Affine Frames of Multivariate Box Splines and Their Affine Duals

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Abstract

We give a simple and explicit construction of primal and dual wavelet filters based on refinable multivariate splines (with respect to various dilation matrices M) such that the corrisponding wavelet functions generate dual affine frames of arbitrarily high regularity. Furthermore, the number of wavelets does not depend on the regularity. We apply the method also to generalized B–splines.

1 Introduction

It is well known that compactly supported wavelets constructed from cardinal box splines play a prominent role in wavelets theory, because spline functions are fairly well understood, allow explicit computations and possess additional properties (see [1] for a general reference on box splines). Moreover, box splines play an important role in applied and computational mathematics. However, unlike the univariate case [8], and leaving aside tensor products, the construction of (biorthogonal, compactly supported) spline wavelets for any given dimension is a difficult task in general and so far no universal explicit algorithm is available; see e.g. [5], [7], [9], [11], [18], [20], [21], [22], [23].

The recent papers [24], [25] lay the foundation for a significant change of perspective. They contain a general theory of affine systems and of affine frames in particular. In these papers a new "unitary extension principle" is introduced, which makes the construction of tight affine frames somewhat easier than the solution of the matrix extension problem which arises in the (bi)-orthogonal wavelets construction. A technique was also derived to implement this principle, starting from a given "scaling" functions $\varphi \in L^2(\mathbb{R}^N)$.

In particular this was applied to univariate box splines, to some multivariate compactly supported box splines [26] and to convolutions of self-similar compact sets in any dimension [17].

Although the application of Ron-Shen's unitary extension principle is easier than the corresponding construction of (bi)-orthogonal wavelets, still multivariate affine tight frames are hard to construct for any given scaling function. For instance, such frames were constructed only for a relatively small class of box splines in higher dimension. A second drawback is that, in the existing examples, the number of mother wavelets (the frame generators) grows proportionally to the regularity. Even though redundancy may be sought in certain applications (e.g. in oversampled filter banks [4]), still it seems desirable to keep some control on the number of generators. For these reasons we feel that although, generally speaking, tight frames would be more desirable, affine frames with explicit affine duals, with number of wavelets independent of regularity and easy to construct, would provide a good substitute.

The paper [25] contains, among other things, a so called "mixed extension principle" i.e., a generalization to the case of not necessarily tight frames of the above mentioned unitary extension principle. However the scope of this extension was so far unexplored. The aim of this paper is exactly to give simple and explicit constructions of affine frames and their affine duals, based on the mixed extension principle. It turns out that our construction can be applied to a large number of cases, where no tight affine frame was previously known. Namely, it holds for any given "reasonable" multivariate box spline φ if the dilation matrix M is an integer multiple of identity, and for a large class of box splines which are refinable with respect to a dilation map M, $M\mathbb{Z}^N \subset \mathbb{Z}^N$, satisfying $M^r = nI$, for some $r > 1$ and $n \geq 2$.

Denote by M^* the adjoint of M. Given a box spline φ_Ξ refinable with respect to M, our technique consists in constructing explicitly primal and dual polynomial masks m_ℓ, $\widetilde{m}_\ell$, $\ell = 0, \ldots, L$ such that the primal scaling function $\widehat{\varphi} = \prod_{j=1}^{\infty} m_0 \left(M^{*-j} \cdot\right)$ is a finite linear combination of translate of φ_Ξ, the dual scaling function $\widehat{\widetilde{\varphi}} = \prod_{j=1}^{\infty} \widetilde{m}_0 \left(M^{*-j} \cdot\right)$ is arbitrarily regular and the wavelets

$$\widehat{\psi}_\ell(M^* \cdot) = m_\ell \widehat{\varphi}, \tag{1}$$

$$\widehat{\widetilde{\psi}}_\ell(M^* \cdot) = \widetilde{m}_\ell \widehat{\widetilde{\varphi}}, \tag{2}$$

$\ell = 1, \ldots, L$ (with L independent of the regularity) are the generators of dual affine frames. Thus, there exist positive constants $C_1, C_2, \widetilde{C}_1, \widetilde{C}_2$ such that for every $f \in L^2(R^N)$ one has

$$C_1 \|f\|_2^2 \leq \sum_{j,k,s} |\langle f, \psi_{j,k,\ell}\rangle|^2 \leq C_2 \|f\|_2^2 , \tag{3}$$

$$\widetilde{C}_1 \|f\|_2^2 \leq \sum_{j,k,s} \left|\left\langle f, \widetilde{\psi}_{j,k,\ell}\right\rangle\right|^2 \leq \widetilde{C}_2 \|f\|_2^2 \tag{4}$$

and the frames are dual to each other. Here we made, as usual,

$$\psi_{j,k,\ell} = |\det M|^{j/2} \psi_\ell(M^j \cdot -k), \qquad \widetilde{\psi}_{j,k,\ell} = |\det M|^{j/2} \widetilde{\psi}_\ell(M^j \cdot -k), \tag{5}$$

with $j \in \mathbb{Z}, k \in \mathbb{Z}^N$, $\ell = 1, \ldots, L$. In most cases the number L of wavelets will be $L = |\det M|$, just one more than the minimum number $|\det M| - 1$.

In this paper we are concerned almost exclusively with spline functions. However our method can be applied in more general or different situations. It is worthwhile to mention the case of the generalized B–spline, introduced [10] and further studied in [12]. These functions are defined as convolutions products of self-similar sets with respect to the same dilation M, but with possibly different digits sets. Thus these functions are more general than the ones studied in [27] or [17]. However our technique works with this generalization as well. We can construct dual affine frames based on generalized B–splines with arbitrary regularity and fixed number of wavelets, for any M and any choice of the digits, thus extending in various way the basic result of [17].

2 Notation and preliminary facts

In this section we first review some of the very basic notions on dilations, scaling functions and wavelets, which are pertinent to the discussion in this paper. Furthermore we will state the mixed extension principle and a lemma which will be used in the paper to apply such a principle to the case of box splines.

A dilation matrix M on $\mathbb{R}^N$ is a $N \times N$ matrix with integer entries, whose spectrum lies outside the closed unit disc. We say that a function $\varphi \in L^2(\mathbb{R}^N)$ is a scaling function with respect to a given dilation matrix M if the following conditions hold: i) φ satisfies a refinement equation of the type

$$\widehat{\varphi}(M^* \cdot) = m_0 \widehat{\varphi}, \tag{6}$$

where m_0 is a 2π–periodic function called the (symbol of the) mask of the refinement equation (6); ii) $\widehat{\varphi}$ is continuous at 0 and $\widehat{\varphi}(0) = 1$. In this paper φ will always be compactly supported, so that m_0 will always be a trigonometric polynomial.

Let denote by V_0 the closed linear span of the translates of φ by means of the vectors in $\mathbb{Z}^N$. For every integer j let us denote by V_j the M^j – dilate of V_0. Clearly the refinement equation (6) implies that the sequence V_j is increasing with j. If $\widehat{\varphi}$ is continuous at 0 and $\widehat{\varphi}(0) = 1$, then it is possible to prove that $\cup_j V_j$ is dense in $L^2(\mathbb{R}^N)$ and $\cap_j V_j = \emptyset$ [3], [19]. As in this paper we will be dealing with integrable box splines, these conditions are automatically satisfied. Moreover, also the dual scaling function $\widetilde{\varphi}$ will satisfy the same conditions. However, it is worth pointing out that, unlike the case of multiresolution analyses, the scaling functions considered in this paper in general do not generate by translation Riesz bases of V_0. In fact they don't even need generating a frame.

Now let $\psi_1, \ldots, \psi_L$ be elements of V_0. Then, there exist 2π–periodic functions m_ℓ, for $\ell = 1, \ldots, L$, such that (1) holds [2, Theorem 2.14]. The same argument can be repeated for $\widetilde{\varphi}$ and the functions $\widetilde{\psi}_\ell \in \widetilde{V}_0$, thus obtaining (2). When the functions ψ_ℓ and $\widetilde{\psi}_\ell$ are dual affine frames generators they are called primal and dual wavelets, respectively. Likewise, m_ℓ and $\widetilde{m}_\ell$, $\ell \geq 1$, are the primal and the dual wavelet masks (or filters). As we will only consider compactly supported wavelets, in this paper the wavelet masks will always be trigonometric polynomials.

We say that $\psi_1, \ldots, \psi_L$ generate a Bessel system if the right hand side inequality in (3) holds true for every $f \in L^2(\mathbb{R}^N)$. Before stating the mixed extension principle mentioned in the introduction we need some more notation.

Let Γ denote the quotient group $\mathbb{Z}^N/M\mathbb{Z}^N$ and let $\Gamma^* = 2\pi(M^{*-1}\mathbb{Z}^N/\mathbb{Z}^N)$ be its dual group. It can be shown that both groups have order $d = |\det M|$. We may choose complete sets of representatives $\gamma_0, \ldots, \gamma_{d-1}$ for Γ and $\gamma_0^*, \ldots, \gamma_{d-1}^*$ for Γ^* (where γ_0 and γ_0^* represent the identity) and identify these sets with the group itself. We then introduce the periodic $(L+1) \times d$

matrices

$$\Delta(\omega) = [m_\ell(\omega + \gamma_k^*)] \tag{7}$$

$$\widetilde{\Delta}(\omega) = [\widetilde{m}_\ell(\omega + \gamma_k^*)] \tag{8}$$

with $\ell = 0, \ldots, L$, $k = 0, \ldots, d-1$.

Theorem 1 *(Mixed extension principle [25, Theorem 3.9]) Let φ and $\widetilde{\varphi}$ be scaling functions with respect to the dilation map M. Let $\psi_1, \ldots, \psi_L$ be elements of V_0 and let $\widetilde{\psi}_1, \ldots, \widetilde{\psi}_L$ be elements of $\widetilde{V}_0$ such that*

(a) both the families $\psi_1, \ldots, \psi_L$ *and* $\widetilde{\psi}_1, \ldots, \widetilde{\psi}_L$ *generate a Bessel system;*
(b) the matrices $\Delta(\omega)$ *and* $\widetilde{\Delta}(\omega)$ *in (7) and (8) satisfy*

$$\Delta^*(\omega)\widetilde{\Delta}(\omega) = I \qquad a.e. \tag{9}$$

Then the families $\psi_1, \ldots, \psi_L$ *and* $\widetilde{\psi}_1, \ldots, \widetilde{\psi}_L$ *generate affine frames which are dual to each other.*

Remark. In the original formulation of the mixed extension principle it is required a mild decay condition on the scaling functions. However, it follows from the result [6, Theorem 2] that the conclusion of Theorem 1 is true also without the decay assumption. In any case, the scaling functions we will be dealing with are box splines, or convolutions of box splines with compactly supported distributions, which satisfy even stronger decay conditions.

Lemma 1 *Let* M *be a dilation matrix. Let* $\tau_0, \ldots, \tau_{L-1}$ *and* $\widetilde{\tau}_0, \ldots, \widetilde{\tau}_{L-1}$ *be trigonometric polynomials satisfying the system*

$$\sum_{s=0}^{L-1} \tau_s(\omega)\overline{\widetilde{\tau}_s(\omega + \gamma_k^*)} = \delta_{0,k}. \tag{10}$$

where $\delta_{0,s}$ *is the Kronecker delta. Set, for any trigonometric polynomial* μ *and* ν*,*

$$m_0(\omega) = \tau_0(\omega)\mu(M^*\omega) \tag{11}$$

$$\widetilde{m}_0(\omega) = \widetilde{\tau}_0(\omega)\overline{\nu(M^*\omega)}\left(2 - \overline{\mu(M^*\omega)\nu(M^*\omega)}\right) \tag{12}$$

$$m_\ell(\omega) = \tau_\ell(\omega), \quad \widetilde{m}_\ell(\omega) = \widetilde{\tau}_\ell(\omega), \quad \ell = 1, \ldots, L-1 \tag{13}$$

$$m_L(\omega) = \tau_0(\omega)\left(1 - \mu(M^*\omega)\nu(M^*\omega)\right) \tag{14}$$

$$\widetilde{m}_L(\omega) = \widetilde{\tau}_0(\omega)(1 - \overline{\mu(M^*\omega)\nu(M^*\omega)}). \tag{15}$$

Then m_ℓ *and* $\widetilde{m}_\ell$ *satisfy the system*

$$\sum_{\ell=0}^{L} m_\ell(\omega)\overline{\widetilde{m}_\ell(\omega + \gamma_k^*)} = \delta_{0,k}. \tag{16}$$

Proof. The proof consists of an elementary computation, after observing that, by the periodicity of μ and ν, one has

$$\mu(M^*(\omega + \gamma_k^*)) = \mu(M^*\omega), \qquad \nu(M^*(\omega + \gamma_k^*)) = \nu(M^*\omega).$$

Hence

$$\sum_{\ell=0}^{L} m_\ell(\omega)\overline{\widetilde{m}_\ell(\omega+\gamma_k^*)}$$
$$= \tau_0(\omega)\overline{\widetilde{\tau}_0(\omega+\gamma_k^*)}\mu(M^*\omega)\nu(M^*\omega)\left(2-\mu(M^*\omega)\nu(M^*\omega)\right)+$$
$$+\sum_{s=1}^{L-1}\tau_s(\omega)\overline{\widetilde{\tau}_j(\omega+\gamma_k^*)}+\tau_0(\omega)\overline{\widetilde{\tau}_0(\omega+\gamma_k^*)}\left(1-\ \mu(M^*\omega)\nu(M^*\omega)\right)^2.$$

Therefore, using (10) we get

$$\sum_{\ell=0}^{L} m_\ell(\omega)\overline{\widetilde{m}_\ell(\omega+\gamma_k^*)} = \tau_0(\omega)\overline{\widetilde{\tau}_0(\omega+\gamma_k^*)}+\left\{\delta_{0,k}-\tau_0(\omega)\overline{\widetilde{\tau}_0(\omega+\gamma_k^*)}\right\}$$
$$= \delta_{0,k}.$$

■

As a consequence, if the masks m_ℓ and $\widetilde{m}_\ell$ are constructed according to Lemma 1 and if the matrices Δ and $\widetilde{\Delta}$ are defined as in (7), (8), then condition (b) in Theorem 1 is satisfied. In the following sections we will give several constructions of filters m_ℓ and $\widetilde{m}_\ell$ by means of equations (11)–(15), in such a way that the corresponding function φ is a finite linear combination of translates of an assigned multivariate box spline (or generalized B-spline in the last section). The function $\widetilde{\varphi}$ will turn out to be the convolution of a box spline and a distribution whose Fourier transform has a good behavior at infinity. Furthermore we will show that, in the cases studied in this paper, ψ_ℓ and $\widetilde{\psi}_\ell$, defined via conditions (1) and (2), satisfy also condition (a) of Theorem 1, thus generating dual affine frames. As already mentioned, in our constructions we will always have $L = d = |\det M|$ (but for one case where $L = 3 = d+1$), and, at the same time, the $\widetilde{\psi}_\ell$ will be of arbitrarily large regularity.

3 Box splines: the case $M = nI$.

Let $\Xi = [\ \xi_1,\ \ \xi_2,\ \ \ldots\ \ ,\xi_K\ ]$, where $K \geq N$, be a full rank matrix with integer entries. The box spline φ_Ξ associated with Ξ is the $L^2(\mathbb{R}^N)$ function whose Fourier transform is

$$\widehat{\varphi}_\Xi(\omega) = \prod_{j=1}^{K}\left(\frac{1-\exp(-i\langle\xi_j,\omega\rangle)}{i\langle\xi_j,\omega\rangle}\right). \tag{17}$$

Note that we do not assume that Ξ is unimodular, so that, in general, φ_Ξ is not the scaling function of a multiresolution analysis. However in this section we will assume that the first N columns are the fundamental vectors of the axes, i.e.

$$\xi_1 = (1,0,\dots,0), \xi_2 = (0,1,\dots,0), \dots, \xi_N = (0,0,\dots,1),$$

so that (17) takes the form

$$\widehat{\varphi}_\Xi(\omega) = \prod_{k=1}^{N} \left(\frac{1-e^{-i\omega_k}}{i\omega_k} \right) \prod_{j=N+1}^{K} \left(\frac{1-\exp(-i\langle \xi_j, \omega\rangle)}{i\langle \xi_j, \omega\rangle} \right). \tag{18}$$

Clearly φ is refinable with respect to the dilation $M = nI$, where $n \geq 2$ is an integer. Letting

$$q_0(t) = \frac{1+e^{-it}+e^{-i2t}+\cdots+e^{-i(n-1)t}}{n} \tag{19}$$

and denoting by m_Ξ the refinement mask of φ_Ξ, we have that

$$\begin{aligned} m_\Xi(\omega) &= \frac{\widehat{\varphi}_\Xi(M^*\omega)}{\widehat{\varphi}_\Xi(\omega)} = \prod_{k=1}^{N} q_0(\omega_k) \prod_{j=N+1}^{K} q_0(\langle \xi_j, \omega\rangle) \\ &= \tau_0(\omega)G(\omega). \end{aligned}$$

Here we set $\tau_0(\omega) = \prod_{k=1}^{N} q_0(\omega_k)$ and $G(\omega) = \prod_{j=N+1}^{K} q_0(\langle \xi_j, \omega\rangle)$. Note that τ_0 is the N–dimensional Haar low–pass filter with respect to the dilation matrix $M = nI$. We set $\tau_0 = \widetilde{\tau}_0$ and we define the filters $\tau_s = \widetilde{\tau}_s$, $s = 1,\dots,n^N-1$, as the corresponding (orthonormal) wavelet filters, provided by the tensor product construction. To describe explicitly these filters, we first write the wavelet filters in the 1–dimensional case. We set, for $s = 1,\dots,n-1$,

$$q_s(t) = \frac{1}{\sqrt{n(s^2+s)}} \left(\sum_{k=1}^{s} \exp(-i(k-1)t) - s\exp(-ist) \right).$$

It is immediate to check that these are the filters looked for in the case $N = 1$. If $N > 1$, we set

$$E = \frac{\mathbb{Z}^N}{M\mathbb{Z}^N} = \{0,1,\dots,n-1\}^N$$

For every $\varepsilon = (\varepsilon_1,\dots,\varepsilon_N) \in E$, with $\varepsilon \neq (0,\dots,0)$, the ε–th wavelet filters are

$$\tau_\varepsilon(\omega) = \widetilde{\tau}_\varepsilon(\omega) = \prod_{k=1}^{N} q_{\varepsilon_k}(\omega_k). \tag{20}$$

Obviously (after indexing the Haar wavelet filters from 1 to $L-1=n^N-1$) these filters satisfy (10). We choose

$$\mu(\omega) = G(\omega) \tag{21}$$

$$\nu(\omega) = \prod_{k=1}^{N} q_0(\omega_k)^{a_k-1} \tag{22}$$

where $a_k > 1$ is any (large) integer. Finally, we define the filters m_ℓ and $\widetilde{m}_\ell$, with $\ell = 0, \ldots, n^N$ according to equations (11)–(15), where μ and ν are as in (21) and (22). Note that

$$m_0(0) = \widetilde{m}_0(0) = 1, \;\; m_\ell(0) = \widetilde{m}_\ell(0) = 0 \text{ for } \ell > 0. \tag{23}$$

Define

$$\widehat{\varphi}(\omega) = \prod_{j=1}^{\infty} m_0(M^{*-j}\omega), \qquad \widehat{\widetilde{\varphi}}(\omega) = \prod_{j=1}^{\infty} \widetilde{m}_0(M^{*-j}\omega).$$

Observe that φ is a finite linear combination of translates of φ_Ξ, as

$$\widehat{\varphi}(\omega) = G(\omega)\widehat{\varphi}_\Xi(\omega).$$

On the other hand $\widetilde{\varphi}$ is the convolution of a box spline with a compactly supported distribution. Let us show that $\widetilde{\varphi}$ can be of arbitrarily high regularity (depending on $a_1, \ldots, a_N$). Actually,

$$\left| \prod_{j=1}^{\infty} \tau_0(M^{*-j}\omega)\nu(M^{*-j+1}\omega) \right| \le C(1+|\omega_1|)^{-a_1} \cdots (1+|\omega_n|)^{-a_N}.$$

By (21) and (22) $|\mu(Mx)\nu(Mx)| \le 1$, so that, a well known argument [13, p. 216–217] gives

$$\prod_{j=1}^{\infty} \left|2-\mu(M^{*-j+1}\omega)\nu(M^{*-j+1}\omega)\right| \le C(1+|\omega|)^{\log_n 3}$$

and

$$\left| \widehat{\widetilde{\varphi}}(x) \right| \le C(1+|\omega|)^{-a+\log_n 3}, \tag{24}$$

where $a = \min_k a_k$. Finally we set, for $\ell = 1, \ldots, n^N$,

$$\widehat{\psi}_\ell(M^*\omega) = m_\ell(\omega)\widehat{\varphi}(\omega), \qquad \widehat{\widetilde{\psi}}_\ell(M^*\omega) = \widetilde{m}_\ell(\omega)\widehat{\widetilde{\varphi}}(\omega). \tag{25}$$

Theorem 2 *Let Ξ be a matrix with integer entries such that its first N columns are the fundamental vectors of the axes. Let m_ℓ and $\widetilde{m}_\ell$ be constructed as above, and let ψ_ℓ and $\widetilde{\psi}_\ell$, with $\ell = 1, \ldots, n^N$, be the L^2 compactly supported functions in (25). Then the affine systems $\{\psi_{j,k,\ell}\}$ and $\left\{\widetilde{\psi}_{j,k,\ell}\right\}$ (with respect to the dilation matrix $M = nI$) are affine frames dual to each other such that $\psi_\ell(M^{-1}\cdot)$ is a finite linear combination of shifts of the spline φ_Ξ and $\widetilde{\psi}_\ell$ is compactly supported and of arbitrarily high regularity (dictated by inequality (24)).*

Proof. Taking into account Theorem 1 we have only to check that $\psi_1, \ldots, \psi_L$ and $\widetilde{\psi}_1, \ldots, \widetilde{\psi}_L$ generate Bessel systems. To this end we apply [5, Theorem 2.11]. Because of (23) we have that $\widehat{\varphi}(0) = \widehat{\widetilde{\varphi}}(0) = 1$ and $\widehat{\psi}_s(0) = \widehat{\widetilde{\psi}}_s(0) = 0$. Furthermore, because of (18) and (24), also the decay assumptions in [5, Theorem 2.11] are satisfied, as soon as $a > \log_n 3$. This concludes the proof. ■

Remark. Clearly a version of the above Theorem holds also under the (apparently) more general assumption that the vectors $\xi_1, \ldots, \xi_N$ form a basis for $\mathbb{Z}^N$. In fact it suffices to perform an unimodular change of variable.

4 Frames with more symmetry

In Theorem 2 we used the Haar system in order to state a general result for every Ξ satisfying the assumptions of the Theorem. In particular situations it can be more convenient to make other wavelets play the role of the Haar wavelets in the above construction. This may happen when one looks for frames enjoying not only regularity, but also good symmetry properties, always with a number of wavelets independent of the regularity. We will illustrate this by means of two examples.

Let us first consider the case of univariate box splines. For simplicity we will confine our discussion to the case $n = 2$, but the argument can be extended without effort to any $n > 2$. We will consider the centered even box splines, whose refinement mask has symbol

$$\tau_0(\omega) = \cos^{2r} \frac{\omega}{2}, \quad r = 1, 2, \ldots .$$

It was noted in [24] that the filters $\tau_\ell(\omega) = \sqrt{\binom{2n}{\ell}} \sin^\ell \omega/2 \cos^{2r-\ell} \omega/2$, $\ell = 0, \ldots, 2r$, satisfy the system (10), so that, by the results proved in that paper, they give rise to a tight fundamental frame whose elements are splines of degree $2r - 1$. In this case we have $2r$ wavelets with continuous derivatives up to the order $2r-2$. Moreover each wavelet is symmetric or antisymmetric. We can show that if we abandon tight frames, we may produce dual frames, generated by only three wavelets, with the desired regularity and symmetry properties. To do this we will make the Ron–Shen filters just mentioned, with $r=1$, play the role of the Haar filters. Namely we choose

$$\begin{aligned}
\tau_0(\omega) &= \widetilde{\tau}_0(\omega) = \cos^2\frac{\omega}{2}\\
\tau_1(\omega) &= \widetilde{\tau}_1(\omega) = \sqrt{2}\cos\frac{\omega}{2}\sin\frac{\omega}{2}\\
\tau_2(\omega) &= \widetilde{\tau}_2(\omega) \ = \sin^2\frac{\omega}{2}.
\end{aligned}$$

Always with the notation of Lemma 1 we set

$$\mu(\omega) = \cos^{2r-2}\omega/2, \qquad \nu(\omega) = \cos^{2\beta-2r+2}\omega/2$$

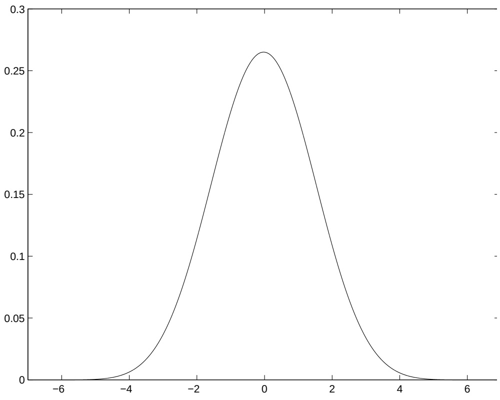
where $\beta \geq r-1$ is any integer. Thus, we obtain

$$\begin{aligned}
m_0(\omega) &= \cos^{2r-2}\omega\cos^2\frac{\omega}{2}, \ \ \widetilde{m}_0(\omega) = \cos^2\frac{\omega}{2}\cos^{2\beta-2r+2}\omega\ \left(2-\cos^{2\beta}\omega\right)\\
m_1(\omega) &= \ \widetilde{m}_1(\omega) = \sqrt{2}\cos\frac{\omega}{2}\sin\frac{\omega}{2}\\
m_2(\omega) &= \ \widetilde{m}_2(\omega) = \sin^2\frac{\omega}{2}\\
m_3(\omega) &= \ \widetilde{m}_3(\omega) = \cos^2\frac{\omega}{2}\,(1-\cos^{2\beta}\omega\ ).
\end{aligned}$$

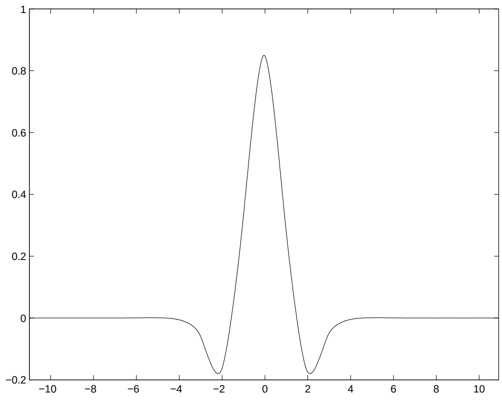
In this case we get

$$\begin{aligned}
\widehat{\varphi}(\omega) &= \cos^{2r-2}(\omega/2)\left(\frac{2\sin(\omega/2)}{\omega}\right)^{2r}\\
\widehat{\widetilde{\varphi}}(t) &= \cos^{2\beta-2r+2}(\omega/2)\left(\frac{2\sin(\omega/2)}{\omega}\right)^{2\beta-2r+4}\prod_{j=1}^{\infty}\left(2-\cos^{2\beta}\left(\omega/2^j\right)\right)
\end{aligned}$$

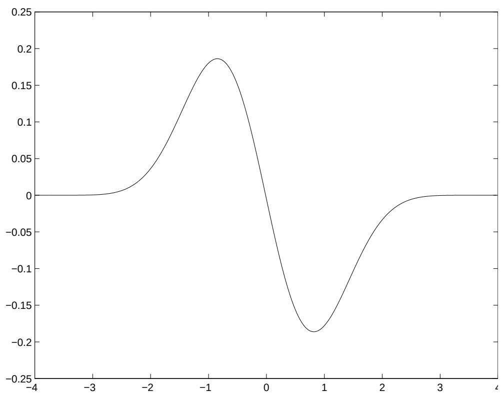
and the wavelets generating the frames are given as usual by $\widehat{\psi}_\ell(2\omega) = m_\ell(\omega)\widehat{\varphi}(\omega)$, $\widehat{\widetilde{\psi}}_\ell(2\omega) = \widetilde{m}_\ell(\omega)\widehat{\widetilde{\varphi}}(\omega)$, for $\ell = 1,2,3$. The dual wavelets can have arbitrarily large regularity, one of them is antisymmetric and the other two are symmetric. The same symmetry properties are enjoyed by the primal wavelets. The figures below show the scaling functions and wavelets in the case $r=\beta=4$.



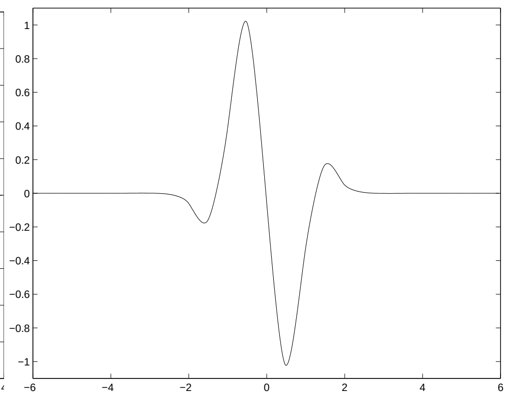
The function φ with $r = 4$



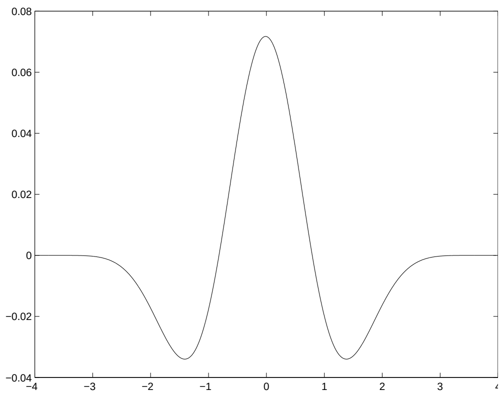
The function $\widetilde{\varphi}$ with $r = 4$, $\beta = 4$



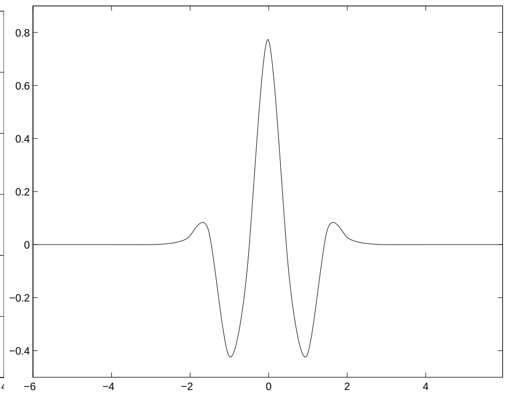
The wavelet ψ_1 with $r = 4$, $\beta = 4$



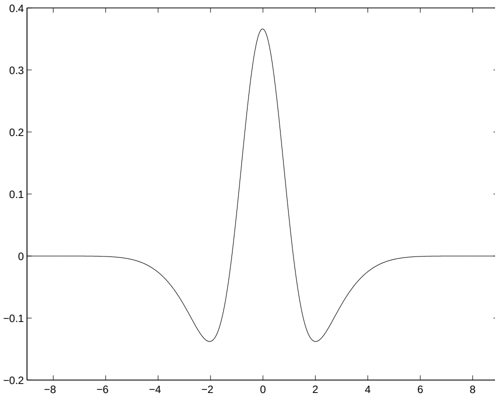
The wavelet $\widetilde{\psi}_1$ with $r = 4$, $\beta = 4$



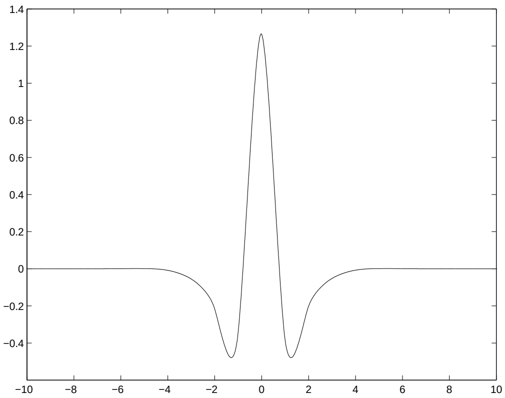
The wavelet ψ_2 with $r = 4$, $\beta = 4$



The wavelet $\widetilde{\psi}_2$ with $r = 4$, $\beta = 4$



The wavelet ψ_3 with $r = 4$, $\beta = 4$



The wavelet $\widetilde{\psi}_3$ with $r = 4$, $\beta = 4$

In the second example we construct bidimensional frames with hexagonal symmetry, arbitrary regularity and fixed number of mother wavelets.

Let τ_0 be a bivariate low–pass filters with dual filter $\widetilde{\tau}_0$. Let $\{\tau_1, \tau_2, \tau_3\}$ and $\{\widetilde{\tau}_1, \widetilde{\tau}_2, \widetilde{\tau}_3\}$ be the corresponding biorthogonal wavelet filters. Assume that each τ_j and each $\widetilde{\tau}_j$, $j = 0, .., 3$ is a function of $(\omega_1, \omega_2, -(\omega_1 + \omega_2))$. Let $\omega_3 = -(\omega_1 + \omega_2)$. According to Cohen and Schlenker [9], the two sets of filters are said to have hexagonal symmetry if

$$\tau_0(\omega_1, \omega_2, \omega_3) = \tau_0(\omega_2, \omega_3, \omega_1) = \tau_0(\omega_3, \omega_1, \omega_2)$$
$$\tau_1(\omega_1, \omega_2, \omega_3) = \tau_2(\omega_2, \omega_3, \omega_1) = \tau_3(\omega_3, \omega_1, \omega_2)$$

and the same relations hold for the $\widetilde{\tau}_j$. If ω_1, ω_2 and ω_3 are interpreted as hexagonal coordinates in the plane [9, p.213], then the above conditions mean that the scaling functions are invariant by a $2\pi/3$ rotation, and the same rotation permutes the mother wavelets. It was shown in the above mentioned paper that the spline filter $\tau_0(\omega_1, \omega_2, \omega_3) = \frac{1}{4}(1 + \cos\omega_1 + \cos\omega_2 + \cos\omega_3)$ admits the dual

$$\widetilde{\tau}_0(\omega_1, \omega_2, \omega_3) = \frac{1}{2}\tau_0(\omega_1, \omega_2, \omega_3)(5 - \cos\omega_1 - \cos\omega_2 - \cos\omega_3).$$

The corresponding dual wavelets filter $\widetilde{\tau}_1$ was shown to be $\widetilde{\tau}_1 = \frac{1}{4}(1 - \exp(-i\omega_1))^2$ and $\widetilde{\tau}_2$ and $\widetilde{\tau}_3$ are obtained by applying the above symmetries to $\widetilde{\tau}_1$. The primal wavelet filters are then obtained by solving the system (10), and it is not hard to see that the primal filters have hexagonal symmetry as well.

It turns out that the primal wavelets belong to the Hölder space $\mathcal{C}^{1-\varepsilon}$ while the dual wavelets only belong to the Sobolev space $\mathcal{H}^s$ with $s < 0.44$.

Thus the Authors of [9] raised the problem of constructing biorthogonal compactly supported wavelet bases of (arbitrarily) high regularity and hexagonal symmetry. We should note that the construction carried out in [18] for general bivariate box splines does not solve the problem, since the wavelet filters deduced by the Hilbert Nullstellensatz need not possess the same symmetries as the primal and the dual scaling functions masks. However we may construct hexagonal frame wavelets of arbitrarily high regularity by using Lemma 1. Namely, let the τ_s and $\widetilde{\tau}_s$ be the Cohen–Schlenker hexagonal filter just described, $s = 0, \ldots, 3$. We choose

$$\mu(\omega_1, \omega_2, \omega_3) = \tau_0^{\beta}\ (\omega_1, \omega_2, \omega_3),$$
$$\nu(\omega_1, \omega_2, \omega_3) = \tau_0^{\delta}\ (\omega_1, \omega_2, \omega_3))$$

where β and δ are positive integer. Set $\omega = (\omega_1, \omega_2, \omega_3)$. Then we have from Lemma 1 and Theorem 2 that the following filters

$$\begin{aligned}
m_0(\omega) &= \ \tau_0(\omega)\tau_0^{\beta}(2\,\omega) \\
\widetilde{m}_0(\omega) &= \widetilde{\tau}_0(\omega)\tau_0^{\delta}(2\,\omega)\left(2 - \tau_0^{\beta+\delta}(2\,\omega)\right) \\
m_\ell(\omega) &= \tau_\ell(\omega), \ \ \widetilde{m}_\ell(\omega) = \widetilde{\tau}_\ell(\omega), \ \ \ell = 1, 2, 3 \\
m_4(\omega) &= \tau_0(\omega)\left(1 - \tau_0^{\beta+\delta}(2\,\omega)\right), \quad \widetilde{m}_4 = \widetilde{\tau}_0(\omega)\left(1 - \tau_0^{\beta+\delta}(2\,\omega)\right).
\end{aligned}$$

give rise to wavelets generating dual frames of arbitrarily high regularity.

These families satisfy a slight generalization of the condition of hexagonal symmetry, the only difference being that the fourth wavelet filters m_4 and $\widetilde{m}_4$ are invariant under permutation of the variables. In the biorthogonal case there are essentially one primal and one dual mother wavelets, the other ones being obtained by a $2\pi/3$ rotation. In the frame case there are essentially two primal and two dual mother wavelets.

5 Box splines: the case $M^r = nI$

In this section we consider affine frames where the $N \times N$ dilation matrix satisfies an equation of the form

$$M^r = nI \tag{26}$$

for some integers $r \geq 2$ and $n \geq 2$, with $n^{N/r}$ an integer. We assume that r is the first integer such that (26) holds.

We start with the case where $N = 2$. In this case $d = |\det M| = n^{2/r}$ is an integer. We notice that not for every value of r and n there are integer 2×2

dilation matrices satisfying (26). We refer to [12, section 4] for classification and admissibility conditions for such matrices. In any case it follows from the results just cited that in all the admissible cases d divides n. Therefore we set

$$n = b \cdot d, \quad \text{where } b \text{ is an integer.}$$

Clearly, not every box spline φ_Ξ is refinable with respect to the dilation matrices considered here. A typical case of refinable spline occurs when the matrix Ξ consists of a cycle for the dilation M i.e., for some integer $K \geq 1$

$$\Xi = \left[\underbrace{\xi_0, \ldots, \xi_0}_{K \text{ times}}, \; \underbrace{\xi_1, \ldots, \xi_1}_{K \text{ times}}, \; \cdots \; \underbrace{\xi_{r-1}, \ldots, \xi_{r-1}}_{K \text{ times}} \right] \tag{27}$$

where $\xi_j = M\xi_{j-1}$, for $j = 1, \ldots, r-1$, and ξ_0 is an integer vector not in $M\mathbb{Z}^2$. The latter assumption will be necessary in order to use Lemma 1, by means of Lemma 2 below. See e.g. [10], [11], [12]. In this case we have

$$\widehat{\varphi}_\Xi(\omega) = \prod_{j=0}^{r-1} \left(\frac{1 - \exp(-i \langle \xi_j, \omega \rangle)}{i \langle \xi_j, \omega \rangle} \right)^K. \tag{28}$$

In order that $\varphi_\Xi \in L^2(\mathbb{R}^2)$ it is clearly necessary and sufficient that ξ_0 is not an eigenvector of M. Denoting by m_Ξ the mask of the refinement equation, we have

$$m_\Xi(\omega) = \left(\frac{1 + e^{-i<\omega,\xi_0>} + e^{-i2<\omega,\xi_0>} + \cdots + e^{-i(n-1)<\omega,\xi_0>}}{n} \right)^K. \tag{29}$$

A moment thought shows that we can factorize m_Ξ as

$$m_\Xi(\omega) = \left(\frac{1 + e^{-i<\omega,\xi_0>} + \cdots + e^{-i(d-1)<\omega,\xi_0>}}{d} \right)^K \rho(\omega)$$

where

$$\rho(\omega) = \left(\frac{1 + e^{-i<d\omega,\xi_0>} + e^{-i2<d\omega,\xi_0>} \cdots + e^{-i(b-1)<d\omega,\xi_0>}}{b} \right)^K. \tag{30}$$

As in section 3 we set

$$q_0(t) = \frac{1 + e^{-it} + e^{-i2t} + \cdots + e^{-i(d-1)t}}{d}. \tag{31}$$

We will follow the same strategy as in section 3. We first construct Haar type wavelet filters and then we invoke Lemma 1 to construct the filters m_ℓ and

$\widetilde{m}_\ell$. We show that we can do this in such a way that the resulting wavelets have arbitrarily large regularity and $L = d$.

As in section 3 we consider the 1–dimensional Haar wavelet filter with respect to the dilation factor d, i.e.

$$q_s(t) = \frac{1}{\sqrt{n(s^2+s)}}\left(\sum_{k=1}^{s} \exp(-i(k-1)t) - s\exp(-ist)\right) \qquad (32)$$

for $s = 1, \ldots, d-1$. We set

$$\tau_s(\omega) = \widetilde{\tau}_s(\omega) = q_s(<\omega, \xi_0>), \qquad s = 0, \ldots, d-1. \qquad (33)$$

Lemma 2 *Let q_0 be as in (31) and let ρ be as in (30), with $\xi_0 \in \mathbb{Z}^2 \setminus M\mathbb{Z}^2$. Then*

1. *the trigonometric polynomials $\tau_s(\omega)$ satisfy (10);*
2. *there exists a trigonometric polynomial μ such that*

$$q_0(<M^*\omega, \xi_0>)^{K-1}\rho(\omega) = \mu(M^*\omega). \qquad (34)$$

Proof. Since $\xi_0 \notin M\mathbb{Z}^2$, arguing as in the proof [15, Lemma 5.1] we see that, as γ_s^* ranges in Γ^*, the exponential $\exp(i <\gamma_s^*, \xi_0>)$ describes a cyclic group of order d. Thus we have, say, $<\gamma_1^*, \xi_0> = 2\pi/d + 2k\pi$, for some $k \in \mathbb{Z}$, and $<\gamma_s^*, \xi_0> = 2\pi s/d + 2k_s\pi$, $s = 0, \ldots, d-1$. This obviously implies the first assertion. As for the second assertion, we may clearly argue only on ρ. Now, for every $\gamma_s^* \in \Gamma^*$ we have

$$1 = \exp(i <d\gamma_s^*, \xi_0>) = \exp(i <\gamma_s^*, d\xi_0>).$$

In other words $d\xi_0 \in M\mathbb{Z}^2$, i.e. there exist $\lambda \in \mathbb{Z}^2$ such that $d\xi_0 = M\lambda$, so that

$$\rho(\omega) = \left(\frac{1 + e^{-i<M^*\omega,\lambda>} + e^{-i2<M^*\omega,\lambda>} \cdots + e^{-i(b-1)<M^*\omega,\lambda>}}{b}\right)^K.$$

■

We now apply Lemma 1 choosing

$$\mu(\omega) = q_0(<\omega, \xi_0>)^{K-1}\rho(M^{*-1}\omega).$$

As for ν we can choose any trigonometric polynomial which guarantees enough decay. A natural choice is

$$\nu(\omega) = q_0^{a-1}(<\omega, \xi_0>) \tag{35}$$

for some large integer exponent $a > 1$. We define the filters m_ℓ, $\widetilde{m}_\ell$ according to Lemma 1 and let ψ_ℓ and $\widetilde{\psi}_\ell$ be as in (25), where

$$\widehat{\varphi}(\omega) = \prod_{j=1}^{\infty} m_0(M^{*-j}\omega) = q_0(<\omega, \xi_0>)^{K-1}\widehat{\varphi}_\Xi(\omega),$$

$$\widehat{\widetilde{\varphi}}(\omega) = \prod_{j=1}^{\infty} \widetilde{m}_0(M^{*-j}\omega).$$

Here again φ is a linear combination of translates of φ_Ξ and $\widetilde{\varphi}$ is the convolution of a box spline with a compactly supported distribution. Arguing as in section 3, we have that

$$\left| \widehat{\widetilde{\varphi}}(x) \right| \leq C(1 + |\omega|)^{-a+\log_c 3} \tag{36}$$

where $c = \|M^{-1}\|^{-1}$. To see this fact it suffices to argue as in the classical case [13, p. 216–217], after noticing that $\|M\| > 1$ and $\|M^{-1}\| < 1$ because M is strictly expansive, replacing M by a power of M if necessary.

¿From Theorem 1 and Lemma 1, in the same way as in section 3 we get the following Theorem.

Theorem 3 *Let M a 2×2 dilation matrix satisfying (26) and let Ξ be of the form (27), where $\xi_j = M\xi_{j-1}$, for $j = 1, \ldots, r-1$, and where $\xi_0 \in \mathbb{Z}^2 \setminus M\mathbb{Z}^2$ is not an eigenvectors of M. Define μ be as in (34) and ν be as in (35). Finally, let the filters τ_ℓ and $\widetilde{\tau}_\ell$ be as in (33), and let m_ℓ and $\widetilde{m}_\ell$, with $\ell = 0, \ldots, d$ be constructed according to Lemma 1. Then the affine systems $\{\psi_{j,k,\ell}\}$ and $\left\{\widetilde{\psi}_{j,k,\ell}\right\}$ are affine frames dual to each other such that $\psi_\ell(M^{-1}\cdot)$ is a finite linear combination of shifts of φ_Ξ and $\widetilde{\psi}_\ell$ is compactly supported and of arbitrarily high regularity (dictated by inequality (36)).*

In general dimension N the structure of the expansion matrices is much more complicated than in dimension 2. We can prove a result similar to the above Theorem under some additional assumptions on M. In first place we will consider only the case where $r = N$, i.e. where the dilation satisfies the equation

$$M^N = nI. \tag{37}$$

Obviously in this case we have $d = |\det M| = n$. Furthermore, we must be able to guarantee that the splines we are dealing with are bona fide L^2

functions. As above we work with vectors ξ_j such that $\xi_j = M\xi_{j-1}$. Thus we have to prove that these vectors are independent. The next theorem gives a sufficient condition.

Proposition 1 *Let M, with $M\mathbb{Z}^N \subset \mathbb{Z}^N$, be a dilation matrix such that $M^N = nI$. Let $\xi_0 \in \mathbb{Z}^N \setminus M\mathbb{Z}^N$ such that $s\xi_0 \notin M\mathbb{Z}^N$ for every integer $s = 1, \dots, n-1$. Then the vectors $\xi_j = M^j\xi$, $j = 0, \dots, N-1$ are linearly independent.*

Proof. Assume on the contrary that the vectors are not independent. Then they are not independent on the rational field, so that there exist integers c_j such that

$$\sum_{j=0}^{N-1} c_j\xi_j = 0. \tag{38}$$

Let $c_j = k_j n^{m_j}$, with $k_j \in \mathbb{Z}$, $n \nmid k_j$ and m_j is a non negative integer. Let $m = \min_j m_j$. We set $\{0, 2, \dots, N-1\} = A \cup B$, where $j \in A$ if and only if $m_j = m$. After simplification (38) becomes

$$\sum_{j\in A} k_j\xi_j = -\sum_{j\in B} k_j n^{m_j - m}\xi_j. \tag{39}$$

Setting $u = -\sum_{j\in B} k_j n^{m_j-m}\xi_j$, we note that $u \in M^N\mathbb{Z}^N$ (in particular $u = 0$ if $B = \emptyset$). Let $A = \{j_0, j_1, \dots, j_h\}$, where $j_i < j_{i+1}$. Then $k_{j_0} = mn + s$ for some integers s and m, with $1 \le s \le n-1$. We have from (39)

$$s\xi_{j_0} = -k_{j_1}\xi_{j_1} - \cdots k_{j_h}\xi_{j_h} - mn\xi_{j_0} + u \in M^{j_1}\mathbb{Z}^N.$$

Therefore, applying M^{-j_0} to both members, we get $s\xi_0 \in M^{j_1-j_0}\mathbb{Z}^N \subseteq M\mathbb{Z}^N$, against the assumption on ξ_0. ■

Thus we are faced with the problem of finding a vector ξ as in the assumption of the Proposition.

Lemma 3 *Let M, with $M\mathbb{Z}^N \subset \mathbb{Z}^N$, be a dilation matrix such that $M^N = nI$. For every vector $\xi \in \mathbb{Z}^N$, $\xi \neq 0$, there exists a unique positive integer $\kappa = \kappa(\xi)$ such that for every integer j one has $j\xi \in M\mathbb{Z}^N$ if and only if j is a multiple of $\kappa(\xi)$. Furthermore, $\kappa(\xi)$ is a divisor of n for every ξ.*

Proof. Given $\xi \neq 0$ let $\kappa(\xi)$ be the least positive integer m such that $m\xi \in \mathbb{Z}^N$. Such a minimum exists because of the assumption on M. Clearly every integer multiple of $\kappa(\xi)\xi$ belongs to $M\mathbb{Z}^N$. Viceversa, let j be such that $j\xi \in M\mathbb{Z}^N$ and write $j = k\kappa(\xi) + r$, where $0 \le r < \kappa(\xi)$. Then r must be

0, otherwise $r\xi = j\xi - \kappa(\xi)\xi \in M\mathbb{Z}^N$, against the definition of $\kappa(\xi)$. Finally, $\kappa(\xi)$ divides n since $n\xi = M^N\xi \in \mathbb{Z}^N$. ■

By this Lemma combined with the Proposition, if n is prime, then for every $\xi \in \mathbb{Z}^N \backslash M\mathbb{Z}^N$ the vectors $\xi_j = M^j\xi$, $j = 0, \ldots, N-1$ are linearly independent. However we can say more.

Lemma 4 *If $n = p_1 p_2 \ldots p_h$, with p_j pairwise distinct primes, then there exists $\xi \in \mathbb{Z}^N \setminus M\mathbb{Z}^N$ such that $\kappa(\xi) = n$.*

Proof. We will show that for every ξ with $\kappa(\xi) < n$ there exists ξ^* such that $\kappa(\xi) < \kappa(\xi^*)$. This clearly implies the thesis.

Let $\xi \in \mathbb{Z}^N \setminus M\mathbb{Z}^N$ be such that, say, $\kappa(\xi) = p_1 \cdots p_i$, with $i < h$ (cfr. Lemma 3). Dividing if necessary by the common factors of the coordinates, we may suppose that ξ is not a integer multiple of any vector $w \in \mathbb{Z}^N$ (note that dividing by the common factors does not decrease $\kappa(\xi)$). It follows that $\kappa(\xi)\xi \in M\mathbb{Z}^N$ but $\kappa(\xi)\xi \notin M^N\mathbb{Z}^N = n\mathbb{Z}^N$. Otherwise there would exist a vector $w \in \mathbb{Z}^N$ such that $\kappa(\xi)\xi = nw$, whence $\xi = p_{i+1} \cdots p_h w$, against our assumption on ξ. Hence there exists a larger j, $j < N$, such that $\kappa(\xi)\xi \in M^j\mathbb{Z}^N \backslash M^{j+1}\mathbb{Z}^N$, and, consequently, a vector $\eta \in \mathbb{Z}^N \backslash M\mathbb{Z}^N$ such that $\kappa(\xi)\xi = M^j\eta$. It follows $\kappa(\xi)M^{N-j}\xi = n\eta$, whence $p_{i+1} \cdots p_h\eta = M^{N-j}\xi \in M\mathbb{Z}^N$. Therefore $\kappa(\eta)$ is of the form, say, $p_{i+1} \cdots p_k$, with $k \leq h$.

Let for instance $\kappa(\eta) > \kappa(\xi)$ (they cannot be equal, by the assumption on n), and let us study $\kappa(\eta + \xi)$. From $\kappa(\eta + \xi)(\eta + \xi) \in M\mathbb{Z}^N$, we get also

$$\kappa(\xi)\kappa(\eta + \xi)\eta + \kappa(\xi)\kappa(\eta + \xi)\xi \in M\mathbb{Z}^N$$

whence $\kappa(\xi)\kappa(\eta + \xi)\eta \in M\mathbb{Z}^N$ and

$$\kappa(\xi)\kappa(\eta + \xi) = \ell\kappa(\eta) \tag{40}$$

for some integer ℓ. As $\kappa(\eta)$ and $\kappa(\xi)$ do not have common factors, $\kappa(\xi)$ must divide ℓ. Now, we note that $\kappa(\eta + \xi) = \kappa(\eta)$ is impossible. Otherwise from

$$\kappa(\eta)\eta + \kappa(\eta)\xi \in M\mathbb{Z}^N$$

we would get $\kappa(\eta)\xi \in M\mathbb{Z}^N$, which is absurd, as $\kappa(\eta)$ is not a multiple of $\kappa(\xi)$. Therefore $\ell > \kappa(\xi)$, whence from (40) we get $\kappa(\eta+\xi) > \kappa(\eta) > \kappa(\xi)$, as desired. If $\kappa(\eta) < \kappa(\xi)$ the same argument leads to $\kappa(\eta + \xi) > \kappa(\xi)$. Letting $\eta + \xi = \xi^*$, this concludes the proof . ■

¿From Lemma 4 and Proposition 1 we conclude that when M satisfy (37) with $n = p_1 p_2 \ldots p_h$, where the p_j are pairwise distinct primes, then for some vector $\xi_0 \in \mathbb{Z}^N \setminus M\mathbb{Z}^N$ the orbit

$$\left\{\xi_0, M\xi_0, M^2\xi_0, \ldots, M^{N-1}\xi_0\right\} \tag{41}$$

is a set of independent generators of $\mathbb{R}^N$, so that the spline φ_Ξ belongs to $L^2(\mathbb{R}^N)$.

Remark. If n is not the product of pairwise distinct primes, then it may be the case that no orbit (41) is a set of independent vectors. Here we give a counterexample.

Let $z > 1$ be any integer and let M be the following 6×6 dilation matrix

$$M = \begin{bmatrix} z & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & z^3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & z^2 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Note that M is the direct sum of three expansion matrices, namely, $M = A \oplus B \oplus C$, where A is 1×1 and satisfies $A^6 = z^6$, B is 3×3 and satisfies $B^2 = z^6 I$ and, finally, C is 2×2 and satisfies $C^3 = z^6 I$. Therefore $M^6 = z^6 I$, and 6 is the first power r such that M^r is diagonal. We see that (38) has nontrivial integer solutions for every ξ. To see this, let ξ be any vector in $\mathbb{Z}^6$ and write $\xi = \alpha + \beta + \gamma$, where

$$\alpha = (\xi_1, 0, \dots, 0),\ \beta = (0, \xi_2, \xi_3, \xi_4, 0, 0),\ \gamma = (0, \dots, 0, \xi_5, \xi_6).$$

Splitting into components, equation (38) becomes

$$\sum_{j=0}^{5} c_j A^j \alpha = \sum_{j=0}^{5} c_j B^j \beta = \sum_{j=0}^{5} c_j C^j \gamma = 0.$$

Thus it suffices that the coefficients c_j satisfy the system

$$\begin{aligned} c_0 + z^3 c_3 &= 0 \\ c_1 + z^3 c_4 &= 0 \\ c_2 + z^3 c_5 &= 0 \\ c_0 + z^2 c_2 + z^4 c_4 &= 0 \\ c_1 + z^2 c_3 + z^4 c_5 &= 0 \\ \sum_{j=0}^{5} c_j z^j &= 0 \end{aligned}$$

which has the integer solutions $c_0 = -z^4 c_4 + z^5 c_5$, $c_1 = -z^3 c_4$, $c_2 = -z^3 c_5$, $c_3 = z c_4 - z^2 c_5$, with c_4 and c_5 arbitrary.

With the exception of μ that we define as

$$\mu(\omega) = q_0(<\omega, \xi_0>)^{K-1}, \tag{42}$$

we now define formally all the relevant quantities as in the previous case (Theorem 3). Then we have the following Theorem, whose proof is identical with the proofs of Theorems 2 and 3.

Theorem 4 *Assume that M, with $M\mathbb{Z}^N \subset \mathbb{Z}^N$, is a dilation matrix satisfying (37) with $n = p_1p_2\dots p_h$, where the p_j are pairwise distinct primes. Let ξ_0 be such that the vectors $\xi_j = M\xi_{j-1}$, for $j = 0, \dots, N-1$ are independent. Let Ξ be of the form (27). Let μ be as in (42) and let the other relevant quantities be as in Theorem 3. Then the affine systems $\{\psi_{j,k,\ell}\}$ and $\left\{\widetilde{\psi}_{j,k,\ell}\right\}$ are affine frames dual to each other such that $\psi_\ell(M^{-1}\cdot)$ is a finite linear combination of shifts of φ_Ξ and $\widetilde{\psi}_\ell$ is compactly supported and has arbitrarily high regularity (dictated by inequality (36)).*

6 Generalized B–splines

In this section we drop assumption (26) and let M be any dilation matrix such that $M\mathbb{Z}^N \subset \mathbb{Z}^N$. As usual we set $d = |\det M|$. In the cosets of $\mathbb{Z}^N/M\mathbb{Z}^N$ we choose $K+1$ different sets of representatives, say, $\Gamma_0, \dots, \Gamma_K$ with $\Gamma_j = \{\gamma_{0,j}, \dots, \gamma_{d-1,j}\}$. To each set Γ_j it is associated a self–affine tile Q_j [16], [15], i.e. a compact set such that

$$MQ_j = \Gamma j + Q_j, \quad j = 0, \dots, K. \tag{43}$$

Let χ_{Q_j} denote the characteristic function of Q_j. We form the convolution product

$$\varphi_{r_0,\dots,r_K} = \underbrace{\chi_{Q_0} * \cdots * \chi_{Q_0}}_{r_0\text{–times}} * \cdots * \underbrace{\chi_{Q_K} * \cdots * \chi_{Q_K}}_{r_K\text{–times}}.$$

The function $\varphi_{r_0,\dots,r_K}$ is a refinable L^2 function called generalized B–spline [10] , [12]. The case of only one self–affine set was considered in [17]. We have

$$\widehat{\varphi}_{r_0,\dots,r_K}(\omega) = \widehat{\chi}_{Q_0}(\omega)^{r_0}\widehat{\chi}_{Q_1}(\omega)^{r_1}\cdots\widehat{\chi}_{Q_K}(\omega)^{r_K}.$$

Let $m_{r_0,\dots,r_K}$ denote the mask of the refinement equation. By (43) we have

$$m_{r_0,\dots,r_K}(\omega) = \prod_{j=0}^{K}\left[d^{-1}\sum_{k=0}^{d-1}\exp(i<\gamma_{k,j},\omega>)\right]^{r_j}.$$

We set

$$\sigma_j(\omega) = d^{-1} \sum_{k=0}^{d-1} \exp(i < \gamma_{k,j}, \omega >), \quad j = 0, \ldots, K.$$

It follows that we can write the mask of the refinement equation as

$$m_{r_0,\ldots,r_K}(\omega) = \tau_0(\omega)\mu(\omega),$$

where

$$\tau_0 = \widetilde{\tau}_0 = \sigma_0(\omega), \quad \mu(\omega) = \sigma_0(\omega)^{r_0-1} \prod_{j=1}^{K} \sigma_j(\omega)^{r_j}.$$

We also set

$$\nu(\omega) = \sigma_0(\omega)^{a-1}$$

for some positive (large) integer $a > 1$. Having defined τ_0 and $\widetilde{\tau}_0$, μ and ν the filters m_0 and $\widetilde{m}_0$ are defined according to Lemma 1. Finally we set

$$\begin{aligned} \widehat{\varphi}(\omega) &= \prod_{j=1}^{\infty} m_0(M^{*-1}\omega) = \mu(\omega)\widehat{\varphi}_{r_0,\ldots,r_K}(\omega) \\ \widehat{\widetilde{\varphi}}(\omega) &= \prod_{j=1}^{\infty} \widetilde{m}_0(M^{*-j}\omega) \\ &= \widehat{\chi}_{Q_0}(\omega) \left(\overline{\widehat{\chi}_{Q_0}\sigma_0}(\omega)\right)^{a-1} \prod_{j=0}^{\infty} (2 - \overline{\mu\nu(M^{*-j}\omega)}). \end{aligned}$$

Clearly the new primal scaling function φ is again a finite linear combination of translates of the original generalized B–spline and the dual scaling function has the same nature, plus a convolution with a compactly supported distribution. For $\widehat{\widetilde{\varphi}}$ we still have a decay similar to (36). In order to complete the definition of the filters m_ℓ and $\widetilde{m}_\ell$ we need only to find the τ_s and $\widetilde{\tau}_s$, $s = 1, \ldots, d-1$. To this end we notice that a construction by means of numerical unitary matrices (see e.g. [27] or [17, Lemma 2.3])) shows easily that do exist $d-1$ trigonometric polynomials $\tau_1, \ldots, \tau_{d-1}$ such that the $d \times d$ matrix

$$\left[\tau_s(\omega + \gamma_k^*)\right], \ \ s = 0, \ldots, d-1, \, k = 0, \ldots, d-1.$$

is unitary (here the γ_k^* describe a complete set of representatives of the dual group $2\pi(M^{*-1}\mathbb{Z}^N/\mathbb{Z}^N)$). Thus the filters τ_s satisfy (10) and they are the wavelets masks relative to the Haar filter $\sigma_0 = \tau_0$. Finally we set $\tau_s = \widetilde{\tau}_s$ for all $s = 0, \ldots, d$.

Having defined all the relevant quantities, the analogue Theorems 2, 3 and 4 follows easily. Thus the d wavelets (1) and (2) are the generators of dual affine frames. Here again the $\psi_\ell(M^{-1}\cdot)$ are finite linear combination of shifts of the generalized B–splines, and the dual wavelets are convolution of a B–spline and a compactly supported distribution.

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