Extremals of some uncertainty inequalities

Carlo Morpurgo

Abstract. We characterize the extremal functions for Heisenberg's type inequalities of the form

$$\|f\|_1^{\gamma}\|f\|_2^{\delta} \le C\,\|\,|x|^2 f\|_1^{\alpha}\|\,|\xi|\,\widehat{f}\,\|_2^{\beta}.$$

Introduction

In [5] we considered the problem of finding the best constant C_0 for the following 1-dimensional uncertainty inequality:

$$\|\,x^2 f\|_1\,\|\,\xi\,\widehat{f}\,\|_2^2 \ge C_0\,\|f\|_1\,\|f\|_2^2. \tag{1}$$

We showed that extremals exist and we computed them explicitly, up to a solution of a certain trigonometric and polynomial equation.

In this note we extend the results of [5] to the n-dimensional version of (1); our result will be in fact slightly more general than (1) even in dimension 1, since we will allow the exponents of the norms to vary.

For $f \in L^1 \cap L^2(\boldsymbol{R}^n)$, we set $\widehat{f}\,(\xi) = \int_{\boldsymbol{R}^n} e^{-2\pi i\xi\cdot x} f(x)dx$; if J_ν denotes the Bessel function of order ν we set, for $r > 0$, $J(r) = r^{1-n/2}J_{n/2-1}(r)$, and for $m > 0$

$$f_m(x) = \begin{cases} J(|x|) - J(m) + \dfrac{J'(m)}{2m}(m^2 - |x|^2) & \text{for } |x| < m \\ 0 & \text{for } |x| \ge m \end{cases} \tag{2}$$

These functions are $C^1(\boldsymbol{R}^n)$ and, as we will see below, they are nonnegative and decreasing in $(0, m)$ if and only if $m \in (0, j_{0,n/2+1})$, where $j_{0,n/2+1}$ is the first positive zero of $J_{n/2+1}$, or, alternatively, the first stationary point (an absolute maximum) of $J'(r)/r$.

Theorem 1. *For* $\dfrac{4}{n+4} < \lambda \le \dfrac{n+2}{2}$, *the best constant* $C(n,\lambda)$ *in the inequality*

$$\|f\|_1^{1-2\lambda/(n+2)}\|f\|_2^{\lambda-4/(n+4)} \le C(n,\lambda)\,\|\,|x|^2 f\|_1^{n/(n+4)}\|\,|\xi|\,\widehat{f}\,\|_2^{n\lambda/(n+2)} \tag{3}$$

is achieved when $f(x) = \alpha f_{\overline{m}}(\beta x)$, *where* $\alpha, \beta \in \boldsymbol{R}\setminus 0$, *and where* $\overline{m} = \overline{m}(n,\lambda)$ *is the only solution in* $(0, j_{0,n/2+1})$ *of the equation*

$$\frac{n\lambda}{n+2}\,\|f_m\|_2^2 = \Big(\lambda - \frac{4}{n+4}\Big)\|\nabla f_m\|_2^2 \tag{4}$$

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Recall that $\|\nabla f\|_2 = 2\pi\| \, |\xi| \, \widehat{f} \, \|_2$.

Estimate (3) is a variation of the well known Heisenberg inequality

$$\|f\|_2^2 \leq \frac{4\pi}{n} \| \, |x| f \|_2 \| \, |\xi| \, \widehat{f} \, \|_2,$$

and it is an alternative formulation of the uncertainty principle: *a nonzero function and its Fourier transform cannot both be sharply localized.* For an excellent survey paper about the uncertainty principle and its mathematical formulations we recommend the one by Folland and Sitaram [4].

Remark 1. Inequality (3) can be derived, without the sharp constant, by multiplying together two known inequalities: Nash's inequality [2] and a special case of a Carlson's type inequality derived by Levin ([6], Chap. VIII). See [5] for a discussion about this point in the 1 dimensional case.

Remark 2. When $\lambda = 2(n+2)/(n+4)$ we obtain

$$\|f\|_1 \|f\|_2^2 \leq C_0(n) \| \, |x|^2 f \|_1 \|\nabla f\|_2^2$$

and condition (4) reduces to $\|f_m\|_2 = \|\nabla f_m\|_2$; this extends the results of [5] in any dimension.

Remark 3. It should be possible, in principle, to write down (4) more explicitly; it is dubious as to whether or not this calculation would be worth the efforts, as the resulting equation appears to have a rather intricate form, in general. However, when $\lambda = (n+2)/2$ the condition expressed by (4) can be rewritten in a simple way, avoiding direct calculations:

Corollary 1. *The best constant $C_1(n)$ in the inequality*

$$\|f\|_2 \leq C_1(n) \, \| \, |x|^2 f \|_1^{2/(n+6)} \| \, |\xi| \, \widehat{f} \, \|_2^{(n+4)/(n+6)} \tag{5}$$

is achieved when $f(x) = \alpha f_{\overline{m}}(\beta x)$, where $\alpha, \beta \in \boldsymbol{R} \setminus 0$, and where $\overline{m} = \overline{m}(n, \lambda)$ is the first positive solution (unique solution in $(0, j_{0,n/2+1})$) of the equation

$$(m^2 - 2n) J'(m) = 2m J(m).$$

In particular, when $n = 1$ the above equation reduces to $(2 - m^2) \tan m = 2m$, and numerical evaluations give $\overline{m} = 2.08157...$ and $C_1(1) = 4.1731....$ Note, however, that it doesn't seem possible to derive a closed form constant $C_1(n)$, even when $n = 1$.

In [3], Cowling and Price derived non-sharp versions of a family of Heisenberg type inequalities, which includes (5) as a special case (see also [4], sec. 3).

Remark 3. The limit case $\lambda = 4/(n+4)$ reduces (3) to

$$\|f\|_1 \leq C_2(n) \, \| \, |x|^2 f \|_1^{(n+4)/(n+6)} \| \, |\xi| \, \widehat{f} \, \|_2^{4/(n+6)}$$

and in fact, we have the following inequalities:

Theorem 2. *For $p > 0$ we have*

$$\|f\|_1^{n+2+2p} \le K(p,n) \| \, |x|^p f\|_1^{n+2} \|\nabla f\|_2^{2p} \tag{6}$$

with

$$K(p,n) = \frac{(n+p)^{n+2}(n+2+2p)^{n+2+p}}{n^{n+2+2p}(n+2)^{n+2+p}(n+2+p)^p}.$$

Equality holds if and only if f is a dilation of the function $|x|^{p+2} - 1 + (1+p/2)(1-|x|^2)$ supported in the unit ball of $\boldsymbol{R}^n$. If $\|f \log |x| \,\|_1 < \infty$ then

$$\|f\|_1 \exp\left(- \frac{n+2}{2\|f\|_1} \int_{\boldsymbol{R}^n} |f| \, \log |x| \, dx \right) \le \alpha_n \, \|\nabla f\|_2 \tag{7}$$

with $\alpha_n^2 = e^{3+2/n} \pi^{n/2} / (\Gamma(n/2) n^2 (n+2))$, and equality holds if and only if f is a dilation of the function $|x|^2 \log |x| + (1 - |x|^2)/2$ supported in the unit ball.

The proof of the first inequality of Theorem 2 is similar to that of Theorem 1 below, and will be omitted. The second inequality follows by taking the $p-$derivative of (6) and its variational equation at $p = 0$, since (6) is an equality at $p = 0$. Details are left to the reader. We notice that (6), with the sharp constant, also may be derived using the above mentioned Carlson's type inequality derived by Levin, and an argument similar to that used by Beckner in [1], Lemma 6.

Proof of Theorem 1

We need to minimize the functional

$$\Lambda(f) = \frac{\| \, |x|^2 f\|_1^\alpha \|\nabla f\|_2^\beta}{\|f\|_1^\gamma \|f\|_2^\delta}$$

over all $f \not\equiv 0$ such that $f, |\nabla f| \in L^2$, $|x|^2 f \in L^1$ (hence $f \in L^1$ by the Poincaré inequality). Here $\alpha = n/(n+4)$, $\beta = n\lambda/(n+2)$, $\gamma = 1 - 2\lambda/(n+2)$ $\delta = \lambda - 4/(n+4)$; with this choice, the functional Λ is invariant under dilations $f(x) \to af(bx)$. Next, note that we can restrict the attention to nonnegative, radially symmetric decreasing functions; the first fact is obvious, and the second follows from $\Lambda(f) \ge \Lambda(f^*)$, where f^* is the radially symmetric decreasing rearrangement of f. Moreover we can take f to be absolutely continuous. With abuse of notation, when f is radial we will write $f(r)$ or $f(x)$, if $|x| = r$.

The proof of the existence of the extremals follows as in [5], with minor changes. For the convenience of the reader we give here an outline of the argument. First take a sequence f_k of nonnegative symmetric decreasing functions, in the class described above, and so that $\Lambda(f_k) \to \Lambda_0 := \inf \Lambda(f)$. Assume that $\Lambda_0 > 0$. After renormalization and scaling we can assume

$$a) \int_0^\infty r^{n-1}(f_k')^2\,dr = 1, \qquad b) \int_0^\infty r^{n+1} f_k\,dr = 1.$$

From b) we obtain $f_k(r) \le cr^{-n-2}$, and from Helly's theorem we can find a subsequence (again denoted by f_k) so that $f_k \to f_0$ a.e., some radial symmetric decreasing f_0. Next, for all $\eta \in S^{n-1}$

$$f_k(0) - f_k(1) = \int_0^1 \nabla f_k(t\eta)\cdot\eta\,dt \le \int_0^1 |\nabla f_k(t\eta)|\,dt.$$

Integrating over η, applying Hölder's inequality and using a) gives $f_k(0) \le C$, for some C independent of k. Thus $f_k(r) \le C\min(1, r^{-n-2})$ which is in $L^1 \cap L^2(r^{n-1}dr)$. By the dominated convergence theorem $\|f_0\|_1^\gamma \|f_0\|^\delta = \Lambda_0^{-1}$ and f_0 is not 0 a.e. At this point one proceeds just as in [5] to show that f_0 satisfies a) and b), and hence that f_0 is an extremal (similarly, follow the argument in [5] to show that $\Lambda_0 \ne 0$).

The characterize the extremals we first let

$$A = \|\,|x|^2 f\|_1 \qquad B = \|\nabla f\|_2^2 \qquad C = \|f\|_1 \qquad D = \|f\|_2^2.$$

In this notation the variational equation for the functional Λ is given by

$$\frac{\alpha}{A}\int |x|^2\phi + \frac{2\beta}{B}\int \nabla f\cdot\nabla\phi = \frac{\gamma}{C}\int\phi + \frac{2\delta}{D}\int f\phi \tag{8}$$

where ϕ is any smooth function with compact support inside the support of f, or any nonnegative and compactly supported function. This implies that f is the classical solution of the equation

$$\frac{2\beta}{B}\Delta f + \frac{2\delta}{D} f = \frac{\alpha}{A}|x|^2 - \frac{\gamma}{C}. \tag{9}$$

Using dilation invariance we can assume that (4) holds, that is,

$$\frac{\beta}{B} = \frac{\delta}{D},$$

and we obtain the equation in radial coordinates

$$f'' + \frac{n-1}{r} f' + f = \frac{\alpha B}{\beta A} r^2 - \frac{\gamma B}{\beta C} \tag{10}$$

The solutions of (9) must be smooth at the origin, so that, as it is well known, the solutions to the homogeneous equation for (10) are constant multiples of $J(r)$

and the general solution is given by $c_1 J(r) + c_2 r^2 + c_3$. Since the critical points f of Λ must be integrable we get that the support of f must be an interval $[0, m]$. Fix such an f, and in (8) plug $\phi = \phi_\epsilon = 1$ in $[0, m]$ and 0 outside $[0, m + \epsilon]$. Using (9) and letting $\epsilon \to 0$ we obtain $f'(m) = 0$ (note that this conclusion is reached by a much simpler argument than the one given in [5]!). Hence, the critical points are dilations of (2), under the condition (4), provided we show that $m \in (0, j_{0,n/2+1})$ if and only if f_m is non increasing and $f_m \geq 0$. To do this, we write $f_m'(r) = r[J'(r)/r - J'(m)/m]$. Next, for $\nu > 0$ let $h_\nu(r) = r^{-\nu} J_\nu$, by a known formula $h_\nu'(r)/r = -h_{\nu+1}$, which has a countable set of zeros. Since $\big(h_\nu'(r)/r\big)' = r h_{\nu+2}$, the stationary points of h_ν'/r are the zeros $\{j_{k,\nu+2}\}$ of $h_{\nu+2}$. It is not hard to see that for every $\sigma > 0$, $|h_\sigma(j_{k,\sigma+1})|$ forms a decreasing sequence in k ($j_{0,\sigma+1}$ are the stationary points of h_σ) ; this follows just like in Watson [7], p. 488, since the function satisfies the equation $\big(h_\nu^2 + (h_\nu')^2\big)' = -(h_\nu')^2(1+2\nu)/r \leq 0$. Moreover, the first stationary point $r_\nu = j_{0,\nu+2}$ of h_ν'/r must be a strict maximum and positive, and hence the function $h_\nu'(r)/r - h_\nu'(m)/m$ is negative in $(0, r_\nu)$ when $m \leq r_\nu$, and when $m > r_\nu$ it has a positive maximum at r_ν, and hence a zero in $(0, r_\nu)$ (since $-h_{\nu+1}(0) < 0$). Thus, the function f_m is positive in $(0, m)$ for $m \leq j_{0,n/2+1}$, and has at least a strict minimum in $(0, j_{0,n/2+1})$ if $m > j_{0,n/2+1}$.

Now we need to show that (4) has one and only one solution $m \in (0, j_{0,n/2+1})$. Observe that we need only show uniqueness of the solution, since an extremal must exist. To do so, define for $m \in (0, j_{0,n/2+1})$

$$\Phi(m) = \frac{\int_0^m r^{n-1} f_m^2 dr}{\int_0^m r^{n-1} (f_m')^2 dr}.$$

To avoid confusion, we shall denote the derivative in the m variable by ∂_m, and the derivative w.r. to r by a prime.

Lemma. *$\partial_m \Phi > 0$ for $m \in (0, j_{0,n/2+1})$.*

Proof. The condition $\partial_m \Phi > 0$ is equivalent to

$$2 \int_0^m r^{n-1} f_m \partial_m f_m dr \int_0^m r^{n-1} (f_m')^2 dr + \int_0^m r^{n-1} f_m^2 dr \int_0^m r^{n-1} f_m \partial_m f_m'' dr > 0.$$

We have

$$\partial_m f_m(r) = \frac{mJ''(m) - J'(m)}{2m^2}(m^2 - r^2), \quad \partial_m f''(r) = -\frac{mJ''(m) - J'(m)}{m^2},$$

moreover, by the Poincaré inequality on the ball with radius m of $\boldsymbol{R}^n$

$$\int_0^m r^{n-1} (f_m')^2 dr \geq \frac{j_{0,n/2-1}^2}{m^2} \int_0^m r^{n-1} f_m^2 dr.$$

We have $mJ''(m) - J'(m) = m^2\partial_m\big(J'(m)/m\big) > 0$, under the hypothesis $m \in (0, j_{0,n/2+1})$, so that we are reduced to showing

$$\frac{j_{0,n/2-1}^2}{m^2}\int_0^m r^{n-1}(m^2-r^2)f_m dr > \int_0^m r^{n-1} f_m dr$$

i.e.

$$\Big(1 - \frac{1}{j_{0,n/2-1}^2}\Big)\int_0^1 r^{n-1} f_m(mr)dr > \int_0^1 r^{n+1} f_m(mr)dr$$

Now we use the following known fact: *If h is nonnegative and decreasing in $[0,1]$, then*

$$\int_0^1 r^{n+1}h(r)dr \le \frac{n}{n+2}\int_0^1 r^{n-1}h(r)dr.$$

Thus, we only need to verify that that $j_{0,n/2-1}^2 > n/2+1$. For $n \ge 6$ this follows from the known fact that $j_{0,n/2-1} > n/2-1$, whereas for $1 \le n \le 5$ one verifies the inequality numerically using the known tabulated values of $j_{0,n/2-1}$.

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Proof of Corollary 1

As we mentioned in Remark 3, we will rewrite an equivalent form of (4) avoiding direct calculations. To do so, notice that the variational equation (10) has $\gamma = 0$, and so the smooth solutions, modulo a multiplicative constant, are given by $f(r) = J(r) + c(r^2 - 2n)$. Since f must be supported in an interval $[0, m]$, with $m < j_{0,n/2+1}$, we must have $J(m) + c(m^2 - 2n) = 0$, and since $\sqrt{2n}$ can't be a zero of $J(r)$ (see [7], p. 485), we must have $m \ne \sqrt{2n}$, and $c = -J(m)/(m^2 - 2n)$. Comparing with (2) gives the result.

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We would like to conclude this note with some remarks concerning possible generalizations of Theorem 1. It should be possible to follow a similar scheme of proof to derive the extremals of the more general inequality

$$\|f\|_1^\gamma \|f\|_2^\delta \le C \|\,|x|^p f\|_1^\alpha \|\,|\xi|\widehat{f}\|_2^\beta \tag{11}$$

with $\alpha+\beta = \gamma+\delta$ and $n\gamma + n\delta/2 = (n+p)\alpha + (n-2)\beta/2$.

In this case the variational equation has still the form given in (10) but with r^2 replaced by r^p. This would involve considering the particular solutions $h_{p,n}$ of the inhomogeneous equation $f'' + (n-1)/rf' + f = r^p$, which are known as Lommel's functions (see [7]). After one writes out the form of the extremals of (11), similar to (2), then the first problem is to verify when such functions are nonnegative. In some cases this is not hard to see, but in general it appears to entail some fine properties of the functions $J'/h'_{n,p}$ (e.g. their maxima form a nonincreasing sequence), which are not immediately deducible from known results.

It should be noted, however, that when $p = 0$ estimate (11) reduces to Nash's inequality, and in this case a proof of its sharp version (alternative to the one in [2]) is actually quite easy, following a variational approach. Indeed, the variational equation in this case is simply $f'' + (n-1)/rf' + f = \text{Const.}$, and therefore the extremals, if they exist, must be dilations of the function $J(|x|) - J(1)$, supported in the unit ball. To show that extremals exist, one proceeds just as in the proof of Theorem 1, with b) replaced by $\int r^{n-1} f_k = 1$. This condition implies that $f_k(r) \leq C \min(1, r^{-n})$ which is in $L^2(r^{n-1}dr)$, and the rest proceeds just like before. Notice that there is no need to use Lemma 1, nor the Poincarè inequality.

If one wants to extend (11) in the case when the norms are taken in more general L^p spaces, then the problem becomes even more difficult, due to the presence of nonlinear terms in the variational equation.

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Carlo Morpurgo
Dipartimento di Matematica e Applicazioni
Università degli Studi di Milano - Bicocca
Via Bicocca degli Arcimboldi, 8
20126 Milano - Italy
morpurgo@matapp.unimib.it