

$$\int_0^1 \int_{\mathbb{T}^n} \left| \left(\sum_{j=1}^{N} \chi_Q(u(j)) \right) - N s^n \right|^2 dt ds \geq c \log^{n-1} N \, . \tag{22}$$

This last result is best possible (the proof of this fact for $n = 2$ has been the motivation for Davenport's theorem (18)).

The lower bound in (22) changes after replacing cubes by balls. Beck and Montgomery independently proved the following result.

Theorem 10 *Let D be a unit ball in $\mathbb{T}^n$. Let $\mathcal{P}_N$ be any distribution of N points in $\mathbb{T}^n$. Then*

$$\int_0^1 \int_{\mathbb{T}^n} \left| \left(\sum_{j=1}^{N} \chi_D(u(j)) \right) - N s^n \left| D \right| \right|^2 dt ds \geq c N^{(n-1)/n}.$$

We shall see that this bound is sharp too.

Again these results depend on estimates for the average decay of the Fourier transform, e.g. the following one.

Lemma 11 *Let B be a convex body in $\mathbb{R}^n$. Then there exist positive constants $\alpha, \beta, \gamma, \delta$ such that*

$$\alpha\rho^{-1} \leq \int_{\gamma\rho\leq|\xi|\leq\delta\rho} \left|\widehat{\chi}_B(\xi)\right|^2 \, d\xi \leq \beta\rho^{-1} \, .$$

The estimate from below has recently been proved by Kolountzakis and Wolff under the mere assumption of (positive) measurability of B.

We prove only the upper bound, which for $n = 2$ is a consequence of Podkorytov's result (11). For $n > 2$ we can argue as follows. Let $h \in \mathbb{R}^n$ satisfy $|h| \approx \rho^{-1}$, then, being B convex,

$$\begin{aligned} |h| \approx \rho^{-1} &\geq c \int_{\mathbb{R}^n} \left|\chi_B(x+h) - \chi_B(x)\right|^2 \, dx \\ &= c \int_{\mathbb{R}^n} \left|e^{2\pi i \xi\cdot h} - 1\right|^2 \left|\widehat{\chi}_B(\xi)\right|^2 \, d\xi \, . \end{aligned}$$

Now suppose $\xi \in \{\xi : \rho \leq |\xi| \leq 2\rho\}$. By choosing different h's one can split this set into pieces where $\left|e^{2\pi i \xi\cdot h} - 1\right| \geq c$.

We can now prove the following general result (the nice proof is due to Beck). For a convex body B in $\mathbb{R}^n$ we define the discrepancy

$$D_N(s,\sigma,t) = \left(\sum_{j=1}^{N} \chi_{s\sigma^{-1}(B)-t}(u(j)) \right) - Ns^n \, |B|$$

Theorem 12 *Let B be a convex body in $\mathbb{T}^n$, with diameter ≤ 1. Then there exist $c > 0$ and $0 < q < 1$ such that, for large N and any $\mathcal{P}_N = \{u(j)\}_{j=1}^N \subset \mathbb{T}^n$,*

$$\int_q^1 \int_{SO(n)} \int_{\mathbb{T}^n} \left|D_N(s,\sigma,t)\right|^2 \, dt d\sigma ds \geq cN^{1-1/n} \, .$$

Proof. Arguing as in (21) we prove that the Fourier transform of the function

$$t \longrightarrow D_N(s,\sigma,t)$$

is

$$\left(\sum_{j=1}^{N} e^{2\pi i u(j)\cdot m} s^n \widehat{\chi}_B(s\sigma(m)) \right) .$$

Let $0 < r < 1$. By the previous lemma

$$r^{-1} \int_{qr}^{r} \int_{SO(n)} \int_{\mathbb{T}^n} |D_N(s,\sigma,t)|^2 \; dt d\sigma ds \tag{23}$$

$$\approx \sum_{m\neq 0} \left| \sum_{j=1}^{N} e^{2\pi i u(j)\cdot m} \right|^2 \frac{r^n}{|m|^n} \int_{qr|m|\leq|\xi|\leq r|m|} |\widehat{\chi}_B(\xi)|^2 \; d\xi$$

$$\approx \sum_{m\neq 0} \left| \sum_{j=1}^{N} e^{2\pi i u(j)\cdot m} \right|^2 \frac{r^n}{|m|^n} \frac{1}{1+r|m|} .$$

We now apply (23) twice. Choosing $r = 1$ we get

$$\int_{q}^{1} \int_{SO(n)} \int_{\mathbb{T}^n} |D_N(s,\sigma,t)|^2 \; dt d\sigma ds \approx \sum_{m\neq 0} \left| \sum_{j=1}^{N} e^{2\pi i u(j)\cdot m} \right| |m|^{-n-1} \; ,$$

which for any r is bounded from below by

$$\left(\inf_{m\neq 0} \left\{ |m|^{-1} r^{-n} (1 + r|m|) \right\} \right) \left(\sum_{m\neq 0} \left| \sum_{j=1}^{N} e^{2\pi i u(j)\cdot m} \right| \left(\frac{r}{|m|} \right)^n \frac{1}{1+r|m|} \right) \tag{24}$$

$$\approx r^{-n+1} \left(r^{-1} \int_{qr}^{r} \int_{SO(n)} \int_{\mathbb{T}^n} |D_N(s,\sigma,t)|^2 \; dt d\sigma ds \right)$$

because of (23). Now we fix $r = kN^{-1/n}$ (k will be chosen later on). Then the last term in (24) is

$$k^{-n} N \int_{qkN^{-1/n}}^{kN^{-1/n}} \int_{SO(n)} \int_{\mathbb{T}^n} |D_N(s,\sigma,t)|^2 \; dt d\sigma ds \; .$$

We observe that s satisfies

$$q^n k^n |B| \leq N s^n |B| \leq k^n |B| \; .$$

inside its domain of integration. We then choose q and k so that

$$0 < \delta < Ns^n |B| < 1 - \delta$$

and we recall that

$$D_N(s, \sigma, t) = \text{an integer number} - Ns^n |B|$$

so that, for s inside its domain of integration (and uniformly in σ and t)

$$|D_N(s, \sigma, t)| > \delta \ .$$

Summarizing,

$$\begin{aligned}
&\int_q^1 \int_{SO(n)} \int_{\mathbb{T}^n} |D_N(s, \sigma, t)|^2 \ dt d\sigma ds \\
&\geq cN \int_{qkN^{-1/n}}^{kN^{-1/n}} \int_{SO(n)} \int_{\mathbb{T}^n} |D_N(s, \sigma, t)|^2 \ dt d\sigma ds \\
&\geq cN^{1-1/n} \ .
\end{aligned}$$

We now want to show that the above lower bound cannot be improved. We consider the case $n = 2$, so that the above theorem reads $\{\}^{1/2} \geq cN^{1/4}$. We forget the integration in s and prove, for any N, the existence of a set $\mathcal{P}_N$ of cardinality N such that

$$\left\{ \int_{SO(2)} \int_{\mathbb{T}^2} \left| \sum_{j=1}^{N} \chi_{\sigma^{-1}(B)-t}(u(j)) - N|B| \right|^2 dt d\sigma \right\}^{1/2} \leq cN^{1/4} \ . \tag{25}$$

Indeed, suppose first that N is a square, $N = \ell^2$. Then we can put the ℓ^2 points in a standard grid in $\mathbb{T}^2$ so that its periodic extension is the grid $\ell^{-1}\mathbb{Z}$ in $\mathbb{R}^2$. We are therefore back to a lattice points problem in $\mathbb{R}^2$. The difference is that now we shrink the lattice by ℓ^{-1} instead of dilating the body by $\ell = \sqrt{N}$. Hence by (20) the L.H.S. in (25) is controlled by $\ell^{1/2} = N^{1/4}$. If N is not a square we write N as a sum of four squares, $N = m^2 + n^2 + p^2 + q^2$ and we dispose the N points in four disjoint grids. Then the bound $N^{1/4}$ comes out also in the general case.

The above result can be rephrased by saying that a standard grid provides the smallest possible L^2 mean for the discrepancy. It is interesting to note that this is not true for L^p means when $p > 2$. Chen proved the following result (which we state in the planar case).

Theorem 13 *Let B be a convex body in $\mathbb{R}^2$ and let $p > 1$. Then for any N there exists a set $\mathcal{P}_N$ of N points in $\mathbb{T}^2$ such that*

$$\left\{ \int_0^1 \int_{SO(2)} \int_{\mathbb{T}^2} |D_N(s, \sigma, t)|^p \ dt d\sigma ds \right\}^{1/p} \leq N^{1/4} \ .$$

This is not an easy generalization of the previous result, since replacing $\mathcal{P}_N$ by the standard grid we get the bound $N^{1/4}$ for the L^p mean only, say, if ∂B has positive curvature, while for a polygon we should only deduce $N^{(p-1)/2p}$ from (19). Since Chen's result says that (for L^p means, $p > 2$ and for polygons) the standard grid is not the best way to dispose N points, it is natural to ask "which one is a better way?". An argument to understand why a better way exists is the following .

If we dispose N points in a standard grid we can say that we put them on straight lines. On the opposite, we could decide to put N points in concentric circles, i.e. we somewhat associate a *curvature* to them. Actually, a large discrepancy depends on the matching between the *curvature* of the set and the curvature of ∂B. This suggests that a small discrepancy for a family of polygons should be achieved by disposing the N points in suitable circles. The difficulty in pursuing this idea is that we no longer have the Fourier series device which is naturally associated to the standard grid.

3 Spherical means for certain operators

The boundedness of certain operators in Fourier analysis depends on precise estimates for the Fourier transforms of suitable functions or distributions. The purpose of this section is twofold. On the one hand we want to show some new applications of the estimates for the Fourier transform studied in the first section, on the other hand we want to use spherical means to point out some connections between certain classical operators. We shall consider Radon transforms, convolution operators associated to singular measures and restriction operators.

3.1 Radon transforms and convolution operators

We consider the classical Radon transform on $\mathbb{R}^2$. For any $\sigma \in \Sigma_1$ and $t \in \mathbb{R}$ we define

$$\mathcal{R}f(\sigma, t) = \int_{x\cdot\sigma=t} f(x)\,dx\ .$$

for suitable functions f. We want to study the boundedness

$$\mathcal{R} : L^p(\mathbb{R}^2) \longrightarrow L^q\left(\Sigma_1 \times \mathbb{R}\right) \tag{26}$$

as p and q vary.

Oberlin and Stein proved that (26) holds true if and only if the point $\left(\frac{1}{p}, \frac{1}{q}\right)$ belongs to the segment of extremes $\left(\frac{2}{3}, \frac{1}{3}\right)$ and $(1, 1)$.

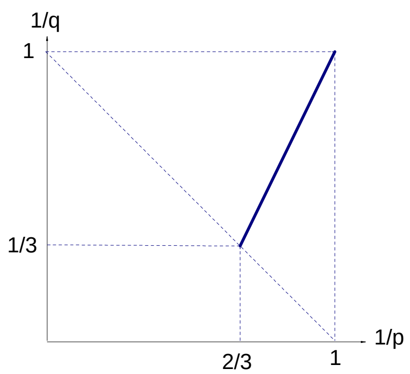

Figure 12

Note that for $p = q = 1$ the boundedness follows from Fubini's theorem. Then the proof of the positive part needs only the bound

$$\|\mathcal{R}f\|_3 \leq c\,\|f\|_{3/2}\ . \tag{27}$$

In order to prove (27) we use Stein's complex interpolation theorem which in our case allows us to deduce (27) from a $L^1 \to L^\infty$ bound for an easier operator (which we are not going to describe) and a $L^2 \to L^2$ bound for a harder operator, which reduces to prove that

$$\left\{\int_{\Sigma_1}\int_{\mathbb{R}}\left||r|^{1/2}\left\{\mathcal{R}f(\sigma,\cdot)\right\}^{\wedge}(r)\right|^2 dr\,d\sigma\right\}^{1/2} \leq c\,\|f\|_2\ . \tag{28}$$

The proof of (28) is not difficult, since the square of its L.H.S. equals

$$\begin{aligned}
&\int_{\Sigma_1}\int_{\mathbb{R}}\left|\int_{\mathbb{R}}\left(\int_{x\cdot\sigma=t} f(x)\,dx\right)e^{-2\pi itr}\,dt\right|^2 |r|\ dr\,d\sigma \qquad (29)\\
&= \int_{\Sigma_1}\int_{\mathbb{R}}\left|\widehat{f}(\sigma r)\right|^2 |r|\ dr\,d\sigma\\
&= 2\int_{\mathbb{R}^2}\left|\widehat{f}(\xi)\right|^2 d\xi\\
&= 2\,\|f\|_2^2\ .
\end{aligned}$$

Now we put the Radon transform aside for a moment and we consider a different problem.

Let μ be the restriction of the Lebesgue measure to the unit circle in $\mathbb{R}^2$. For suitable functions f let

$$Tf = f * \mu\ .$$

Being μ finite we have

$$\|Tf\|_{L^p(\mathbb{R}^2)} \leq c\,\|f\|_{L^p(\mathbb{R}^2)}$$

for any p. Actually more is true, since Strichartz proved that

$$\|Tf\|_{L^q(\mathbb{R}^2)} \leq c\,\|f\|_{L^p(\mathbb{R}^2)} \tag{30}$$

if and only if the pair $\left(\frac{1}{p}, \frac{1}{q}\right)$ belongs to the following triangle

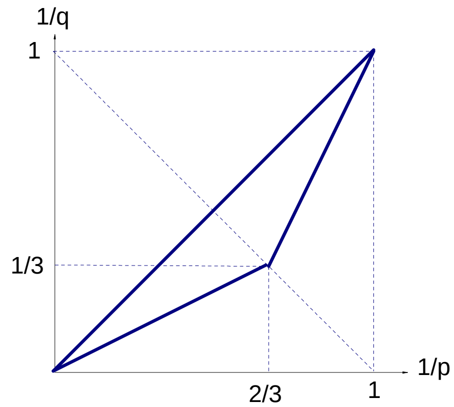

Figure 13

and again the non trivial bound is

$$\|Tf\|_{L^3(\mathbb{R}^2)} \leq c \|f\|_{L^{3/2}(\mathbb{R}^2)} \ . \tag{31}$$

Note that (31) has a very simple statement. Let $f \in L^{3/2}(\mathbb{R}^2)$, for every $x_0 \in \mathbb{R}^2$ replace $f(x_0)$ by the integral of f over a circle of center x_0 and radius 1. Then you get a function in $L^3(\mathbb{R}^2)$.

Using complex interpolation one can obtain (31) as a consequence of the inequality

$$\left\{ \int_{\mathbb{R}^2} \left| \widehat{f}(\xi)\widehat{\mu}(\xi) \right|^2 |\xi| \ d\xi \right\}^{1/2} \leq c \|f\|_{L^2(\mathbb{R}^2)} \tag{32}$$

which is true since (e.g. through Bessel functions) one knows that $|\widehat{\mu}(\xi)| \leq c |\xi|^{-1/2}$.

At this point it is interesting to compare (29) and (32). In (29) the term $|r|$ is used as a jacobian, so that one can pass from $\Sigma_1 \times \mathbb{R}$ to $\mathbb{R}^2$, while in (32) the same term, i.e. $|\xi|$ is balanced by the decay of $\widehat{\mu}$. In both cases a curvature is involved and we can essentially recover the two results in the following way.

Theorem 14 *Let Γ be a convex curve (i.e. it locally coincides with the graph of a convex function in suitable coordinates). Let μ^Γ be the restriction of the Lebesgue measure to Γ and let $\mu_\sigma = \mu_\sigma^\Gamma$ be defined by $\mu_\sigma(E) = \mu(\sigma^{-1}(E))$ for any measurable set E and $\sigma \in SO(2)$. Then*

$$\left\{ \int_{SO(2)} \int_{\mathbb{R}^2} \left| (f * \mu_\sigma)(x) \right|^3 \ dx d\sigma \right\}^{1/3} \le c \left\| f \right\|_{L^{3/2}(\mathbb{R}^2)} . \tag{33}$$

Note that if C is a circle (which we may assume centered at the origin), then $\mu^C = \mu_\sigma^C$ for any σ and therefore (33) reduces to (31). On the other hand, if S is a segment, $f * \mu_\sigma^S$ is a Radon transform (with respect to all segments of a given length, in place of straight lines) and (33) reduces to an analog of (27). The proof of (33) is similar to the proofs of (27) and (31) and here the $L^2\left(SO(2) \times \mathbb{R}^2\right) \to L^2\left(\mathbb{R}^2\right)$ bound reads

$$\left\{ \int_{SO(2)} \int_{\mathbb{R}^2} \left| \widehat{f}(\xi) \widehat{\mu}_\sigma(\xi) \right|^2 |\xi| \ d\xi d\sigma \right\}^{1/2} \le c \left\| f \right\|_{L^2(\mathbb{R}^2)}$$

which holds true since the above L.H.S. is

$$\left\{ \int_{\mathbb{R}^2} \left| \widehat{f}(\xi) \right|^2 |\xi| \left(\int_{SO(2)} \left| \widehat{\mu}_\sigma(\xi) \right|^2 d\sigma \right) d\xi \right\}^{1/2} \le c \left\| f \right\|_{L^2(\mathbb{R}^2)}$$

by the L^2 average decay result for curves (12).

One can obtain sharper results by replacing the $L^2\left(SO(2) \times \mathbb{R}^2\right)$ norm with a mixed norm and applying the results in Theorem 3 to prove inequalities of the form

$$\left\{ \int_{SO(2)} \left\{ \int_{\mathbb{R}^2} \left| (f * \mu_\sigma)(x) \right|^s \ dx \right\}^{p'/s} d\sigma \right\}^{1/p'} \le c \left\| f \right\|_{L^p(\mathbb{R}^2)} .$$

3.2 Spherical means for the restriction phenomenon

If $f \in L^1(\mathbb{R}^2)$, then its Fourier transform $\widehat{f}$ is continuous and therefore well defined on sets of measure zero. On the other hand, if $f \in L^2(\mathbb{R}^2)$, then $\widehat{f}$ is an arbitrary function in L^2 and therefore defined only almost everywhere. For $1 < p < 2$ the Hausdorff-Young theorem leads us to think $\widehat{f}$ as an $L^{p'}$ function. The "restriction phenomenon" says that for certain $1 < p < 2$ the

Fourier transform $\widehat{f}$ can be integrated when restricted to suitable curves. Let Γ be a curve in $\mathbb{R}^2$ with everywhere positive curvature. Let $\mu = \mu^\Gamma$ be the restriction of the Lebesgue measure to Γ. The restriction theorem says that for suitable functions f

$$\left\{ \int_\Gamma \left| \widehat{f}(\xi) \right|^2 \, d\mu(\xi) \right\}^{1/2} \leq c \, \|f\|_{L^{6/5}(\mathbb{R}^2)} \ . \tag{34}$$

Note that some assumptions on the curvature of Γ are necessary. Indeed, let S be a segment, say with extremes $(-1,0)$ and $(1,0)$. Let

$$f_\varepsilon(x_1, x_2) = \frac{\sin 2\pi x_1}{\pi x_1} \frac{\sin 2\pi \varepsilon x_2}{\pi x_2} \ .$$

Then $\widehat{f_\varepsilon}$ is the characteristic function of the rectangle having vertices $(\pm 1, \pm \varepsilon)$ and $\|f_\varepsilon\|_{L^p(\mathbb{R}^2)} = c_p \varepsilon^{1-1/p}$ if $p > 1$. Restricting $\widehat{f_\varepsilon}$ to S we get

$$\left\{ \int_S \left| \widehat{f}(\xi) \right|^2 \, d\mu(\xi) \right\}^{1/2} = \sqrt{2}$$

which cannot be controlled by $\varepsilon^{1-1/p}$.

The restriction phenomenon is not only a deep geometric feature of the Fourier transform in several variables. Strichartz has shown, e.g., that considering the Schrodinger equation

$$\frac{\partial u}{\partial t} = i \frac{\partial^2 u}{\partial x^2} \ , \qquad u(x,0) = f(x) \ , \qquad (x,t) \in \mathbb{R}^2$$

(34) implies (when applied to a parabola)

$$\|u\|_{L^6(\mathbb{R}^2)} \leq c \, \|f\|_{L^2(\mathbb{R})} \ . \tag{35}$$

(35) can be appreciated when compared to the identity $\|u(\cdot, t)\|_{L^2(\mathbb{R})} = \|f\|_{L^2(\mathbb{R})}$ coming from the Plancherel theorem.

The proof of (34) depends on the decay of $\widehat{\mu}$. Note that in the case of a segment there is no decay in one direction.

When the segment (or the curve) is not given and it is randomly chosen, the results are different. We take a convex compact curve Γ and we consider the following inequality.

$$\left\{ \int_{SO(2)} \left\{ \int_{\mathbb{R}^2} \left| \widehat{f}(\xi) \right|^2 \, d\mu_\sigma(\xi) \right\}^{s/2} d\sigma \right\}^{1/s} \leq c \, \|f\|_{L^p(\mathbb{R}^2)} \tag{36}$$

where $\mu_\sigma = \mu_\sigma^\Gamma$ is defined by $\mu_\sigma(E) = \mu(\sigma^{-1}(E))$ and $s \geq 1$, $1 < p \leq 2$.

When $s = \infty$ we are back to (34). When $s < \infty$ the result depends on several facts: the average decay of $\widehat{\mu}$, the location of the centers of rotation, the degree to which the arc in question is different from a circle. After all, if the arc coincides locally with a circle, and if the point of rotation is the center of the circle, then the process of averaging over the rotations does not help and we are left with the result in (34). To summarize the results we need the following figure

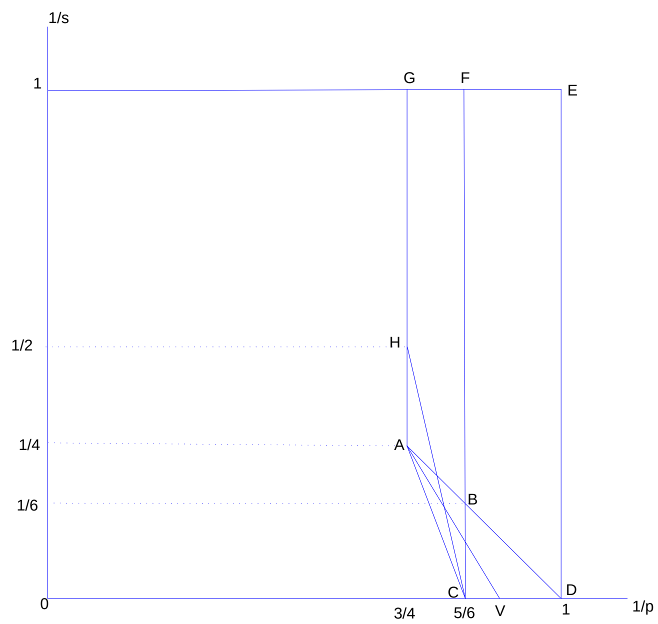

Figure 14

where $A = \left(\frac{3}{4}, \frac{1}{4}\right)$, $B = \left(\frac{5}{6}, \frac{1}{6}\right)$, $H = \left(\frac{3}{4}, \frac{1}{2}\right)$, $V = \left(\frac{2\gamma+1}{2\gamma+2}, 0\right)$.

Then

i)

$$\text{(36) holds for any } \Gamma$$
$$\Updownarrow$$
$$\left(\frac{1}{p}, \frac{1}{s}\right) \in BDEF\ .$$

ii)

$$\text{(36) holds for any } \Gamma \text{ such that the origin is the center of rotation for no point in } \Gamma$$
$$\Updownarrow$$
$$\left(\frac{1}{p}, \frac{1}{s}\right) \in ADEG\ .$$

Note that the assumption in *ii*) includes the segments.

iii)

$$\text{(36) holds whenever } \Gamma \text{ is the graph of the function } y = x^\gamma$$
$$\Updownarrow$$
$$\left(\frac{1}{p}, \frac{1}{s}\right) \in AVDEG\ .$$

iv)

$$\text{(36) holds whenever } \Gamma \text{ has order of contact}$$
$$\text{at most 4with any circle centered at the origin}$$
$$\Updownarrow$$
$$\left(\frac{1}{p}, \frac{1}{s}\right) \in HCDEG\ .$$

Here the typical example is the parabola.

The proof of *i*) depends on a complex interpolation argument and it is natural to expect that (12) plays a role. Indeed one can show that if $\widehat{\mu}$ satisfies

$$\left\{\int_0^{2\pi} |\widehat{\mu}(\rho\Theta)|^q \ d\theta\right\}^{1/q} \leq c\rho^{-a}\ ,$$

then

$$\left\{\int_{SO(2)}\left\{\int_{\mathbb{R}^2}\left|\widehat{f}(\xi)\right|^2\ d\mu_\sigma(\xi)\right\}^{q(a+1)} d\sigma\right\}^{1/(2q(a+1))} \leq c\,\|f\|_{L^{2(a+1)/(a+2)}(\mathbb{R}^2)}\ .$$

However, for the other cases the study of the average decay is not enough and certain ad hoc arguments are necessary, like the one we shall outline for the following two special cases. Let us consider (36) when $s=2$, i.e.

$$\left\{\int_{SO(2)}\int_{\mathbb{R}^2}\left|\widehat{f}(\xi)\right|^2\ d\mu_\sigma(\xi)d\sigma\right\}^{1/2} \leq c\,\|f\|_{L^p(\mathbb{R}^2)}\ . \tag{37}$$

We have

v)

(37) holds whenever Γ passes through the origin

$$\Updownarrow$$

$$p \leq \frac{4}{3}\ .$$

vi)

(37) holds whenever Γ does not pass through the origin and
it has order of contact $N \geq 1$ with a circle centered at the origin

$$\Updownarrow$$

$$p \leq \frac{6N}{5N-2}\ .$$

In the case *v)* we can assume Γ to be a piece of the graph of a smooth convex function $h(x)$ such that $h(0)=h'(0)=0$. Then the L.H.S. in (37) becomes

$$\left\{\int_0^{2\pi}\int_{\mathbb{R}^2}\left|\widehat{f}\left(\left[\begin{array}{cc} x & h(x)\end{array}\right]\left[\begin{array}{cc}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{array}\right]\right)\right|^2\ d\mu(x)d\theta\right\}^{1/2}\ . \tag{38}$$

We change variables by setting

$$\xi = \left[\begin{array}{cc} x & h(x)\end{array}\right]\left[\begin{array}{cc}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{array}\right]\ , \tag{39}$$

so that the jacobian of this transformation is

$$x + h(x)h'(x) \approx x$$

and (38) reduces to

$$\left\{ \int_{|\xi| \le c} \left| \widehat{f}(\xi) \right|^2 \frac{d\xi}{|\xi|} \right\}^{1/2} = \left\{ \int_{|\xi| \le c} \left| \widehat{f}(\xi) \frac{1}{|\xi|^{1/2}} \right|^2 \, d\xi \right\}^{1/2} ,$$

(i.e. a fractional integral) bounded by $\|f\|_{L^p(\mathbb{R}^2)}$ if $\frac{1}{2} = \frac{1}{p} - \frac{1}{4}$, that is $p = \frac{4}{3}$.

We finally consider $vi)$. We can assume Γ to be a piece of the graph of the function $k(x)$, where

$$k(x) = \sqrt{1 - x^2} + cx^N + \dots$$

$c \neq 0$. We perform the change of variables as in (39). The jacobian satisfies

$$x + k(x)k'(x) \approx x^{N-1}$$

since

$$x + \left(\sqrt{1 - x^2} + cx^N + \dots \right) \left(\frac{-x}{\sqrt{1 - x^2}} + cNx^{N-1} + \dots \right) = cNx^{N-1} + \dots .$$

Also observe that

$$|\xi|^2 - 1 \approx x^N$$

since

$$|\xi|^2 = x^2 + k^2(x) = x^2 + \left(1 - x^2 + 2cx^N \sqrt{1 - x^2} + \dots \right) .$$

Then the L.H.S. in (37) takes the form

$$\left\{ \int_{c_1 \le |\xi| \le c_2} \left| \widehat{f}(\xi) \right|^2 \frac{d\xi}{\left(|\xi|^2 - 1 \right)^{(N-1)/N}} \right\}^{1/2}$$

$$= \left\{ \int_{c_1 \le |\xi| \le c_2} \left| \widehat{f}(\xi) \left(|\xi|^2 - 1 \right)^{(1-N)/2N} \right|^2 d\xi \right\}^{1/2}$$

As an example, assume that Γ is a segment with extremes $(\pm 1, 1)$, then $N = 2$ and we have

$$\left\{ \int_{1 \le |\xi| \le \sqrt{2}} \left| \widehat{f}(\xi) \left(|\xi|^2 - 1 \right)^{-1/4} \right|^2 d\xi \right\}^{1/2}$$

which differs from a Bocher Riesz mean of order $-\frac{1}{4}$ mainly because we are *outside* the unit disc. However it is possible to use the techniques developed for Bochner Riesz means of negative order and deduce that the last norm is bonded by $\|f\|_{L^{6N/(5N-2)}(\mathbb{R}^2)}$.